Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2018) 42: 2707 – 2723
© TÜBİTAK
doi:10.3906/mat-1706-17

Relative stable (co)homology

Chaoling HUANG*, **Li YUAN**
Department of Mathematics, Finance, and Economics, Hanjiang Normal University, Shiyan, P.R. China

Received: 05.06.2017 • **Accepted/Published Online:** 17.08.2018 • **Final Version:** 27.09.2018

Abstract: For a fixed precovering class $\mathcal{X}$ and a fixed preenveloping class $\mathcal{Y}$, we first introduce the notions of relative stable cohomology $\widetilde{Ext}_{\mathcal{X}}(-,\ -)$ and relative stable homology $\widetilde{Tor}_{\mathcal{X}\mathcal{Y}}(-,\ -)$. Then we consider their properties and, more importantly, we study the stable (co)homology under the case of $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$ and the case of $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, and $\mathcal{I}_C(R)$. Finally, we generalize relative stable (co)homology from the case of R-modules to the case of R-complexes.

Key words: Precover, preenvelope, relative stable cohomology, relative stable homology, semidualizing module

1. Introduction

Vogel introduced a homology theory in the 1980s, but it was not published by Vogel. This theory first appeared in print in Goichot's paper in 1992, where it was called Tate–Vogel homology as a generalization of Tate homology for modules over finite group rings; see [13]. Avramov and Veliche also studied a generalization of Tate cohomology, which was called stable cohomology, and developed general techniques for computing stable cohomology; see [3]. Similar to Avramov and Veliche, the authors of [5] called Tate–Vogel homology stable homology. In that paper, they considered the finiteness of homological dimensions and the vanishing of stable homologies $\widetilde{Tor}(M,\ -)$ and $\widetilde{Tor}(-,\ N)$ for any R°-module M and R-module N. The balancedness of stable homology and the comparison of Tate homology were also considered.

The idea of relative homological algebra was first introduced by Eilenberg and Moore [8], and it was reinvigorated by Enochs, Jenda, and Torrecillas [9–11]. To date, many authors have studied related subjects; see [2, 4, 6, 12, 14, 15, 17, 18, 22, 23, 25].

The aim of this paper is to investigate the relative stable (co)homology. More precisely speaking, in Section 3, for a precovering class $\mathcal{X}$ and a preenveloping class $\mathcal{Y}$, we first give definitions of relative unbounded homology, relative stable homology, relative bounded cohomology, and relative stable cohomology.

We give many examples about precovering and preenveloping classes to illustrate how wide the definitions above are, and then we consider the properties of relative stable (co)homology.

Theorem 1.1 *Let* $\mathcal{W} \subseteq \mathcal{X}$ *be two precovering classes and* $\mathcal{Y}$ *be a preenveloping class. Assume that* $\mathcal{X}$ *and*

*Correspondence: clhuang1978@live.cn
2010 *AMS Mathematics Subject Classification:* 13D02, 13D07, 16E05, 16E30
The first author was supported by a guidance project of the scientific research plan of the Department of Education of Hubei Province (No. B2016162) the plan of science and technology innovation team of excellent young and middle-age of Hubei province (No. T201731) and the second author was supported by a project of the scientific research plan of the Department of Education of Hubei Province (No. D20163101).

$\mathcal{Y}$ satisfy that $X \otimes_R A$ is acyclic for any acyclic complex $X \in C_{\sqsupset}^{\mathcal{X}}(R^{\circ})$ and any R-module $A \in \mathcal{Y}$. Then $\overline{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N) \cong \overline{Tor}_i^{\mathcal{W}\mathcal{Y}}(M,\ N)$ for any R°-module M and R-module N.

In Section 4 and 5, we concretely consider relative stable (co)homology under the case of $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$ and the case of $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, and $\mathcal{I}_C(R)$, respectively. The following are given:

Proposition 1.2 *Let R be a right coherent ring. Let M be an R°-module and N be an R-module. Then $\overline{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N) = Tor_i^{\mathcal{F}\mathcal{F}}(M,\ N) \cong \widetilde{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N) = 0$ for any $i \geq 1$.*

Proposition 1.3 *Let R be a communicative Noetherian ring and C be faithfully semidualizing. Let M be an R°-module and N be an R-module. Then $\overline{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = Tor_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) \cong \widetilde{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = 0$ for any $i \geq 1$.*

In the last section, we investigate the relative stable (co)homology of complexes as a generalization of the relative stable (co)homology of modules.

Proposition 1.4 *Let M be an R°-complex and N be a homologically bounded above R-complex. Then $\overline{Tor}_i^{E(R)\mathcal{C}^{\mathcal{S}\mathcal{I}}(R)}(M,\ N) = 0$ and $Tor_i^{E(R)\mathcal{C}^{\mathcal{S}\mathcal{I}}(R)}(M,\ N) = \widetilde{Tor}_i^{E(R)\mathcal{C}^{\mathcal{S}\mathcal{I}}(R)}(M,\ N)$ for any $i \in \mathbb{Z}$.*

2. Preliminaries

In this section we first recall the definitions of the unbounded tensor product, the stable tensor product, the bounded Hom complex, and the stable Hom complex. Then, as in absolute homological algebra, we consider the isomorphisms about them in order to use them freely later.

Notation 2.1 *In this note, rings are all to be associative with a unit. For a ring R, by $\mathcal{M}(R)$ we denote the category of R-modules. $\mathcal{P} = \mathcal{P}(R)$, $\mathcal{F} = \mathcal{F}(R)$, and $\mathcal{I} = \mathcal{I}(R)$ are the subcategories of projective, flat, and injective R-modules, respectively.*

Definition 2.2 *An R-complex is a sequence of homomorphisms in $\mathcal{M}(R)$*

$$A = \ \cdots \xrightarrow{\partial_{n+1}^A} A_n \xrightarrow{\partial_n^A} A_{n-1} \xrightarrow{\partial_{n-1}^A} \cdots$$

such that $\partial_{n-1}^A \partial_n^A = 0$ for all n. The nth homology module is $H_n(A) = Ker(\partial_n^A)/Im(\partial_{n+1}^A)$. An R-complex A is exact or acyclic if $Ker(\partial_n^A) = Im(\partial_{n+1}^A)$ for all n.

For an R°-complex X and an R-complex Y, the tensor product $X \otimes_R Y$ is the $\mathbb{Z}$-complex with degree n term $(X \otimes_R Y)_n = \coprod_{i\in\mathbb{Z}}(X_i \otimes_R Y_{n-i})$ and differential given by $\partial(x \otimes y) = \partial_i^X(x) \otimes y + (-1)^i x \otimes \partial_{n-i}^y(y)$ for $x \in X_i$ and $y \in Y_{n-i}$. There are another two tensor products called the unbounded tensor product and the stable tensor product, respectively, which first appeared in [13].

Definition 2.3 *For an R°-complex X and an R-complex Y, the $\mathbb{Z}$-complex $X \overline{\otimes}_R Y$ with degree n term $(X \overline{\otimes}_R Y)_n = \prod_{i\in\mathbb{Z}}(X_i \otimes_R Y_{n-i})$ and differential on elementary tensors given by $\partial(x \otimes y) = \partial_i^X(x) \otimes y +$*

$(-1)^i x \otimes \partial^y_{n-i}(y)$ for $x \in X_i$ and $y \in Y_{n-i}$ is called the unbounded tensor product. It contains the tensor product $X \otimes_R Y$ as a subcomplex. Their quotient complex $(X \overline{\otimes}_R Y)/(X \otimes_R Y)$, denoted by $X \widetilde{\otimes}_R Y$, is called the stable tensor product.

For R-complexes X and Y, the Hom complex $Hom_R(X,Y)$ is the complex with degree l term $(Hom_R(X,Y))_l = \prod_{p\in\mathbb{Z}}(Hom_R(X_p, Y_{p+l}))$ and for $\psi = (\psi_p)_{p\in\mathbb{Z}} \in (Hom_R(X,Y))_l$ differential given by $\partial_l(\psi)_p = \partial^Y_{p+l}\psi_p - (-1)^l \psi_{p-1}\partial^X_p$. There are also another two Hom complexes called the bounded Hom complex and the stable Hom complex, respectively; see [3, 5, 13].

Definition 2.4 *For R-complexes X and Y, the bounded Hom complex $\overline{Hom}_R(X,Y)$ is the subcomplex of $Hom_R(X,Y)$ with degree l term $(\overline{Hom}_R(X,Y))_l = \coprod_{p\in\mathbb{Z}}(Hom_R(X_p, Y_{p+l}))$. The stable Hom complex $\widetilde{Hom}_R(X,Y)$ is the quotient complex $Hom_R(X,Y)/\overline{Hom}_R(X,Y)$.*

Now we consider the isomorphisms of complexes about $X\overline{\otimes}_R Y$, $X\widetilde{\otimes}_R Y$, $\overline{Hom}_R(X,Y)$, and $\widetilde{Hom}_R(X,Y)$, which are analogs of the absolute cases; see [10, 20].

Proposition 2.5 *Let X be a complex of finitely generated R-modules and $\{A^i, i \in \lambda\}$ be a class of complexes of R-modules. Then there are isomorphisms of complexes:*

(1) $\overline{Hom}_R(X, \coprod_{i\in\lambda} A^i) \cong \coprod_{i\in\lambda} \overline{Hom}_R(X, A^i)$;

(2) $\widetilde{Hom}_R(X, \coprod_{i\in\lambda} A^i) \cong \coprod_{i\in\lambda} \widetilde{Hom}_R(X, A^i)$.

Proof (1) For every $n \in \mathbb{Z}$,

$$(\overline{Hom}_R(X, \coprod_{i\in\lambda} A^i))_n = \coprod_{j\in\mathbb{Z}} Hom_R(X_j, (\coprod_{i\in\lambda} A^i)_{n+j}) \cong \coprod_{j\in\mathbb{Z}}\coprod_{i\in\lambda} Hom_R(X_j, A^i_{n+j}),$$

where the isomorphism holding for X_j is finitely generated for any $j \in \mathbb{Z}$. On the other hand, for every $n \in \mathbb{Z}$,

$$(\coprod_{i\in\lambda} \overline{Hom}_R(X, A^i))_n = \coprod_{i\in\lambda}(\overline{Hom}_R(X, A^i))_n \cong \coprod_{j\in\mathbb{Z}}\coprod_{i\in\lambda} Hom_R(X_j, A^i_{n+j}).$$

(2) First we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{Hom}_R(X, \coprod_{i\in\lambda} A^i) & \longrightarrow & Hom_R(X, \coprod_{i\in\lambda} A^i) & \longrightarrow & \widetilde{Hom}_R(X, \coprod_{i\in\lambda} A^i) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & & & \\
0 & \longrightarrow & \coprod_{i\in\lambda} \overline{Hom}_R(X, A^i) & \longrightarrow & \coprod_{i\in\lambda} Hom_R(X, A^i) & \longrightarrow & \coprod_{i\in\lambda} \widetilde{Hom}_R(X, A^i) & \longrightarrow & 0,
\end{array}$$

where the first square is commutative and the rows are short exact sequences of R-complexes. By diagram-chasing, there is a morphism from $\widetilde{Hom}_R(X, \coprod_{i\in\lambda} A^i)$ to $\coprod_{i\in\lambda} \widetilde{Hom}_R(X, A^i)$, which is an isomorphism.
□

One can proceed as in the above proof to prove the following results.

Proposition 2.6 *Let X be a complex of finitely presented R°-modules and $\{A^i, i\in\lambda\}$ be a class of complexes of R-modules. Then there are isomorphisms of complexes:*

(1) $X\overline{\otimes}_R(\prod_{i\in\lambda}A^i)\cong\prod_{i\in\lambda}(X\overline{\otimes}_R A^i)$;

(2) $X\widetilde{\otimes}_R(\prod_{i\in\lambda}A^i)\cong\prod_{i\in\lambda}(X\widetilde{\otimes}_R A^i)$.

Proposition 2.7 *Let R and S be commutative rings and S be a flat R-algebra. Let X be a complex of finitely presented R-modules and Y be a complex of R-modules. Then there are isomorphisms of complexes:*

(1) $\overline{Hom}_R(X,\ Y)\otimes_R S\cong\overline{Hom}_S(X\otimes_R S,\ Y\otimes_R S)$;

(2) $\widetilde{Hom}_R(X,\ Y)\otimes_R S\cong\widetilde{Hom}_S(X\otimes_R S,\ Y\otimes_R S)$.

Proof (1) For every $n\in\mathbb{Z}$,

$$\begin{aligned}(\overline{Hom}_R(X,\ Y)\otimes_R S)_n &= \coprod_{p\in\mathbb{Z}}(Hom_R(X_p,\ Y_{p+n}))\otimes_R S\\ &\cong \coprod_{p\in\mathbb{Z}}(Hom_R(X_p,\ Y_{p+n})\otimes_R S)\\ &\cong \coprod_{p\in\mathbb{Z}} Hom_R(X_p\otimes_R S,\ Y_{p+n}\otimes_R S)\\ &\cong \coprod_{p\in\mathbb{Z}} Hom_R((X\otimes_R S)_p,\ (Y\otimes_R S)_{p+n})\\ &= (\overline{Hom}_S(X\otimes_R S,\ Y\otimes_R S))_n,\end{aligned}$$

where the second isomorphism holds as every X_p is finitely presented by [10, Lemma 3.2.4].

(2) It is routine as in the proof of Proposition 2.5. □

3. Relative stable (co)homology

In this section, we first define the relative unbounded homology, the bounded cohomology, and the relative stable (co)homology. Then we consider their properties. Now we begin by recalling the following definition of precovers.

Definition 3.1 *Let $\mathfrak{F}$ be a class of R-modules. A homomorphism $\varphi:\ F\to M$ is called an $\mathfrak{F}$-precover of M if $F\in\mathfrak{F}$ and $Hom(F',\ F)\to Hom(F',\ M)\to 0$ is exact for all $F'\in\mathfrak{F}$. If every R-module admits an $\mathfrak{F}$-precover, then we say $\mathfrak{F}$ is a precovering class. An augmented proper $\mathfrak{F}$-resolution of an R-module M is a complex*

$$\mathbf{X}^+ = \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{\gamma} M \longrightarrow 0$$

with all $X_i\in\mathfrak{F}$ such that $Hom(F',\ X^+)$ is exact for any $F'\in\mathfrak{F}$. The truncated complex

$$\mathbf{X} = \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \longrightarrow 0$$

is called a proper $\mathfrak{F}$-resolution of M. Clearly, if $\mathfrak{F}$ is precovering, every R-module M has a proper $\mathfrak{F}$-resolution.

Lemma 3.2 *([10, Ex. 8.1.2, P. 169]) Let $\mathfrak{F}$ be a precovering class. Consider the diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & F_1' & \longrightarrow & F_0' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where the rows are augmented proper $\mathfrak{F}$-resolutions of M and M', respectively. Then $f : M \to M'$ induces a chain map of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & F_1' & \longrightarrow & F_0' & \longrightarrow & M' & \longrightarrow & 0, \end{array}$$

which is unique up to homotopy.

Dually, one can define an $\mathfrak{F}$-preenvelope, preenveloping class, and augmented proper $\mathfrak{F}$-coresolution. We surely have the dual case of the above lemma.

Now we give the following definitions as we promised to do.

Definition 3.3 *Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class. For any R°-module M and R-module N, there is an augmented proper $\mathcal{X}$-resolution of M*

$$\mathbf{X}^{+} = \cdots \longrightarrow X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\gamma} M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$, and an augmented proper $\mathcal{Y}$-coresolution of M

$$\mathbf{Y}^{+} = 0 \longrightarrow N \xrightarrow{\delta} Y_0 \xrightarrow{\partial_0^Y} Y_1 \longrightarrow \cdots$$

with all $Y_i \in \mathcal{Y}$. The ith homology $\mathbb{Z}$-module of complex $X\overline{\otimes}_R Y$, denoted by $\overline{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N)$, is called the ith relative unbounded homology module of M and N. $\widetilde{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N) = H_i(X\widetilde{\otimes}_R Y)$ is called the ith relative stable homology module of M and N.

Definition 3.4 *Let $\mathcal{X}$ be a precovering class. For any R-module M and N, there are augmented proper $\mathcal{X}$-resolutions of M and N*

$$\mathbf{X}^{+} = \cdots \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\gamma} M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$, and

$$\mathbf{Z}^{+} = \cdots \longrightarrow Z_1 \xrightarrow{\partial_1} Z_0 \xrightarrow{\gamma} N \longrightarrow 0$$

with all $Z_i \in \mathcal{X}$. The ith cohomology $\mathbb{Z}$-module of complex $\overline{Hom}_R(X,\ Z)$, denoted by $\overline{Ext}_{\mathcal{X}}^i(M,\ N)$, is called the ith relative bounded cohomology module of M and N. $\widetilde{Ext}_{\mathcal{X}}^i(M,\ N) = H^{i+1}(\widetilde{Hom}_R(X,\ Z))$ is called the ith relative stable cohomology module of M and N.

Since projective resolutions and injective resolutions (in absolute homology) are proper, if we set $\mathcal{X} = \mathcal{P}(R)$ and $\mathcal{Y} = \mathcal{I}(R)$, $\overline{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N)$ and $\widetilde{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N)$ are just $\overline{Tor}_i^{R}(M,\ N)$ and $\widetilde{Tor}_i^{R}(M,\ N)$, respectively, and $\overline{Ext}^i_{\mathcal{X}}(M,\ N)$ and $\widetilde{Ext}^i_{\mathcal{X}}(M,\ N)$ are just $\overline{Ext}^i_{R}(M,\ N)$ and $\widetilde{Ext}^i_{R}(M,\ N)$, respectively; see [3, 5, 13]. Since functors $-\overline{\otimes}_R-$, $-\widetilde{\otimes}_R-$, $\overline{Hom}_R(-,\ -)$, and $\widetilde{Hom}_R(-,\ -)$ preserve homotopy, by Lemma 3.2 and its duality, the above definitions are independent of the choices of (co)resolutions.

Here are some examples of precovering classes and preenveloping classes that can illustrate how wide the definitions above are.

Example 3.5 *(1) For any ring R, $\mathcal{P}(R)$ and $\mathcal{F}(R)$ are precovering, and $\mathcal{I}(R)$ is preenveloping. If R is a left Noetherian ring, then $\mathcal{I}(R)$ is precovering; see [10, Proposition 5.4.1]. If R is a right coherent ring, then $\mathcal{F}(R)$ is preenveloping by [10, Proposition 6.5.1].*

(2) Let R be a commutative Noetherian ring and C be a semidualizing module over R. Set $\mathcal{P}_C = \mathcal{P}_C(R) = \{M|M \cong P \otimes_R C,\ P \in \mathcal{P}(R)\}$; $\mathcal{F}_C = \mathcal{F}_C(R) = \{M|M \cong F \otimes_R C,\ F \in \mathcal{F}(R)\}$, and $\mathcal{I}_C = \mathcal{I}_C(R) = \{M|M \cong Hom_R(C,\ I)),\ I \in \mathcal{I}(R)\}$. Then $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are precovering and $\mathcal{I}_C(R)$ is preenveloping. If C is faithfully semidualizing, then $\mathcal{I}_C(R)$ are precovering and $\mathcal{F}_C(R)$ is preenveloping; see [16, Proposition 5.10].

(3) Let R be an n-Gorenstein ring. $\mathcal{GI}(R)$ is preenveloping by [10, Theorem 11.2.1]; $\mathcal{GP}(R)$ are precovering by [10, Theorem 11.5.1] and $\mathcal{GF}_C(R)$ are precovering by [24, Theorem A].

(4) Let R be a left coherent ring. Set $\mathcal{W} = \{M|fd_R(M) < \infty\}$. By [19, Theorem 3.8], $(^{\perp}\mathcal{W},\ \mathcal{W})$ is a complete cotorsion pair, so $^{\perp}\mathcal{W}$ is precovering, and $\mathcal{W}$ is preenveloping.

(5) Let $\mathcal{P}^n$ be the class of modules whose projective dimensions are less than or equal to n. By [1, Theorem 4.2], $(\mathcal{P}^n,\ \mathcal{P}^{n\perp})$ is a complete cotorsion pair. Thus, $\mathcal{P}^n$ is precovering, and $\mathcal{P}^{n\perp}$ is preenveloping.

(6) Let R be a ring with $glGpd_R(R) < \infty$. By [17, Theorem 5.1], $(\mathcal{GP}_C(R),\ \mathcal{GP}_C(R)^{\perp})$ is a complete cotorsion pair. Thus, $\mathcal{GP}_C(R)$ is precovering, and $\mathcal{GP}_C(R)^{\perp}$ is preenveloping.

It is easy to prove the next four propositions, but they are important.

Proposition 3.6 *Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class, which are closed under finite sums.*

(1) Let $0 \to A \to B \to C \to 0$ be a $Hom_R(-,\ \mathcal{Y})$-exact complex of R-modules. Then for any R°-module M, there is a long exact sequence

$$\cdots \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(M,\ A) \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(M,\ B) \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(M,\ C) \to \widetilde{Tor}_{n-1}^{\mathcal{X}\mathcal{Y}}(M,\ A) \to \cdots.$$

(2) Let $0 \to A \to B \to C \to 0$ be a $Hom_R(\mathcal{X},\ -)$-exact complex of R°-modules. Then for any R-module N, there is a long exact sequence

$$\cdots \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(A,\ N) \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(B,\ N) \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(C,\ N) \to \widetilde{Tor}_{n-1}^{\mathcal{X}\mathcal{Y}}(A,\ N) \to \cdots.$$

Proposition 3.7 *Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class. For any R°-module M and R-module N, there is a long exact sequence*

$$\cdots \to \widetilde{Tor}_n^{\mathcal{X}\mathcal{Y}}(M,\ N) \to Tor_n^{\mathcal{X}\mathcal{Y}}(M,\ N) \to \overline{Tor}_n^{\mathcal{X}\mathcal{Y}}(M,\ N) \to \widetilde{Tor}_{n-1}^{\mathcal{X}\mathcal{Y}}(M,\ N) \to \cdots.$$

Proposition 3.8 *Let $\mathcal{X}$ be a precovering class that is closed under finite sums.*

(1) Let $0 \to A \to B \to C \to 0$ be a $Hom_R(\mathcal{X},\ -)$-exact complex of R-modules. Then for any R-module M there is a long exact sequence

$$\cdots \to \widetilde{Ext}^n_{\mathcal{X}}(M,\ A) \to \widetilde{Ext}^n_{\mathcal{X}}(M,\ B) \to \widetilde{Ext}^n_{\mathcal{X}}(M,\ C) \to \widetilde{Ext}^{n+1}_{\mathcal{X}}(M,\ A) \to \cdots .$$

(2) Let $0 \to A \to B \to C \to 0$ be a $Hom_R(\mathcal{X},\ -)$-exact complex of R-modules. Then for any R-module N there is a long exact sequence

$$\cdots \to \widetilde{Ext}^n_{\mathcal{X}}(C,\ N) \to \widetilde{Ext}^n_{\mathcal{X}}(B,\ N) \to \widetilde{Ext}^n_{\mathcal{X}}(A,\ N) \to \widetilde{Ext}^{n+1}_{\mathcal{X}}(C,\ N) \to \cdots .$$

Proposition 3.9 *Let $\mathcal{X}$ be a precovering class. For any R-module M and R-module N, there is a long exact sequence*

$$\cdots \to \overline{Ext}^n_{\mathcal{X}}(M,\ N) \to Ext^n_{\mathcal{X}}(M,\ N) \to \widetilde{Ext}^n_{\mathcal{X}}(M,\ N) \to \overline{Ext}^{n+1}_{\mathcal{X}}(M,\ N) \to \cdots .$$

Theorem 3.10 *Let $\mathcal{W} \subseteq \mathcal{X}$ be two precovering classes and $\mathcal{Y}$ be a preenveloping class. Assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy that $X \otimes_R A$ is acyclic for any acyclic complex $X \in C^{\mathcal{X}}_{\sqsupset}(R^\circ)$ and $A \in \mathcal{Y}$. Then $\overline{Tor}^{\mathcal{X}\mathcal{Y}}_i(M,\ N) \cong \overline{Tor}^{\mathcal{W}\mathcal{Y}}_i(M,\ N)$ and $\widetilde{Tor}^{\mathcal{X}\mathcal{Y}}_i(M,\ N) \cong \widetilde{Tor}^{\mathcal{W}\mathcal{Y}}_i(M,\ N)$ for any R°-module M and R-module N.*

Proof (1) Let

$$\mathbf{X}^+ = \cdots \longrightarrow X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\gamma} M \longrightarrow 0$$

be an augmented proper $\mathcal{X}$-resolution of M and

$$\mathbf{W}^+ = \cdots \longrightarrow W_1 \xrightarrow{\partial_1^W} W_0 \xrightarrow{\beta} M \longrightarrow 0$$

be an augmented proper $\mathcal{W}$-resolution of M, and

$$\mathbf{Y}^+ = 0 \longrightarrow N \xrightarrow{\delta} Y_0 \xrightarrow{\partial_0^Y} Y_1 \longrightarrow \cdots$$

be an augmented proper $\mathcal{Y}$-coresolution of N. Then there is a quasi-isomorphism $\alpha : W \to X$, so $cone(\alpha)$ is acyclic. By the hypothesis and [5, Proposition 1.7], $cone(\alpha\overline{\otimes}_R Y) \cong cone(\alpha)\overline{\otimes}_R Y$ is acyclic. Then $\alpha\overline{\otimes}_R Y$ is a quasi-isomorphism. Therefore, $\overline{Tor}^{\mathcal{X}\mathcal{Y}}_i(M,\ N) = H_i(X\overline{\otimes}_R Y) \cong H_i(W\overline{\otimes}_R Y) = \overline{Tor}^{\mathcal{W}\mathcal{Y}}_i(M,\ N)$.

(2) First we have the following diagram of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W \otimes_R Y & \xrightarrow{\iota} & W\overline{\otimes}_R Y & \longrightarrow & W\widetilde{\otimes}_R Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha\otimes_R Y} & & \downarrow{\scriptstyle \alpha\overline{\otimes}_R Y} & & & & \\ 0 & \longrightarrow & X \otimes_R Y & \xrightarrow{\kappa} & X\overline{\otimes}_R Y & \longrightarrow & X\widetilde{\otimes}_R Y & \longrightarrow & 0, \end{array}$$

where the first square is commutative such that $\alpha \otimes_R Y$ and $\alpha\overline{\otimes}_R Y$ are quasi-isomorphisms by [7, Proposition 2.14] and the rows are short exact sequences of complexes. Since $Cone(\iota) \simeq Coker\iota = W\widetilde{\otimes}_R Y$ and $Cone(\kappa) \simeq$

$Coker\kappa = X\widetilde{\otimes}_R Y$, the square above induces a morphism of triangles in the homotopy category

$$\begin{array}{ccccccc} W\otimes_R Y & \xrightarrow{1_{W_0}} & W\overline{\otimes}_R Y & \longrightarrow & W\widetilde{\otimes}_R Y & \longrightarrow & \Sigma(W\otimes_R Y) \\ {\scriptstyle \alpha\otimes_R Y}\downarrow & & {\scriptstyle \alpha\overline{\otimes}_R Y}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle \Sigma(\alpha\otimes_R Y)}\downarrow \\ X\otimes_R Y & \xrightarrow{g_0'} & X\overline{\otimes}_R Y & \xrightarrow{\alpha_0'} & X\widetilde{\otimes}_R Y & \longrightarrow & \Sigma(X\otimes_R Y). \end{array}$$

Thus, θ is a quasi-isomorphism and $\widetilde{Tor}_i^{\mathcal{XY}}(M,\ N)\cong\widetilde{Tor}_i^{\mathcal{WY}}(M,\ N)$. □

Theorem 3.11 *Let $\mathcal{X}$ be a precovering class that is closed under direct sums. Let A be an R-module admitting a degree-wise finitely generated proper $\mathcal{X}$-resolution. $\{A^j, j\in\lambda\}$ is a class of R-modules such that the direct sum of whose proper $\mathcal{X}$-resolutions is a proper $\mathcal{X}$-resolution of $\coprod_{j\in\lambda}A^j$. Then $\overline{Ext}^i_{\mathcal{X}}(A,\ \coprod_{j\in\lambda}A^j)\cong\coprod_{j\in\lambda}\overline{Ext}^i_{\mathcal{X}}(A,\ A^j)$ and $\widetilde{Ext}^i_{\mathcal{X}}(A,\ \coprod_{j\in\lambda}A^j)\cong\coprod_{j\in\lambda}\widetilde{Ext}^i_{\mathcal{X}}(A,\ A^j)$.*

Proof Let X be a degree-wise finitely generated proper $\mathcal{X}$-resolution of the R-module A and X_{A^i} be a proper $\mathcal{X}$-resolution of the R-module A^i for every $i\in\lambda$. By the hypothesis, $\coprod_{i\in\lambda}X_{A^i}$ is a proper $\mathcal{X}$-resolution of $\coprod_{i\in\lambda}A^i$. By Proposition 2.5, $\overline{Hom}_R(X,\ \coprod_{i\in\lambda}X_{A^i})\cong\coprod_{i\in\lambda}\overline{Hom}_R(X,\ X_{A^i})$. Thus,

$$\begin{aligned}\overline{Ext}^i_{\mathcal{X}}(A,\ \coprod_{i\in\lambda}A^i) &= H^i(\overline{Hom}_R(X,\ \coprod_{i\in\lambda}X_{A^i}))\\ &\cong H^i(\coprod_{i\in\lambda}\overline{Hom}_R(X,\ X_{A^i}))\\ &\cong\coprod_{i\in\lambda}H^i(\overline{Hom}_R(X,\ X_{A^i}))\\ &=\coprod_{i\in\lambda}\overline{Ext}^i_{\mathcal{X}}(A,\ A^i).\end{aligned}$$

The proof of the second isomorphism is routine. □

One can prove the next result by proceeding as in the proof of Theorem 3.11 using Proposition 2.6.

Theorem 3.12 *Let R be a Noetherian ring. Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class that is closed under direct products. Let A be an R-module admitting a degree-wise finitely generated proper $\mathcal{X}$-resolution, and let $\{A^i, i\in\lambda\}$ be a class of R-modules such that direct products of whose proper $\mathcal{Y}$-coresolutions are a proper $\mathcal{Y}$-coresolution of $\prod_{i\in\lambda}A^i$. Then $\overline{Tor}_i^{\mathcal{XY}}(A,\ \prod_{i\in\lambda}A^i)\cong\prod_{i\in\lambda}\overline{Tor}_i^{\mathcal{XY}}(A,\ A^i)$ and $\widetilde{Tor}_i^{\mathcal{XY}}(A,\ \prod_{i\in\lambda}A^i)\cong\prod_{i\in\lambda}\widetilde{Tor}_i^{\mathcal{XY}}(A,\ A^i)$.*

Proposition 3.13 *Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class. Let M be an R°-module and $n\in\mathbb{Z}$. The following are equivalent:*

(i) The connecting morphism $\widetilde{Tor}_i^{\mathcal{XY}}(M,\ -)\to Tor_i^{\mathcal{XY}}(M,\ -)$ is an isomorphism for $i\geq n$.

(ii) $Tor_i^{\mathcal{XY}}(M,\ Y)=0$ for any $Y\in\mathcal{Y}$ and any $i\geq n$.

(iii) $\overline{Tor}_i^{\mathcal{XY}}(M,\ -)=0$ for any $i\geq n$.

Proof $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear.

$(ii) \Rightarrow (iii)$: One can prove it by using the same argument of $(ii) \Rightarrow (iii)$ in the proof of [5, Proposition 2.9]. □

Proposition 3.14 *Let $\mathcal{X}$ be a precovering class. For an R-module M and $n \in \mathbb{Z}$, the following are equivalent:*

(i) *The connecting morphism $Ext^i_{\mathcal{X}}(M, -) \to \widetilde{Ext}^i_{\mathcal{X}}(M, -)$ is an isomorphism for $i \geq n$.*

(ii) *$Ext^i_{\mathcal{X}}(M, X) = 0$ for any $X \in \mathcal{X}$ and any $i \geq n$.*

(iii) *$\overline{Ext}^{\mathcal{X}}_i(M, -) = 0$ for any $i \geq n$.*

Proof $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear.

$(ii) \Rightarrow (iii)$ can be proved by using the same method of $(ii) \Rightarrow (iii)$ in the proof of [3, Theorem 2.2]. □

4. The case of $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$

In this section we mainly investigate the isomorphic relations about $\overline{Ext}$, $\widetilde{Ext}$, $\overline{Tor}$, and $\widetilde{Tor}$ in the case of $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$ as the special parts of common ones. Under certain circumstances, they agree with their corresponding absolute counterparts. We first give the following result, which is a direct consequence of Theorem 3.11.

Proposition 4.1 *Let A be an R-module admitting a degree-wise finitely generated projective resolution and $\{A^i, i \in \lambda\}$ be a set of R-modules. Then $\overline{Ext}^i_R(A, \coprod_{i\in\lambda} A^i) \cong \coprod_{i\in\lambda} \overline{Ext}^i_R(A, A^i)$ and $\widetilde{Ext}^i_R(A, \coprod_{i\in\lambda} A^i) \cong \coprod_{i\in\lambda} \widetilde{Ext}^i_R(A, A^i)$.*

If R is a Noetherian ring, the next result is a direct consequence of Theorem 3.10. Note that any finitely generated projective R-module is finitely presented, so the next result holds by using Proposition 2.6.

Proposition 4.2 *Let A be an R-module admitting a degree-wise finitely generated projective resolution and $\{A^i, i \in \lambda\}$ be a class of R-modules. Then $\overline{Tor}^R_i(A, \prod_{i\in\lambda} A^i) \cong \prod_{i\in\lambda} \overline{Tor}^R_i(A, A^i)$ and $\widetilde{Tor}^R_i(A, \prod_{i\in\lambda} A^i) \cong \prod_{i\in\lambda} \widetilde{Tor}^R_i(A, A^i)$.*

The next result is essentially from [3, Proposition 3.2], which we still list for completeness, and we point here that we do not need the assumption "Notherian".

Proposition 4.3 *Let R and S be commutative rings and S be a flat R-algebra. Let M be an R-module admitting a degree-wise finitely generated projective resolution, and let N be any R-module. Then there are isomorphisms:*

(1) $\overline{Ext}^i_R(M, N) \otimes_R S \cong \overline{Ext}^i_S(M \otimes_R S, N \otimes_R S)$;

(2) $\widetilde{Ext}^i_R(M, N) \otimes_R S \cong \widetilde{Ext}^i_S(M \otimes_R S, N \otimes_R S)$ *for all $i \geq 0$.*

Proof (1) Let P_M be a degree-wise finitely generated projective resolution of M and P_N be a projective resolution of N. $P_M \otimes_R S$ is a projective resolution of $M \otimes_R S$, and $P_N \otimes_R S$ is a projective resolution of

$N \otimes_R S$, since S is a flat R-algebra. Thus,

$$\begin{aligned} \overline{Ext}^i_R(M,\ N) \otimes_R S &= H^i(\overline{Hom}_R(P_M,\ P_N)) \otimes_R S \\ &\cong H^i(\overline{Hom}_R(P_M,\ P_N) \otimes_R S) \\ &\cong H^i(\overline{Hom}_S(M \otimes_R S,\ N \otimes_R S)) \\ &= \overline{Ext}^i_S(M \otimes_R S,\ N \otimes_R S). \end{aligned}$$

(2) is routine. □

Corollary 4.4 *Let R and S be commutative rings. Let M be an R-module admitting a degree-wise finitely generated projective resolution, and let N be an R-module. Then there are isomorphisms:*

(1) $\overline{Ext}^i_R(M,\ N)_{\mathfrak{p}} \cong \overline{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\ N_{\mathfrak{p}})$;

(2) $\widetilde{Ext}^i_R(M,\ N)_{\mathfrak{p}} \cong \widetilde{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\ N_{\mathfrak{p}})$ for all $i \geq 0$ and all $\mathfrak{p} \in SpecR$.

Proposition 4.3 is the analog of absolute cohomology; see [10, Theorem 3.2.5]. For the Tor functor, the following result holds. Let R and S be commutative rings and let $R \to S$ be a ring homomorphism that makes S into a flat R-algebra. If M and N are R-modules, then $Tor^R_i(M,\ N) \otimes_R S \cong Tor^S_i(M \otimes_R S,\ N \otimes_R S)$; see [10, Theorem 2.1.11]. However, we do not know whether the analogs of functors $\overline{Tor}$ and $\widetilde{Tor}$ hold.

Next we consider the analog of [10, Theorem 3.2.13].

Proposition 4.5 *Let R and S be commutative rings and $\theta : S \to R$ be a ring homomorphism. Then any R-module M may be viewed as an S-module. Let X be an R-module admitting a degree-wise finitely generated projective resolution, and let Y be an R-module and C be an injective S-module. Then there are isomorphisms:*

(1) $\overline{Tor}^R_i(X,\ Hom_S(Y,\ C)) \cong Hom_S(\overline{Ext}^i_R(X,\ Y),\ C))$;

(2) $\widetilde{Tor}^R_i(X,\ Hom_S(Y,\ C) \cong Hom_S(\widetilde{Ext}^i_R(X,\ Y),\ C))$ for all $i \geq 0$.

Proof For any projective R-module P, we first show that $Hom_S(P,\ C)$ is an injective R-module. Let $0 \to A \to B$ be exact of R-modules, so $0 \to P \otimes_R A \to P \otimes_R B$ is exact for P being a projective R-module. Then $Hom_S(P \otimes_R B,\ C) \to Hom_S(P \otimes_R A,\ C) \to 0$ is exact, since C is an injective S-module. On the other hand, there is a commutative diagram

$$\begin{array}{ccccc} Hom_S(P \otimes_R B,\ C) & \longrightarrow & Hom_S(P \otimes_R A,\ C) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & \\ Hom_R(B,\ Hom_S(P,\ C)) & \longrightarrow & Hom_R(A,\ Hom_S(P,\ C)) & \longrightarrow & 0. \end{array}$$

Thus, $Hom_S(P,\ C)$ is an injective R-module.

Let P_X be a degree-wise finitely generated projective resolution of X and P_Y be a projective resolution of Y. $Hom_S(P_Y, C)$ is an injective resolution of $Hom_S(Y, C)$. Thus,

$$\begin{aligned} Hom_S(\overline{Ext}^i_R(X, Y), C)) &= Hom_S(H^i(\overline{Hom}_R(P_X, P_Y)), C) \\ &\cong H_i(Hom_S(\overline{Hom}_R(P_X, P_Y), C)) \\ &\cong H_i(P_X \overline{\otimes}_R Hom_S(P_Y, C)) \\ &= \overline{Tor}^R_i(X, Hom_S(Y, C)), \end{aligned}$$

where the second isomorphism holds by [5, Proposition A. 6]. The proof of (2) is routine. □

Corollary 4.6 *Let R be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue k. Let X be an R-module admitting a degree-wise finitely generated projective resolution and Y be an R-module. Then there are isomorphisms:*

(1) $\overline{Tor}^R_i(X, Y^v) \cong \overline{Ext}^i_R(X, Y)^v$;

(2) $\widetilde{Tor}^R_i(X, Y^v) \cong \widetilde{Ext}^i_R(X, Y)^v$ for all $i \geq 0$, where $-^v$ denotes the Matlis dual $Hom_R(-, E(k))$.

Next we consider the analog of [10, Theorem 3.2.15].

Proposition 4.7 *Let R and S be commutative rings and $\theta : S \to R$ be a ring homomorphism. Let X be an R-module admitting a degree-wise finitely generated projective resolution, and let Y be an R-module and C be a projective S-module. Then there are isomorphisms:*

(1) $\overline{Ext}^i_R(X, Y) \otimes_S C \cong \overline{Ext}^i_R(X, Y \otimes_S C)$;

(2) $\widetilde{Ext}^i_R(X, Y) \otimes_S C \cong \widetilde{Ext}^i_R(X, Y \otimes_S C)$ for all $i \geq 0$.

Proof For any projective R-module P, we first show that $P \otimes_S C$ is a projective R-module. Let $A \to B \to 0$ be exact of R-modules, so $Hom_R(P, A) \to Hom_R(P, B) \to 0$ is exact for P being a projective R-module. Then $Hom_S(C, Hom_R(P, A)) \to Hom_S(C, Hom_R(P, B)) \to 0$ is exact, since C is a projective S-module. On the other hand, there is a commutative diagram

$$\begin{array}{ccccc} Hom_R(P \otimes_S C, A) & \longrightarrow & Hom_R(P \otimes_S C, B) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & \\ Hom_S(C, Hom_R(P, A)) & \longrightarrow & Hom_S(C, Hom_R(P, B)) & \longrightarrow & 0. \end{array}$$

Thus, the upper row is exact and $P \otimes_S C$ is a projective R-module.

Let P_X be a degree-wise finitely generated projective resolution of X and P_Y be a projective resolution of Y. $P_Y \otimes_S C$ is a projective resolution of $P \otimes_S C$. Thus,

$$\begin{aligned} \overline{Ext}^i_R(X, Y) \otimes_S C &= H^i(\overline{Hom}_R(P_X, P_Y)) \otimes_S C \\ &\cong H^i(\overline{Hom}_R(P_X, P_Y) \otimes_S C) \\ &\cong H^i(\overline{Hom}_R(P_X, P_Y \otimes_S C)) \\ &= \overline{Ext}^i_R(X, Y \otimes_S C), \end{aligned}$$

where the second isomorphism holds by [5, Proposition A. 10]. The proof of (2) is routine. □

Proposition 4.8 *Let R be a right coherent ring. Let M be an R°-module and N be an R-module. Then $\overline{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N)=0$ and $Tor_i^{\mathcal{F}\mathcal{F}}(M,\ N)\cong\widetilde{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N)$ for any $i\geq 1$.*

Proof Let

$$\mathbf{F}^{+}=\cdots\longrightarrow F_1\xrightarrow{\partial_1^F}F_0\xrightarrow{\gamma}M\longrightarrow 0$$

be an augmented proper $\mathcal{F}$-resolution of M and

$$\mathbf{Y}^{+}=0\longrightarrow N\xrightarrow{\delta}Y_0\xrightarrow{\partial_0^Y}Y_{-1}\longrightarrow\cdots$$

be an augmented proper $\mathcal{F}$-coresolution of N. Thus, $\mathbf{F}^{+}\otimes_R Y_{-j}$ is exact for any $j\geq 0$. By [5, Proposition 1.7], $\mathbf{F}^{+}\overline{\otimes}_R\mathbf{Y}$ is exact. For any $i\geq 1$, $\overline{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N)=H_i(\mathbf{F}\overline{\otimes}_R\mathbf{Y})=H_i(\mathbf{F}^{+}\overline{\otimes}_R\mathbf{Y})=0$. By [7, Lemma 2.13], using the same argument above, one can prove that $Tor_i^{\mathcal{F}\mathcal{F}}(M,\ N)=0$. Then $\widetilde{Tor}_i^{\mathcal{F}\mathcal{F}}(M,\ N)=0$ for any $i\geq 1$. □

5. The case of $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, and $\mathcal{I}_C(R)$

As an application of Theorem 3.10, we give the following result:

Theorem 5.1 *Let $\mathcal{P}_C(R)\subseteq\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ be as in Example 3.5 (2). Then $\overline{Tor}_i^{\mathcal{P}_C(R)\mathcal{I}_C(R)}(M,\ N)\cong\overline{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(M,\ N)$ and $\widetilde{Tor}_i^{\mathcal{P}_C(R)\mathcal{I}_C(R)}(M,\ N)\cong\widetilde{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(M,\ N)$ for any R°-module M and R-module N.*

Proof By Theorem 3.10, we just need to check that $(C\otimes_R F)\otimes_R Hom(C,\ I)$ is acyclic for any acyclic complex $C\otimes_R F\in C_{\sqsupset}^{\mathcal{F}_C(R)}(R)$ and $I\in\mathcal{I}(R)$. Let

$$(\mathbf{C}\otimes_{\mathbf{R}}\mathbf{F})=\cdots\xrightarrow{f_2}C\otimes_R F_1\xrightarrow{f_1}C\otimes_R F_0\longrightarrow 0$$

and set $K_i=Kerf_i$. Since $C\otimes_R F_i\in\mathcal{B}_C(R)$, so is K_1. Then $Ext_R^i(C,\ K_1)=0$, so

$$0\longrightarrow Hom_R(C,\ K_1)\longrightarrow Hom_R(C,\ C\otimes_R F_1)\longrightarrow Hom_R(C,\ C\otimes_R F_0)\longrightarrow 0\ .$$

Continuing the process, we have that $F\cong Hom_R(C,\ C\otimes_R F)$ is exact. Therefore, $(C\otimes_R F)\otimes_R Hom(C,\ I)\cong F\otimes_R(C\otimes_R Hom(C,\ I))\cong F\otimes_R I$ is acyclic. □

As an application of Theorem 3.11, we give the following:

Theorem 5.2 *Let R be a commutative Noetherian ring, A be a finitely generated R-module, and $\{A^i,i\in\lambda\}$ be a class of R-modules. Then $\overline{Ext}^i_{\mathcal{P}_C(R)}(A,\ \coprod_{i\in\lambda}A^i)\cong\coprod_{i\in\lambda}\overline{Ext}^i_{\mathcal{P}_C(R)}(A,\ A^i)$ and $\widetilde{Ext}^i_{\mathcal{P}_C(R)}(A,\ \coprod_{i\in\lambda}A^i)\cong\coprod_{i\in\lambda}\widetilde{Ext}^i_{\mathcal{P}_C(R)}(A,\ A^i)$.*

Proof It is clear that $\mathcal{P}_C(R)$ is closed under direct sums. Let X_{A^i} be a proper $\mathcal{P}_C(R)$-resolution of the R-module A^i for every $i \in \lambda$. By [21, Lemma 3.2], since for any projective R-module P, $Hom_R(C \otimes_R P,\ X_{A^i})$ is exact, so is $Hom_R(C,\ X_{A^i})$. Thus, $Hom_R(C \otimes_R P,\ \coprod_{i\in\lambda} X_{A^i}) \cong Hom_R(P,\ Hom_R(C,\ \coprod_{i\in\lambda} X_{A^i})) \cong Hom_R(P,\ \coprod_{i\in\lambda} Hom_R(C,\ X_{A^i}))$ is exact. By Theorem 3.8, the result holds. □

As an application of Theorem 3.12, we give the following:

Theorem 5.3 *Let R be a commutative Noetherian ring, A be a finitely generated R-module, and $\{A^i, i \in \lambda\}$ be a class of R-modules. Then $\overline{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(A,\ \prod_{i\in\lambda} A^i) \cong \prod_{i\in\lambda} \overline{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(A,\ A^i)$ and $\widetilde{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(A,\ \prod_{i\in\lambda} A^i) \cong \prod_{i\in\lambda} \widetilde{Tor}_i^{\mathcal{F}_C(R)\mathcal{I}_C(R)}(A,\ A^i)$.*

Proposition 5.4 *Let R be a commutative ring. For any $i \in \mathbb{Z}$ and for R-modules M and N, there are isomorphisms $\overline{Tor}_i^{\mathcal{F}_C\mathcal{I}_C}(M,\ N) \cong \overline{Tor}_i^{R}(Hom_R(C,\ M),\ C \otimes_R N) \cong \overline{Tor}_i^{\mathcal{P}_C\mathcal{I}_C}(M,\ N)$ and $\widetilde{Tor}_i^{\mathcal{F}_C\mathcal{I}_C}(M,\ N) \cong \widetilde{Tor}_i^{R}(Hom_R(C,\ M),\ C \otimes_R N) \cong \widetilde{Tor}_i^{\mathcal{P}_C\mathcal{I}_C}(M,\ N)$.*

Proof Let $C \otimes_R F$ be a proper $\mathcal{P}_C$- or $\mathcal{F}_C$-resolution of M and $Hom_R(C,\ I)$ be a proper $\mathcal{I}_C$-coresolution of N. $F \cong Hom_R(C,\ C \otimes_R F)$ is a proper projective or flat resolution of $Hom_R(C,\ M)$, and $I \cong C \otimes_R Hom_R(C,\ I)$ is a proper injective resolution of $C \otimes_R N$. Thus,

$$\begin{aligned}
\overline{Tor}_i^{\mathcal{P}_C(\mathcal{F}_C)\mathcal{I}_C}(M,\ N) &= H_i((C \otimes_R F)\overline{\otimes}_R Hom_R(C,\ I)) \\
&\cong H_i(F\overline{\otimes}_R(C \otimes_R Hom_R(C,\ I))) \\
&\cong H_i(F\overline{\otimes}_R I) \\
&= \overline{Tor}_i^{R}(Hom_R(C,\ M),\ C \otimes_R N).
\end{aligned}$$

In view of [21, Theorem 3.10] and Proposition 3.7, the other two isomorphisms hold. □

We do not know whether the isomorphisms $\overline{Ext}^i_{\mathcal{P}_C}(M,\ N) \cong \overline{Ext}^i_R(Hom_R(C,\ M),\ Hom_R(C,\ N))$ and $\widetilde{Ext}^i_{\mathcal{P}_C}(M,\ N) \cong \widetilde{Ext}^i_R(Hom_R(C,\ M),\ Hom_R(C,\ N)$ hold.

Proposition 5.5 *Let R be a commutative Noetherian ring and C be faithfully semidualizing. Let M be an R°-module and N be an R-module. Then $\overline{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = 0$ and $Tor_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) \cong \widetilde{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N)$ for any $i \geq 1$.*

Proof Let F be an augmented proper $\mathcal{F}$-resolution of $Hom_R(C,\ M)$ and $C \otimes_R F'$ be an augmented proper $\mathcal{F}_C$-coresolution of N. $C \otimes_R F$ is an augmented proper $\mathcal{F}_C$-resolution of M. Thus, $F^+ \otimes_R F'_{-i}$ is exact for any $i \geq 0$. By [5, Proposition 1.7], $F^+\overline{\otimes}_R F'$ is exact. For any $i \geq 1$, $\overline{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = H_i(C \otimes_R F\overline{\otimes}_R C \otimes_R F') = H_i(F\overline{\otimes}_R C \otimes_R C \otimes_R F') = H_i(F^+\overline{\otimes}_R F') = 0$. By [7, Lemma 2.13], using the same argument above, one can prove that $Tor_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = 0$. Then $\widetilde{Tor}_i^{\mathcal{F}_C\mathcal{F}_C}(M,\ N) = 0$. □

6. Relative stable (co)homology of complexes

In this section, we investigate the relative stable (co)homology of complexes as a generalization of the relative stable (co)homology of modules.

Definition 6.1 *(1) Let $\mathcal{X}$ be a precovering class and $\mathcal{Y}$ be a preenveloping class for complexes. Let X be a precover of R°-complex M and let Y be a preenvelope of R-complex N. The ith homology $\mathbb{Z}$-module of complex $X\overline{\otimes}_R Y$, also denoted by $\overline{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N)$, is called the ith relative unbounded homology module of M and N. $\widetilde{Tor}_i^{\mathcal{X}\mathcal{Y}}(M,\ N) = H_i(X\widetilde{\otimes}_R Y)$ is called the ith relative stable homology module of M and N.*

(2) Let $\mathcal{X}$ be a precovering class for complexes. Let X and Z be precovers of R-complexes M and N, respectively. The ith cohomology $\mathbb{Z}$-module of complex $\overline{Hom}_R(X,\ Z)$, denoted by $\overline{Ext}_{\mathcal{X}}^i(M,\ N)$, is called the ith relative bounded cohomology module of M and N. $\widetilde{Ext}_{\mathcal{X}}^i(M,\ N) = H^{i+1}(\widetilde{Hom}_R(X,\ Z))$ is called the ith relative stable cohomology module of M and N.

We remark that the long exact sequences and the criteria that judge the connecting morphisms of stable (co)homology modules for complexes being isomorphisms are still correct, whereas we do not intend to list them here in order to avoid more white elephants.

Recall that complex P is called a semiprojective complex if each P_n is projective and $Hom(P, E)$ is exact for any exact complex E. Let $\mathcal{C}^{\mathcal{SP}}(R)$ denote the class of all semiprojective complexes. Then $\mathcal{C}^{\mathcal{SP}}(R)$ is a precovering class. Dually, the class of all semiinjective complexes $\mathcal{C}^{\mathcal{SI}}(R)$ is a preenveloping class. Let $E(R)$ be the class of exact complexes of flat R-modules such that they remain exact after applying $I \otimes_R -$ for any injective R°-module. $E(R)$ is precovering, and $E(R)^\perp$ is preenveloping by [24, Lemma 3.5]. The following results, which are analogs of absolute cases, are given.

Proposition 6.2 *Let R and S be commutative rings and S be a flat R-algebra. Let M be an R-complex admitting a degree-wise finitely generated semiprojective resolution, and let N be any R-complex. Then there are isomorphisms:*

(1) $\overline{Ext}_R^i(M,\ N) \otimes_R S \cong \overline{Ext}_S^i(M \otimes_R S,\ N \otimes_R S)$;

(2) $\widetilde{Ext}_R^i(M,\ N) \otimes_R S \cong \widetilde{Ext}_S^i(M \otimes_R S,\ N \otimes_R S)$ for all $i \geq 0$.

Proof (1) Let P_M be a degree-wise finitely generated semiprojective resolution of M and P_N be a semiprojective resolution of N. $P_M \otimes_R S$ is a semiprojective resolution of $M \otimes_R S$, and $P_N \otimes_R S$ is a semiprojective resolution of $N \otimes_R S$, since S is a flat R-algebra. Thus,

$$\begin{aligned}
\overline{Ext}_R^i(M,\ N) \otimes_R S &= H^i(\overline{Hom}_R(P_M,\ P_N)) \otimes_R S \\
&\cong H^i(\overline{Hom}_R(P_M,\ P_N) \otimes_R S) \\
&\cong H^i(\overline{Hom}_S(M \otimes_R S,\ N \otimes_R S)) \\
&= \overline{Ext}_S^i(M \otimes_R S,\ N \otimes_R S).
\end{aligned}$$

(2) is routine. □

Corollary 6.3 *Let R and S be commutative rings. Let M be an R-complex admitting a degree-wise finitely generated semiprojective resolution, and let N be any R-complex. Then there are isomorphisms:*

(1) $\overline{Ext}^i_R(M,\ N)_{\mathfrak{p}} \cong \overline{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\ N_{\mathfrak{p}})$;

(2) $\widetilde{Ext}^i_R(M,\ N)_{\mathfrak{p}} \cong \widetilde{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},\ N_{\mathfrak{p}})$ *for all* $i \geq 0$ *and all* $\mathfrak{p} \in SpecR$.

Proposition 6.4 *Let R and S be commutative rings and $\theta : S \to R$ be a ring homomorphism. Let X be an R-complex admitting a degree-wise finitely generated semiprojective resolution, Y be any R-complex, and C be an injective S-module. Then there are isomorphisms:*

(1) $\overline{Tor}^R_i(X,\ Hom_S(Y,\ C)) \cong Hom_S(\overline{Ext}^i_R(X,\ Y),\ C))$;

(2) $\widetilde{Tor}^R_i(X,\ Hom_S(Y,\ C) \cong Hom_S(\widetilde{Ext}^i_R(X,\ Y),\ C))$ *for all* $i \geq 0$.

Proof For any projective R-module P, $Hom_S(P,\ C)$ is an injective R-module. Let P_X be a degree-wise finitely generated semiprojective resolution of X and P_Y be a semiprojective resolution of Y. $Hom_S(P_Y,\ C)$ is an semiinjective resolution of $Hom_S(Y,\ C)$. Thus,

$$\begin{aligned} Hom_S(\overline{Ext}^i_R(X,\ Y),\ C)) &= Hom_S(H^i(\overline{Hom}_R(P_X,\ P_Y)),\ C) \\ &\cong H_i(Hom_S(\overline{Hom}_R(P_X,\ P_Y),\ C)) \\ &\cong H_i(P_X \overline{\otimes}_R Hom_S(P_Y,\ C)) \\ &= \overline{Tor}^R_i(X,\ Hom_S(Y,\ C)), \end{aligned}$$

where the second isomorphism holds by [5, Proposition A. 6]. The proof of (2) is routine. □

Corollary 6.5 *Let R be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue k. Let X be an R-complex admitting a degree-wise finitely generated semiprojective resolution, and let Y be any R-complex. Then there are isomorphisms:*

(1) $\overline{Tor}^R_i(X,\ Y^v) \cong \overline{Ext}^i_R(X,\ Y)^v$;

(2) $\widetilde{Tor}^R_i(X,\ Y^v) \cong \widetilde{Ext}^i_R(X,\ Y)^v$ *for all* $i \geq 0$, *where* $-^v$ *denotes the Matlis dual* $Hom_R(-,\ E(k))$.

Next we consider the analog of [10, Theorem 3.2.15].

Proposition 6.6 *Let R and S be commutative rings and $\theta : S \to R$ be a ring homomorphism. Let X be an R-complex admitting a degree-wise finitely generated semiprojective resolution, Y be any R-complex, and C be a projective S-module. Then there are isomorphisms:*

(1) $\overline{Ext}^i_R(X,\ Y) \otimes_S C \cong \overline{Ext}^i_R(X,\ Y \otimes_S C)$;

(2) $\widetilde{Ext}^i_R(X,\ Y) \otimes_S C \cong \widetilde{Ext}^i_R(X,\ Y \otimes_S C)$ *for all* $i \geq 0$.

Proof For any projective R-module P, $P \otimes_S C$ is a projective R-module. Let P_X be a degree-wise finitely generated semiprojective resolution of X and P_Y be a semiprojective resolution of Y. $P_Y \otimes_S C$

is a semiprojective resolution of $P \otimes_S C$. Thus,

$$\begin{aligned}\overline{Ext}^i_R(X,\ Y) \otimes_S C &= H^i(\overline{Hom}_R(P_X,\ P_Y)) \otimes_S C \\ &\cong H^i(\overline{Hom}_R(P_X,\ P_Y) \otimes_S C) \\ &\cong H^i(\overline{Hom}_R(P_X,\ P_Y \otimes_S C)) \\ &= \overline{Ext}^i_R(X,\ Y \otimes_S C),\end{aligned}$$

where the second isomorphism holds by [5, Proposition A. 10]. The proof of (2) is routine. □

We end this section by giving the following:

Proposition 6.7 *Let M be an R°-complex and N be a homologically bounded above R-complex. Then $\overline{Tor}_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N) = 0$ and $Tor_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N) = \widetilde{Tor}_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N)$ for any $i \in \mathbb{Z}$.*

Proof Let E be a semiprojective resolution of M and I be a semiinjective resolution of N. By [5, Proposition 1.7], $E \overline{\otimes}_R I$ is exact. For any $i \in \mathbb{Z}$, $\overline{Tor}_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N) = H_i(E \overline{\otimes}_R I) = 0$. Then $Tor_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N) = \widetilde{Tor}_i^{E(R)\mathcal{C}^{\mathcal{SI}}(R)}(M,\ N)$ for any $i \in \mathbb{Z}$. □

7. Acknowledgement

The authors thank the anonymous referees and editors for several corrections and values comments that improved the presentation at several points.

References

[1] Aldrich ST, Enochs EE. Envelopes and covers by modules of finite injective and projective dimensions. J Algebra 2001; 242: 447-459.

[2] Asadollahi J, Salarian S. Tate cohomology and Gorensteinness for triangulated categories. J Algebra 2006; 299: 480-502.

[3] Avramov LL, Veliche O. Stable cohomology over local rings. Adv Math 2007; 213: 93-139.

[4] Beligiannis A. Relative homological algebra and purity in triangulated categories. J Algebra 2000; 227: 268-361.

[5] Celikbas O, Christensen LW, Liang L, Piepmeyer G. Stable homology over associative rings. T Am Math Soc 2017; 369: 8061-8086.

[6] Christensen LW. Gorenstein dimensions. Lecture Notes in Mathematics 1747. Berlin, Germany: Springer-Verlag, 2000.

[7] Christensen LW, Frankild A, Holm H. On Gorenstein projective, injective and flat dimensions-A functorial description with applications. J Algebra 2006; 302: 231-279.

[8] Eilenberg S, Moore JC. Foundations of Relative Homological Algebra. Providence, RI, USA: AMS, 1965.

[9] Enochs EE, Jenda OMG. Gorenstein injective and projective modules. Math Z 1995; 220: 611-633.

[10] Enochs EE, Jenda OMG. Relative Homological Algebra. de Gruyter Expositions in Mathematics, Vol. 30. Berlin, Germany: Walter de Gruyter, 2000.

[11] Enochs EE, Jenda OMG, Torrecillas B. Gorenstein flat modules. Nanjing Daxue Xuebao Shuxue Bannian Kan 1993; 10: 1-9.

[12] Geng Y, Ding N. $\mathcal{W}$-Gorenstein modules. J Algebra 2011; 325: 132-146.

[13] Goichot F. Homologie de Tate-Vogel équivariante. J Pure Appl Algebra 1992; 82: 39-64 (in French).

[14] Holm H. Gorenstein homological dimensions. J Pure Appl Algebra 2004; 189: 167-193.

[15] Holm H, Jørgensen P. Semi-dualizing modules and related Gorenstein homological dimensions. J Pur App Alg 2006; 205: 423-445.

[16] Holm H, White D. Foxby equivalence over associative rings. J Math Kyto Univ 2007; 47: 781-808.

[17] Huang C. G_C-projective, injective and flat modules under change of rings. J Algebra Appl 2012; 11: 1-14.

[18] Huang C, Huang Z. Gorenstein syzygy modules. J Algebra 2010; 324: 3408-3419.

[19] Mao L, Ding N. Relative fp-projective modules. Comm Algebra 2005; 33: 1587-1602.

[20] Rotoman JJ. An Introduction to Homological Algebra. New York, NY, USA: Academic Press, 1978.

[21] Salimi M, Sather-Wagstaff S, Tavasoli E, Yassemi S. Relative Tor functors with respect to a semidualizing module. Algebra Represent Theor 2014; 17: 103-120.

[22] Trlifaj J. Covers, Envelopes and Cotorsion Theorie. Lecture Notes for the Workshop Homological Methods in Module Theory. Cortona, Italy, 2000.

[23] White D. Gorenstein projective dimension with respect to a semidualizing module. J Commut Algebra 2010; 2: 111-137.

[24] Yang G, Liang L. All modules have Gorenstein flat precovers. Commun Algebra 2014; 42: 3078-3085.

[25] Yang X, Liu Z. Strongly Gorenstein projective, injective and flat modules. J Algebra 2008; 320: 2659-2674.

[26] Zhu X. Resolving resolution dimensions. Algebra Represent Theor 2013; 16: 1165-1191.