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Convergence and Gundy's decomposition for noncommutative quasi-martingales

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Abstract: In this paper, we prove the bilaterally almost uniformly convergence of bounded $L_1(\mathcal{M})$-noncommutative quasi-martingales. We also prove Gundy's decomposition for noncommutative quasi-martingales. As an application, we prove that every relatively weakly compact quasi-martingale difference sequence in $L_1(\mathcal{M},\tau)$ whose sequence of norms is bounded away from zero is 2-co-lacunary.

Key words: Convergence, Gundy's decomposition, noncommutative quasi-martingales

1. Introduction

Inspired by quantum mechanics and probability, noncommutative probability has become an independent field of mathematical research. The study of noncommutative martingales originated at the beginning of the 70s. Today, the theory has achieved a satisfactory development and many classical martingale results have been transferred to the noncommutative setting.

The noncommutative quasi-martingale is a generalization of noncommutative martingales and the noncommutative analogue of classical quasi-martingales. In [4], we studied the duality theorems for some special quasi-martingale spaces. In [5], we studied interpolation in the noncommutative quasi-martingale setting. In the present paper, we continue to examine the noncommutative quasi-martingale. One of our main results is Theorem 3.3, which is the convergence of noncommutative quasi-martingales. Our proof uses Cuculescu's result in [Proposition 6, 2] and Doob's decomposition in [4]. The main novelty of our approach is Lemma 3.4, which extends the classical Doob maximal weak type (1,1) inequality for martingales to the quasi-martingale setting.

The other main result of this paper is Theorem 4.1, which concerns Gundy's decomposition of noncommutative quasi-martingales. Such kind of result of noncommutative martingales was first obtained by Parcet and Randrianantoanina [6]. Note that we can obtain our result by using Doob's decomposition in [4] and the result of Parcet and Randrianantoanina. However, this decomposition is not useful for our next proof. Hence we will give a direct decomposition of quasi-martingales in Theorem 4.1.

The paper is organized as follows. In Section 2, we set some basic definitions concerning noncommutative martingale and noncommutative quasi-martingale. In Section 3, we first prove Cuculescu's inequality for noncommutative quasi-martingales. Using the inequality, we prove the bilaterally almost uniformly convergence of bounded $L_1(\mathcal{M})$ quasi-martingales. In Section 4, we present Gundy's decomposition for noncommutative quasi-martingales and its application.

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2. Preliminary

Let $\mathcal{M}$ be a finite von Neumann algebra with a normal faithful finite trace τ. For $1 \leq p \leq \infty$, we denote by $L_p(\mathcal{M})$ the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. Note that if $p = \infty$, $L_\infty(\mathcal{M})$ is just $\mathcal{M}$ with the usual operator norm; also recall that for $1 \leq p < \infty$ the norm on $L_p(\mathcal{M})$ is defined by

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}, \quad x \in L_p(\mathcal{M}),$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is the usual modulus of x.

Let us recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$ and $\mathcal{E}_n$ (with $\mathcal{E}_0 = 0$) the conditional expectation with respect to $\mathcal{M}_n$. A noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all} \quad n \geq 1.$$

If additionally, $x = (x_n)_{n\geq 1} \subset L_p(\mathcal{M})$ for some $1 \leq p <\propto$, we call x an $L_p(\mathcal{M})$-martingale. In this case, we set $\|x\|_p = \sup_n \|x_n\|_p$. If $\|x\|_p < \infty$, then x is called a bounded $L_p(\mathcal{M})$-martingale. Note that the space of all bounded L_p-martingales, equipped with $\|\cdot\|_p$, is isometric to $L_p(\mathcal{M})$ for $p > 1$. This permits us to not distinguish a martingale and its final value x_∞ (if the latter exists). For more details on noncommutative martingales see [3].

Now we turn to the definition of noncommutative quasi-martingales, which is a generalization of noncommutative martingales.

Definition 2.1 *(see [4]).* *Let $1 \leq p \leq \infty$. An adapted sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$ is called a p-quasi-martingale with respect to $(\mathcal{M}_n)$ (or simply a quasi-martingale for $p = 1$) if*

$$V_p(x) := \sum_{n=1}^{\infty} \|\mathcal{E}_{n-1}(dx_n)\|_p < \infty.$$

If additionally, $x = (x_n)_{n\geq 1} \subset L_p(\mathcal{M})$ for some $1 \leq p < \infty$, we call x an $L_p(\mathcal{M})$-quasi-martingale. In this case, we set

$$\|x\|_p := \sup_n \|x_n\|_p + V_p(x).$$

If $\|x\|_p < \infty$, then x is called a bounded L_p-quasi-martingale.

The following decomposition plays an important role in this paper.

Lemma 2.2 (Doob's decomposition) *(see [4]).* *Let $1 \leq p \leq \infty$. Each p-quasi-martingale $x = (x_n)_{n\geq 1}$ can be uniquely decomposed as a sum of two sequences $y = (y_n)_{n\geq 1}$ and $z = (z_n)_{n\geq 1}$, where $y = (y_n)_{n\geq 1}$ is a martingale and $z = (z_n)_{n\geq 1}$ is a predictable p-quasi-martingale with $z_1 = 0$. Moreover, when $x = (x_n)_{n\geq 1}$ is $L_p(\mathcal{M})$-bounded, $y = (y_n)_{n\geq 1}$ and $z = (z_n)_{n\geq 1}$ are also $L_p(\mathcal{M})$-bounded.*

3. Convergence of noncommutative quasi-martingales

In this section, we focus on the convergence of noncommutative quasi-martingales. Of course, pointwise convergence does not make sense in the noncommutative setting. Note that in the commutative case almost every convergence and almost uniformly convergence are equivalent when the measure space is finite by Egoroff's theorem. The convergence bilaterally almost uniformly defined in the following is a replacement of almost every convergence when $\mathcal{M}$ is finite.

Definition 3.1 *(see [2]). Let $(x_n)_{n\geq 1}$ be a sequence in $L_0(\mathcal{M})$ and $x \in L_0(\mathcal{M})$. We say that $(x_n)_{n\geq 1}$ converges bilaterally almost uniformly (b.a.u. in short) to x, if for every $\varepsilon > 0$, there is a projection $p \in \mathcal{M}$ such that $\tau(1-p) < \varepsilon$ and*

$$\lim_{n\to\infty} \|p(x_n - x)p\| = 0.$$

As for the convergence of noncommutative martingales, we have the following results that we will need later.

Lemma 3.2 *(see [2]).*

(i) *Let $x = (x_n)_{n\geq 1}$ be a bounded L_p-martingale with $1 < p \leq \infty$. Then there exists $x_\infty \in L_p(\mathcal{M})$ such that x_n converges to x_∞ in $L_p(\mathcal{M})$ (in w^*-topology for $p = \infty$).*

(ii) *Let $x = (x_n)_{n\geq 1}$ be a bounded L_1-martingale. Then there exists $x_\infty \in L_1(\mathcal{M})$ such that $x_n \to x_\infty$ b.a.u.*

We extend the results of Lemma 3.2 to the case of quasi-martingales.

Theorem 3.3 (i) *Let $x = (x_n)_{n\geq 1}$ be a bounded L_p-quasi-martingale with $1 < p \leq \infty$. Then there exists $x_\infty \in L_p(\mathcal{M})$ such that x_n converges to x_∞ in $L_p(\mathcal{M})$ (in w^*-topology for $p = \infty$).*

(ii) *Let $x = (x_n)_{n\geq 1}$ be a bounded L_1-quasi-martingale. Then there exists $x_\infty \in L_1(\mathcal{M})$ such that $x_n \to x_\infty$ b.a.u.*

The following lemma is the key ingredient of our proof.

Lemma 3.4 (Cuculescu's inequality) *Let $x = (x_n)_{n\geq 1}$ be a bounded positive L_1-quasi-martingale and $s > 0$. Then there exists a decreasing sequence $(e_n)_{n\geq 0}$ of projections in $\mathcal{M}$ such that for every $n \geq 1$*

(i) *$e_n \in \mathcal{M}_n$;*

(ii) *e_n commutes with $e_{n-1}x_n e_{n-1}$;*

(iii) *$e_n x_n e_n \leq s e_n$;*

(iv) *moreover, if $e = \wedge_{n\geq 1} e_n$, then*

$$e x_n e \leq s e \text{ for all } n \geq 1 \text{ and } \tau(e^\perp) \leq \frac{2}{s}\|x\|_1.$$

Proof We define the required sequence $(e_n)_{n\geq 0}$ by induction. First let $e_0 = 1$. Then for every $n \geq 1$ define

$$e_n = \chi_{(0,s]}(e_{n-1}x_n e_{n-1}).$$

It is clear that properties (i), (ii), and (iii) are satisfied. We also have $ex_n e \leq se$ for all $n \geq 1$. Thus it remains to show the trace estimate (iv). It is easy to see that

$$(e_{k-1} - e_k)(e_{k-1}x_k e_{k-1})(e_{k-1} - e_k) \geq s(e_{k-1} - e_k).$$

Let $x_n = y_n + z_n (n \geq 1)$ be its Doob's decomposition. Then we deduce that

$$\begin{aligned} s\tau(1 - e_n) &= s\sum_{k=1}^{n} \tau(e_{k-1} - e_k) \\ &\leq \sum_{k=1}^{n} \tau((e_{k-1} - e_k)(e_{k-1}x_k e_k)(e_{k-1} - e_k)) \\ &= \sum_{k=1}^{n} \tau((e_{k-1} - e_k)x_k) \\ &= \sum_{k=1}^{n} \tau((e_{k-1} - e_k)y_k) + \sum_{k=1}^{n} \tau((e_{k-1} - e_k)z_k) \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Noting that $y = (y_n)_{n\geq 1}$ is a martingale, by the trace preserving of $\mathcal{E}_k$ we have

$$\mathrm{I} = \sum_{k=1}^{n} \tau(\mathcal{E}_k((e_{k-1} - e_k)y_n)) = \sum_{k=1}^{n} \tau((e_{k-1} - e_k)y_n) = \tau((1 - e_n)y_n).$$

Also

$$\begin{aligned} \mathrm{II} &= \sum_{k=1}^{n} \tau(e_{k-1}z_k) - \sum_{k=1}^{n} \tau(e_{k-1}z_{k-1}) - \tau(e_n z_n) \\ &= \sum_{k=1}^{n} \tau(e_{k-1}dz_k) - \tau(e_n z_n) \\ &\leq \sum_{k=1}^{n} \|dz_k\|_1 - \tau(e_n z_n). \end{aligned}$$

Combining the preceding estimates, we obtain

$$\begin{aligned} s\tau(1-e_n) &\leq \tau((1-e_n)(x_n-z_n)) + \sum_{k=1}^{n} \|dz_k\|_1 - \tau(e_n z_n) \\ &= \tau((1-e_n)x_n) - \tau(z_n) + \sum_{k=1}^{n} \|dz_k\|_1 \\ &\leq \|x_n\|_1 + 2\sum_{k=1}^{n} \|dz_k\|_1 \\ &\leq 2(\sup_n \|x_n\|_1 + V_1(x)). \end{aligned}$$

Letting $n \to \infty$, we get $\tau(e^{\perp}) \leq \frac{2}{s}(\sup_n \|x_n\|_1 + V_1(x))$. Thus the theorem is proved. □

Remark 3.5 *A general bounded L_1-quasi-martingale $x = (x_n)_{n\geq 1}$ can be decomposed into a sum of four bounded positive L_1-quasi-martingales. Indeed, let $x_n = y_n + z_n (n \geq 1)$ be Doob's decomposition. We decompose $(y_n)_{n\geq 1}$ into a sum of four positive martingales: $y_n = y_n^1 - y_n^2 + i(y_n^3 - y_n^4)(n \geq 1)$ and $(dz_n)_{n\geq 1}$ into a sum of four positive parts: $dz_n = dz_n^1 - dz_n^2 + i(dz_n^3 - dz_n^4)(n \geq 1)$. Noting that $(dz_n^k)_{n\geq 1}$ is predictable, we have*

$$\sum_{n=1}^{\infty} \|\mathcal{E}_{n-1} dz_n^k\|_1 = \sum_{n=1}^{\infty} \|dz_n^k\|_1 \leq \sum_{n=1}^{\infty} \|dz_n\|_1 < \infty.$$

Thus for each k, $(z_n^k)_{n\geq 1}$ is a positive quasi-martingale with $z_n^k = \sum_{j=1}^{n} dz_j^k$. Let $x_n^k = y_n^k + z_n^k (n \geq 1)$. Then $(x_n^k)_{n\geq 1}$ is a positive quasi-martingale for each k and $x_n = x_n^1 - x_n^2 + i(x_n^3 - x_n^4)(n \geq 1)$. Thus Lemma 3.4 is still valid for not necessarily positive quasi-martingales.

Proof of Theorem 3.3. Let $x = (x_n)_{n\geq 1}$ be a bounded L_p-martingale with $1 \leq p \leq \infty$ and $x_n = y_n + z_n (n \geq 1)$ its Doob's decomposition. Then $y = (y_n)_{n\geq 1}$ is a bounded $L_p(\mathcal{M})$-martingale. Suppose that $m > n$. Since

$$\|z_m - z_n\|_p \leq \sum_{k=n+1}^{m} \|dz_k\|_p \to 0 \quad \text{as} \quad m, n \to \infty, \tag{3.1}$$

$(z_n)_{n\geq 1}$ is a Cauchy sequence in $L_p(\mathcal{M})$ for $1 \leq p \leq \infty$.

(i) Let $1 < p \leq \infty$. Since there exists $y_\infty \in L_p(\mathcal{M})$ such that y_n converges to y_∞ in $L_p(\mathcal{M})$ (in w*-topology for $p = \infty$)by Lemma 3.2, it suffices to prove that $(z_n)_{n\geq 1}$ has the same convergence. This is true since $(z_n)_{n\geq 1}$ is a Cauchy sequence in $L_p(\mathcal{M})$.

(ii) It is a little more complicated for the case of $p = 1$. Since there exists $y_\infty \in L_1(\mathcal{M})$ such that $y_n \to y_\infty$ b.a.u. by Lemma 3.2, it suffices to prove the b.a.u. convergence of $(z_n)_{n\geq 1}$. For any $\varepsilon > 0$, there exists an increasing sequence (n_k) of nonnegative integers such that

$$\sum_{n\geq n_k} \|dz_n\|_1 < 4^{-k}\varepsilon. \tag{3.2}$$

For any nonnegative integer k, define

$$u_n^k = \begin{cases} 0, & n \le n_k \\ z_n - z_{n_k}, & n > n_k \end{cases}$$

and let $u^k = (u_n^k)_{n\ge1}$. For any fixed k, it follows from (3.2) that

$$\sum_{n=1}^{\infty} \|\mathcal{E}_{n-1} du_n^k\|_1 = \sum_{n\ge n_k} \|dz_n\|_1 < 4^{-k}\varepsilon \tag{3.3}$$

and

$$\sup_n \|u_n^k\|_1 = \sup_{n>n_k} \|z_n - z_{n_k}\|_1 \le \sum_{n\ge n_k} \|dz_n\|_1 < 4^{-k}\varepsilon. \tag{3.4}$$

Thus u^k is a bounded L_1-quasi-martingale. Then by Remark 3.5, for each k, there exists a projection $e^k \in \mathcal{M}$ such that $\sup\limits_n \|e^k u_n^k e^k\| \le 2\cdot 2^{-k}$ and

$$\tau(1-e^k) \le \frac{c}{2^{-k}}\|u^k\|_1 = c2^k(\sup_n \|u_n^k\|_1 + \sum_{n=1}^{\infty} \|\mathcal{E}_{n-1} du_n^k\|_1) < 2c\cdot 2^{-k}\varepsilon$$

by using (3.3) and (3.4). Letting $e = \wedge_{k\ge1} e_k$, we have that $\tau(1-e) \le \sum\limits_{k=1}^{\infty} \tau(1-e^k) < 2c\varepsilon$ and

$$\|e(z_n - z_{n_k})e\| = \|eu_n^k e\| \le \|e^k u_n^k e^k\| \le \frac{1}{2^{k-1}} \quad \text{for any } n \ge n_k.$$

Thus $(ez_ne)_{n\ge1}$ is a Cauchy sequence in $\mathcal{M}$ and hence there exists $v \in \mathcal{M}$ such that

$$\|ez_ne - v\| \to 0 \quad \text{as } n \to \infty.$$

On the other hand, it follows that from (3.1) there exists $z_\infty \in L_1(\mathcal{M})$ such that

$$\|z_n - z_\infty\|_1 \to 0$$

and hence

$$\|ez_ne - ez_\infty e\|_1 \to 0 \text{ as } n \to \infty.$$

Therefore, $v = ez_\infty e$ and

$$\|ez_ne - ez_\infty e\| \to 0 \text{ as } n \to \infty.$$

Thus z_n b.a.u. converges to z_∞. The proof is completed.

□

4. Gundy's decomposition and its application

In this section, we first prove Gundy's decomposition for bounded L_1-quasi-martingales. We should point out that this result can be simply obtained by using Doob's decomposition in [4] and Theorem 3.6 in [6]. However, we will give a direct proof of Theorem 4.1 since by this way we can get Equation (4.1) which is useful for the proof of Theorem 4.2 (that is the application of Theorem 4.1).

Theorem 4.1 *Let* $x = (x_n)_{n\geq 1}$ *be a bounded positive* L_1*-quasi-martingale and* $s > 0$*. Then there exist a bounded* L_1*-martingale* $y = (y_n)_{n\geq 1}$ *and three bounded* L_1*-quasi-martingales* $z = (z_n)_{n\geq 1}$*,* $v = (v_n)_{n\geq 1}$ *and* $w = (w_n)_{n\geq 1}$ *satisfying the following properties:*

(i) $x_n = y_n + z_n + v_n + w_n$ *for every* $n \geq 1$*;*

(ii) $y = (y_n)_{n\geq 1}$ *is a bounded* L_2*-martingale such that*

$$\|y\|_1 \leq 15\|x\|_1 \quad \textit{and} \quad \|y\|_2^2 \leq 8s\|x\|_1;$$

(iii) $\sum_{n=1}^{\infty} \|dz_n\|_1 \leq 12\|x\|_1$*;*

(iv) $\max(\tau(\vee_n r(dv_n)), \tau(\vee_n l(dw_n))) \leq \frac{2}{s}\|x\|_1$.

Proof

(i) Let $s > 0$ and $(e_n)_{n\geq 0}$ be the sequence of projections associated with x as in the proof of Lemma 3.4. We define the four required sequences as follows (with $\mathcal{E}_0 = 0$):

$$dy_n = e_n dx_n e_n - \mathcal{E}_{n-1}(e_n dx_n e_n), \tag{4.1}$$

$$dz_n = e_{n-1} dx_n e_{n-1} - dy_n,$$

$$dv_n = e_{n-1} dx_n e_{n-1}^{\perp}, \quad dw_n = e_{n-1}^{\perp} dx_n$$

for every $n \geq 1$. It is easy to see that (i) holds and $y = (y_n)_{n\geq 1}$ is a martingale.

(ii) Using the orthogonality of $(dy_k)_{k\geq 1}$ in $L_2(\mathcal{M})$ and the contractivity of $\mathcal{E}_{k-1}$, we get for every $n \geq 1$

$$\|y_n\|_2^2 = \sum_{k=1}^{n} \|dy_k\|_2^2 \leq 2\sum_{k=1}^{n} \|e_k dx_k e_k\|_2^2.$$

Since

$$e_k dx_k e_k = e_k(e_k x_k e_k - e_{k-1} x_{k-1} e_{k-1}) e_k,$$

we have

$$\|e_k dx_k e_k\|_2^2 \leq \|e_k x_k e_k - e_{k-1} x_{k-1} e_{k-1}\|_2^2 := \|a_k - a_{k-1}\|_2^2.$$

Now using the identity

$$(a_k - a_{k-1})^2 = a_k^2 - a_{k-1}^2 + a_{k-1}(a_{k-1} - a_k) + (a_{k-1} - a_k)a_{k-1}$$

and the tracial property of τ, we get that for any $k \geq 2$

$$\begin{aligned} \|a_k - a_{k-1}\|_2^2 &= \tau(a_k^2) - \tau(a_{k-1}^2) + 2\tau[a_{k-1}(a_{k-1} - a_k)] \\ &= \tau(a_k^2) - \tau(a_{k-1}^2) + 2\tau[a_{k-1}(a_{k-1} - \mathcal{E}_{k-1}(a_k))]. \end{aligned}$$

By the commutation of e_k and $e_{k-1}x_ke_{k-1}$, we have for any $k \geq 2$

$$\begin{aligned}\mathcal{E}_{k-1}(a_k) &= \mathcal{E}_{k-1}(e_ke_{k-1}x_ke_{k-1}e_k) \\ &\leq \mathcal{E}_{k-1}(e_{k-1}x_ke_{k-1}) \\ &= e_{k-1}\mathcal{E}_{k-1}(x_{k-1}+dx_k)e_{k-1} \\ &= a_{k-1}+e_{k-1}\mathcal{E}_{k-1}(dx_k)e_{k-1}.\end{aligned}$$

Set $a_k' = a_{k-1} - \mathcal{E}_{k-1}(a_k) + e_{k-1}\mathcal{E}_{k-1}(dx_k)e_{k-1} (k \geq 2)$. It follows that $a_k' \geq 0$ for any $k \geq 2$. Consequently, by $a_{k-1} \leq s$, we get for any $k \geq 2$

$$\begin{aligned}\tau[a_{k-1}(a_{k-1}-\mathcal{E}_{k-1}(a_k))] &= \tau(a_{k-1}a_k') - \tau(a_{k-1}e_{k-1}\mathcal{E}_{k-1}(dx_k)e_{k-1}) \\ &\leq \tau((a_k')^{\frac{1}{2}}a_{k-1}(a_k')^{\frac{1}{2}}) + s\|\mathcal{E}_{k-1}(dx_k)\|_1 \\ &\leq s\tau(a_k') + s\|\mathcal{E}_{k-1}(dx_k)\|_1 \\ &\leq s\tau(a_{k-1}-a_k) + 2s\|\mathcal{E}_{k-1}(dx_k)\|_1.\end{aligned}$$

Combining all preceding inequalities, we deduce that

$$\begin{aligned}\|y_n\|_2^2 &\leq 2\sum_{k=1}^{n}[\tau(a_k^2)-\tau(a_{k-1}^2)] + 4s\sum_{k=2}^{n}[\tau(a_{k-1}-a_k)+2\|\mathcal{E}_{k-1}(dx_k)\|_1] \\ &\leq 2\tau(a_n^2) + 4s\tau(a_1) - 4s\tau(a_n) + 8s\sum_{k=1}^{\infty}\|\mathcal{E}_{k-1}(dx_k)\|_1 \\ &\leq 8s\|x\|_1.\end{aligned}$$

Therefore, $\|y\|_2^2 \leq 8s\|x\|_1$. This is the second inequality of (ii). The first inequality is postponed after the proof of (iii).

(iii) Set

$$a_n = e_{n-1}x_ne_{n-1} - e_nx_ne_n \quad \text{and} \quad b_n = e_{n-1}x_{n-1}e_{n-1} - e_nx_{n-1}e_n$$

for any $n \geq 1$. Then $dz_n = a_n - b_n - \mathcal{E}_{n-1}(a_n - b_n) + \mathcal{E}_{n-1}(e_{n-1}dx_ne_{n-1})(n \geq 1)$. It follows that

$$\sum_{n=1}^{\infty}\|dz_n\|_1 \leq 2\sum_{n=1}^{\infty}(\|da_n\|_1 + \|db_n\|_1) + \sum_{n=1}^{\infty}\|\mathcal{E}_{n-1}(dx_n)\|_1.$$

Using the commutation of e_n and $e_{n-1}x_ne_{n-1}$, we have that

$$a_n = (e_{n-1}-e_n)e_{n-1}x_ne_{n-1} = (e_{n-1}-e_n)^{\frac{1}{2}}e_{n-1}x_ne_{n-1}(e_{n-1}-e_n)^{\frac{1}{2}} \geq 0.$$

Thus $\|a_n\|_1 = \tau(a_n)$ for any $n \geq 1$. Therefore,

$$\begin{aligned}\sum_{n=1}^{N} \|a_n\|_1 = \tau(\sum_{n=1}^{N} a_n) &= \tau[\sum_{n=1}^{N}(e_{n-1}x_n e_{n-1}) - \sum_{n=2}^{N}(e_{n-1}x_{n-1}e_{n-1})] - \tau(e_N x_N e_N) \\ &= \sum_{n=2}^{N} \tau(e_{n-1}\mathcal{E}_{n-1}(dx_n)e_{n-1}) + \tau(e_0 x_1 e_0) - \tau(e_N x_N e_N) \\ &\leq \sum_{n=2}^{N} \|\mathcal{E}_{n-1}dx_n\|_1 + \|x_1\|_1 + \|x_N\|_1,\end{aligned}$$

whence $\sum\limits_{n=1}^{\infty} \|a_n\|_1 \leq 2\|x\|_1$. Pass to the sum on b_n. Writing b_n as

$$b_n = e_{n-1}x_{n-1}e_{n-1}(e_{n-1} - e_n) + (e_{n-1} - e_n)e_{n-1}x_{n-1}e_n (n \geq 1)$$

and using that $e_{n-1}x_{n-1}e_{n-1} \leq s$, we get, for any $n \geq 1$, $\|b_n\|_1 \leq 2s\tau(e_{n-1} - e_n)$. Thus by Lemma 3.4,

$$\sum_{n=1}^{\infty} \|b_n\|_1 \leq 2s\tau(e^{\perp}) \leq 4\|x\|_1.$$

Putting the preceding inequalities together, we obtain $\sum\limits_{n=1}^{\infty} \|dz_n\|_1 \leq 12\|x\|_1$.

Now return to the first inequality of (ii). Note that

$$\begin{aligned}\sum_{n=1}^{N} e_{n-1}dx_n e_{n-1} &= \sum_{n=1}^{N}(e_{n-1}x_n e_{n-1} - e_{n-1}x_{n-1}e_{n-1}) \\ &= \sum_{n=1}^{N-1}(e_{n-1}x_n e_{n-1} - e_n x_n e_n) + e_{N-1}x_N e_{N-1} \\ &= \sum_{n=1}^{N-1} a_n + e_{N-1}x_N e_{N-1}.\end{aligned}$$

Thus $\| \sum\limits_{n=1}^{N} e_{n-1}dx_n e_{n-1}\|_1 \leq 3\|x\|_1$. Therefore,

$$\|y_N\|_1 \leq \|\sum_{n=1}^{N} e_{n-1}dx_n e_{n-1}\|_1 + \|z_N\|_1 \leq 15\|x\|_1,$$

whence the first inequality of (ii) holds.

(iv) By the definition of dv_n, for any $n \geq 1$, $r(dv_n) \leq e_{n-1}^{\perp} \leq e^{\perp}$. Therefore $\vee_n r(dv_n) \leq e^{\perp}$. Thus, by Lemma 3.4,

$$\tau(\vee_n r(dv_n)) \leq \frac{2}{s}\|x\|_1.$$

The second estimate on dw_n is proved in the same way. Thus the proof of Theorem 4.1 is complete. □

Theorem 4.1 is still valid for not necessarily positive quasi-martingales by Remark 3.6.

Now we give an application of Theorem 4.1 that concerns 2-co-lacunary sequences in noncommutative quasi-martingale spaces. We need the sequence of 2-co-lacunary, which we recall briefly below. We refer to [1] for more details. Let X be a Banach space. A sequence $(x_n)_{n\geq 1} \subset X$ is called 2-co-lacunary if there is a constant $\delta > 0$ such that for any finite sequence $(a_n)_{n\geq 1}$ of scalars,

$$\delta(\sum_{n\geq 1} |a_n|^2)^{\frac{1}{2}} \leq \| \sum_{n\geq 1} a_n x_n \|_X.$$

The following is the application of Theorem 4.1.

Theorem 4.2 *Let $(d_k)_{k\geq 1}$ be a relatively weakly compact quasi-martingale difference sequence in $L_1(\mathcal{M}, \tau)$ whose sequence of norms is bounded away from zero. Then $(d_n)_{n\geq 1}$ is a 2-co-lacunary sequence in $L_1(\mathcal{M}, \tau)$.*

Note that our proof mostly follows the proof of Theorem 3.6 in [6]. Our main novelty are Lemma 3.4 and Theorem 4.1, which extend Cuculescu's inequality and Gundy's decomposition for noncommutative martingales to the quasi-martingale setting.

Proof of Theorem 4.2 Set

$$y_n = \sum_{k=1}^{n} a_k d_k (n \geq 1),$$

then $y = (y_n)_{n\geq 1}$ is a quasi-martingale. By assumption that the series $\sum_k a_k d_k$ is convergent in $L_1(\mathcal{M}, \tau)$, thus $y = (y_n)_{n\geq 1}$ is L_1-bounded. Let (e_k) be the sequence of Cuculescu's projections associated with y as in the proof of Lemma 3.4. Imitating the proof of Theorem 3.6 in [6], we can obtain that

$$\inf\{\|e_k d_k e_k - \mathcal{E}_{k-1}(e_k d_k e_k)\|_2 : k \geq 1\} \geq \frac{1}{5}\sigma, \tag{4.2}$$

where

$$\sigma := \inf\{\|d_k\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} : k \geq 1\} > 0.$$

Let $y_n = b_n + c_n + v_n + w_n (n \geq 1)$ be Gundy's decomposition of $y = (y_n)_{n\geq 1}$ as in Theorem 4.1. Then we have $\|b\|_2^2 \leq c\lambda \|y\|_1$ and

$$db_k = e_k dy_k e_k - \mathcal{E}_{k-1}(e_k dy_k e_k).$$

Thus

$$db_k = e_k dy_k e_k - \mathcal{E}_{k-1}(e_k dy_k e_k) = a_k(e_k d_k e_k - \mathcal{E}_{k-1}(e_k d_k e_k)).$$

This gives

$$\sum_{k=1}^{\infty} |a_k|^2 \|e_k d_k e_k - \mathcal{E}_{k-1}(e_k d_k e_k)\|_2^2 \leq c\lambda \|y\|_1,$$

and therefore by (4.2) we conclude that

$$\sigma^2 \sum_{k=1}^{\infty} |a_k|^2 \leq 25 c\lambda \|y\|_1 < \infty.$$

The proof is complete. □

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