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Delta-shocks and vacuums as limits of flux approximation for the pressureless type system

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Abstract: In this paper, we investigate the phenomena of concentration and cavitation and the formation of delta-shocks and vacuum states in solutions of the pressureless type system with flux approximation. First, the Riemann problem of the pressureless type system with a flux perturbation is considered. A parameterized delta-shock and generalized constant density solution are obtained. Then we rigorously prove that, as the flux perturbation vanishes, they converge to the delta-shock and vacuum state of the pressureless type system, respectively. Secondly, by adding an artificial pressure term in the pressureless type system, we solve the Riemann problem of the system with a double parameter flux approximation including pressure. It is shown that, as the flux perturbations vanish, any two-shock Riemann solution tends to a delta-shock solution to the pressureless type system; any two-rarefaction-wave Riemann solution tends to a two-contact-discontinuity solution to the pressureless type system and the intermediate nonvacuum state in between tends to a vacuum state.

Key words: Pressureless type system, Riemann problem, delta-shock, vacuum state, flux approximation

1. Introduction

The pressureless type system reads as

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + (\rho u f(u))_x = 0, \end{cases} \tag{1.1}$$

where ρ and u represent the density and velocity, and $f(u)$ is given to be a smooth and strictly monotone function. The Riemann solutions of (1.1) were obtained in [12], which comprise two kinds: delta-shock and vacuum. Under the generalized δ-Rankine–Hugoniot relation and entropy condition, all of the existence, uniqueness, and stability of Riemann solutions of (1.1) to viscous perturbations were also proved in [12]. The Riemann problem of (1.1) with initial data containing Dirac delta functions was discussed in [15]. Huang [6] studied the Cauchy problem of (1.1). Moreover, when $f(u) = u$, the system (1.1) coincides with the Euler equations for pressureless fluids:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.2}$$

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which has been analyzed extensively; see [1, 2, 5, 7, 10, 12], etc. It has also been shown that δ-shocks and vacuum states do occur in the Riemann solutions. Since the two eigenvalues of (1.2) coincide, the occurrence of δ-shocks and vacuum states can be regarded as a result of resonance between the two characteristic fields. Such phenomena can also be regarded as the phenomena of concentration and cavitation in solutions to the Euler equations for compressible fluids as the pressure vanishes; for instance, see [3] for isentropic Euler equations, [8] for isothermal Euler equations, [4] for nonisentropic fluids, etc. Recently, motivated partly by [3, 4, 8], Yang and Liu [13, 14] introduced a two-parameter flux approximation including pressure in the isentropic Euler equations and adiabatic Euler equations to study the phenomena of concentration and cavitation as the flux approximation vanishes.

In this paper, we continue the topic of formation of delta-shocks and vacuum states in solutions and study the two-parameter flux perturbation problem in a pressureless type system. For this purpose, we add an artificial pressure term in the pressureless type system and consider the flux approximation system

$$\begin{cases} \rho_t + (\rho f(u) - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u f(u) - \epsilon_1 u^2 + \epsilon_2 p(\rho))_x = 0, \end{cases} \tag{1.3}$$

which is strictly hyperbolic and genuinely nonlinear, where $\epsilon_1, \epsilon_2 > 0$ are small parameters, ρ and u are in the physical region $\{(\rho, u) : \rho \geq \frac{2\epsilon_1}{f'(u)} > 0, |u| \leq V_0\}$ for some V_0, the pressure function $p(\rho)$ is taken to be the polytropic gas

$$p(\rho) = A\rho^\gamma, \quad \gamma > 1, \tag{1.4}$$

and $A > 0$ is a constant. For convenience, the constant A is chosen as $A = \frac{1}{\gamma}$ in the present paper. In addition, we assume $f''(u) \geq 0$ for the sake of convenience and the rest of the case can be discussed in a similar way.

We first investigate a pure flux approximation

$$\begin{cases} \rho_t + (\rho f(u) - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u f(u) - \epsilon_1 u^2)_x = 0, \end{cases} \tag{1.5}$$

which is the special case $\epsilon_2 = 0$ in (1.3). Taking initial data as

$$(u, \rho)(t = 0, x) = \begin{cases} (u_-, \rho_-), & x < 0, \\ (u_+, \rho_+), & x > 0, \end{cases} \tag{1.6}$$

where $(u_\pm, \rho_\pm)$ are constants, we solve the Riemann problem of (1.5). The Riemann solutions contain a parameterized delta-shock when $u_- > u_+$ and a generalized constant density solution ($\rho = \frac{2\epsilon_1}{f'(u)}$) when $u_- < u_+$. As $\epsilon_1 \to 0$, we show that any parameterized delta-shock of (1.5) converges to the delta-shock of the system (1.1). By contrast, any generalized constant density solution tends to the vacuum of the system (1.1).

Then we solve the Riemann problem (1.3) and (1.6) and analyze the limits of solutions as $\epsilon_1, \epsilon_2 \to 0$. It is shown that when $u_- > u_+$, any two-shock Riemann solution of (1.3) converges to the delta-shock solution of the system (1.1) as $\epsilon_1, \epsilon_2 \to 0$. It is also shown that when $u_- < u_+$, any two-rarefaction-wave Riemann solution of (1.3) tends to a two-contact-discontinuity solution of (1.1), and the nonvacuum intermediate state in between tends to a vacuum state as $\epsilon_1, \epsilon_2 \to 0$. Besides, when $\epsilon_1 = 0$ and $\epsilon_2 \to 0$ in (1.3), the limits of solutions were considered in [9]. Compared with [13], one can find that the results from [13] are recovered.

The organization of this paper is as follows. Section 2 reviews the solutions of (1.1) and (1.6). In Section 3, we solve the Riemann problem (1.5) and (1.6) and study the limits of solutions as $\epsilon_1 \to 0$. Section 4 solves the Riemann problem for the system (1.3) and investigates the limits of solutions as $\epsilon_1, \epsilon_2 \to 0$.

2. Riemann solutions of the pressureless type system

In this section, we briefly recall the Riemann solutions to the system (1.1). We refer to [12] for more details. The Riemann problem (1.1) and (1.6) can be solved by the following two cases under the assumption $f'(u) > 0$.

When $u_- < u_+$, the solution includes a vacuum state, two contact discontinuities, and constant states $(u_\pm, \rho_\pm)$. It can be expressed as

$$(u,\rho)(\xi) = \begin{cases} (u_-, \rho_-), & -\infty < \xi < f(u_-), \\ (f^{-1}(\xi), 0,), & f(u_-) \leq \xi \leq f(u_+), \\ (u_+, \rho_+), & f(u_+) < \xi < +\infty. \end{cases} \tag{2.1}$$

When $u_- > u_+$, the solution contains a δ-shock. A two-dimensional weighted δ-function $w(s)\,\delta_S$ supported on a smooth curve S parameterized as $t = t(s)$, $x = x(s) (c \leq s \leq d)$ can be defined by

$$\langle w(t(s))\delta_S, \varphi(t(s), x(s))\rangle = \int_c^d w(t(s))\varphi(t(s), x(s))\sqrt{t'(s)^2 + x'(s)^2}ds \tag{2.2}$$

for all test functions $\varphi(t,x) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R} = (-\infty, +\infty)$.

With this definition, we can introduce a δ-shock solution of (1.1) as follows:

$$\rho(t,x) = \rho_0(t,x) + w(t)\delta_S, \qquad u(t,x) = u_0(t,x), \tag{2.3}$$

where $S = \{(t, \sigma t) : 0 \leq t < \infty\}$,

$$\rho_0(t,x) = \rho_- + [\rho]\chi(x - \sigma t), u_0(t,x) = u_- + [u]\chi(x - \sigma t), w(t) = \frac{(\sigma[\rho] - [\rho f(u)])t}{\sqrt{1+\sigma^2}}, \tag{2.4}$$

in which $[h] = h_+ - h_-$ denotes the jump of function h across the discontinuity, σ is the velocity of the δ-shock, and $\chi(x)$ the characteristic function that is 0 when $x < 0$ and 1 when $x > 0$.

For any $\varphi(t,x) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, as shown in [12], the δ-shock solution constructed above satisfies

$$\begin{cases} \langle \rho, \varphi_t \rangle + \langle \rho f(u), \varphi_x \rangle = 0, \\ \langle \rho u, \varphi_t \rangle + \langle \rho u f(u), \varphi_x \rangle = 0, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} \langle \rho, \varphi \rangle &= \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho_0 \varphi dx dt + \langle w\delta_S, \varphi \rangle, \\ \langle \rho u, \varphi \rangle &= \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho u_0 \varphi dx dt + \langle u_\delta w \delta_S, \varphi \rangle, \end{aligned} \tag{2.6}$$

and $u|_S = u_\delta$, $f(u)|_S = \sigma$.

Then we can deduce the generalized Rankine-Hugoniot relation

$$\frac{dx}{dt}=\sigma, \qquad \frac{d(w\sqrt{1+\sigma^2})}{dt}=\sigma[\rho]-[\rho f(u)], \qquad \frac{d(wu_\delta\sqrt{1+\sigma^2})}{dt}=\sigma[\rho u]-[\rho u f(u)], \tag{2.7}$$

and

$$\sigma=f(u_\delta). \tag{2.8}$$

In addition, the entropy condition is supplemented as

$$f(u_+)<\sigma=f(u_\delta)<f(u_-), \tag{2.9}$$

which is equivalent to

$$u_+<u_\delta<u_- \tag{2.10}$$

under the condition $f'(u)>0$.

3. Riemann solutions and limit analysis of (1.5) as $\epsilon_1\to 0$

In this section, we solve the Riemann problem (1.5) and (1.6), then analyze the limit of Riemann solutions as $\epsilon_1\to 0$.

3.1. Riemann solutions of (1.5)

The system (1.5) provides two eigenvalues $\lambda_i=f(u)$ with the associated eigenvectors $r_i=(1,0)^\top$ satisfying $\nabla\lambda_i\cdot r_i=0$, where $i=1,2$, which means that it is full linear degenerate. Therefore, the elementary waves of (1.5) only involve contact discontinuities.

Performing the self-similar transformation $\xi=x/t$, we can find that the system (1.5) has the singular solution

$$\begin{cases}\rho=\dfrac{2\epsilon_1}{f'(u)},\\ \xi=f(u),\end{cases} \tag{3.1}$$

which is called generalized constant density. The elementary wave is contact discontinuity

$$J:\quad \omega=\xi=f(u_-)=f(u_+). \tag{3.2}$$

In the (u,ρ)-plane, two states (u_-,ρ_-) and (u_+,ρ_+) can be connected by the contact discontinuity if and only if they are located on the line $u=u_-=u_+$.

Now we can construct the Riemann solutions by the following two cases.

For the case $u_-<u_+$, the solutions of Riemann problem (1.5) and (1.6) can be solved by a generalized constant density and two contact discontinuities besides two constant states (see Figure 1), and can be given as

$$(u,\rho)(\xi)=\begin{cases}(u_-,\rho_-), & -\infty<\xi<f(u_-),\\ (f^{-1}(\xi),\frac{2\epsilon_1}{f'(u)}), & f(u_-)\le\xi\le f(u_+),\\ (u_+,\rho_+), & f(u_+)<\xi<+\infty.\end{cases} \tag{3.3}$$

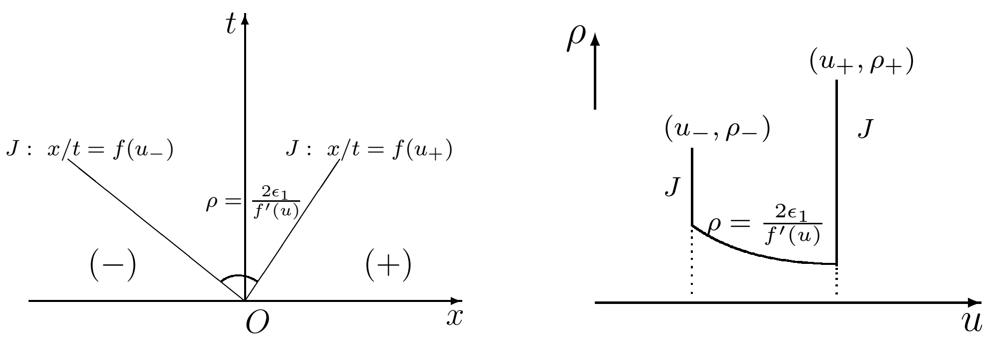

Figure 1. Generalized constant density.

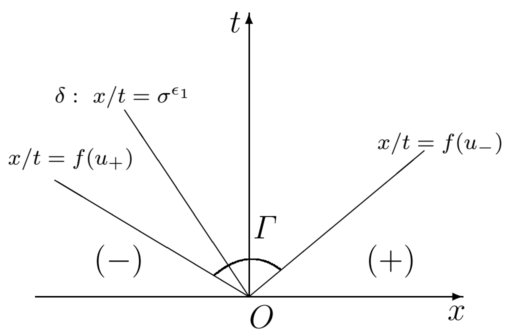

Figure 2. Delta-shock.

For the case $u_- > u_+$, as shown in Figure 2, the singularity of solutions must develop in the region Γ due to the overlap of the characteristic lines. Therefore, we use a delta-shock to construct the Riemann solution. We seek a delta-shock solution of (1.5) with discontinuity $x = x(t)$ in the form

$$(u,\rho)(t,x) = \begin{cases} (u_-,\rho_-)(t,x), & x < x(t), \\ \Big(u_\delta^{\epsilon_1}, w^{\epsilon_1}(t)\delta(x-x(t))\Big), & x = x(t), \\ (u_+,\rho_+)(t,x), & x > x(t), \end{cases} \tag{3.4}$$

where $\delta(\cdot)$ is the Dirac measure and $x(t) \in C^1$. Then we can get that (3.4) satisfies the generalized Rankine–Hugoniot relation

$$\begin{cases} \dfrac{dx}{dt} = \sigma^{\epsilon_1}, \\ \dfrac{d(w^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2})}{dt} = \sigma^{\epsilon_1}[\rho] - [\rho f(u) - 2\epsilon_1 u], \\ \dfrac{d(w^{\epsilon_1}u_\delta^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2})}{dt} = \sigma^{\epsilon_1}[\rho u] - [\rho u f(u) - \epsilon_1 u^2], \end{cases} \tag{3.5}$$

and

$$\sigma^{\epsilon_1} = f(u_\delta^{\epsilon_1}). \tag{3.6}$$

The generalized Rankine–Hugoniot relation describes the relationship among the location, propagation speed, weight, and assignment of u on the discontinuity. In addition, the discontinuity should satisfy the entropy condition

$$u_+ < u_\delta^{\epsilon_1} < u_-. \tag{3.7}$$

Thus, this Riemann problem is reduced to solving (3.5) and (3.6) with the initial conditions

$$t = 0: \ x(0) = 0, \ w^{\epsilon_1}(0) = 0. \tag{3.8}$$

According to the knowledge about delta-shocks in [12], we find that $w^{\epsilon_1}(t)$ is a linear function of t, σ^{ϵ_1} and $u_\delta^{\epsilon_1}$ are constants. Therefore, a delta-shock of (1.5) and (1.6) can be assumed to take the form

$$\delta: \quad x(t) = \sigma^{\epsilon_1} t, \quad w^{\epsilon_1}(t) = w_0^{\epsilon_1} t, \quad u_\delta^{\epsilon_1}(t) = u_\delta^{\epsilon_1}, \tag{3.9}$$

where σ^{ϵ_1}, $w_0^{\epsilon_1}$, and $u_\delta^{\epsilon_1}$ are to be determined constants. Substituting (3.9) into (3.5) and (3.6) yields

$$\begin{cases} w_0^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2} = [\rho]\sigma^{\epsilon_1} - [\rho f(u) - 2\epsilon_1 u], \\ w_0^{\epsilon_1} u_\delta^{\epsilon_1}\sqrt{1+(\sigma^{\epsilon_1})^2} = [\rho u]\sigma^{\epsilon_1} - [\rho u f(u) - \epsilon_1 u^2], \\ \sigma^{\epsilon_1} = f(u_\delta^{\epsilon_1}), \end{cases} \tag{3.10}$$

which gives

$$([\rho]u_\delta^{\epsilon_1} - [\rho u]) f(u_\delta^{\epsilon_1}) - [\rho f(u) - 2\epsilon_1 u] u_\delta^{\epsilon_1} + [\rho u f(u) - \epsilon_1 u^2] = 0. \tag{3.11}$$

Taking the entropy condition (3.7) into account, we analyze the solutions of function equation (3.11). Set

$$F(u_\delta^{\epsilon_1}) = ([\rho]u_\delta^{\epsilon_1} - [\rho u]) f(u_\delta^{\epsilon_1}) - [\rho f(u) - 2\epsilon_1 u] u_\delta^{\epsilon_1} + [\rho u f(u) - \epsilon_1 u^2]. \tag{3.12}$$

One can calculate that

$$\begin{aligned} F(u_+) &= ([\rho]u_+ - [\rho u]) f(u_+) - [\rho f(u) - 2\epsilon_1 u] u_+ + [\rho u f(u) - \epsilon_1 u^2] \\ &= \rho_-(u_- - u_+)(f(u_+) - f(u_-)) + \epsilon_1 (u_+ - u_-)^2 \\ &= -\rho_-[u][f(u)] + \epsilon_1 [u]^2, \end{aligned} \tag{3.13}$$

which yields $F(u_+) < 0$ for $\epsilon_1 < \dfrac{\rho_-[f(u)]}{[u]}$.

Similarly,

$$F(u_-) = \rho_+[u][f(u)] - \epsilon_1 [u]^2, \tag{3.14}$$

and $F(u_-) > 0$ for $\epsilon_1 < \dfrac{\rho_+[f(u)]}{[u]}$.

Thus, we have

$$F(u_+)F(u_-) < 0 \quad \text{for} \quad \epsilon_1 < \min\Big(\frac{\rho_-[f(u)]}{[u]}, \frac{\rho_+[f(u)]}{[u]}\Big) := a. \tag{3.15}$$

Furthermore, differentiating $F(u_\delta^{\epsilon_1})$ in (3.12) with respect to $u_\delta^{\epsilon_1}$ leads to

$$\begin{aligned} F'(u_\delta^{\epsilon_1}) =& ([\rho]u_\delta^{\epsilon_1} - [\rho u]) f'(u_\delta^{\epsilon_1}) + [\rho] f(u_\delta^{\epsilon_1}) - [\rho f(u) - 2\epsilon_1 u] \\ =& \rho_-(f(u_-) - f(u_\delta^{\epsilon_1})) + \rho_+(f(u_\delta^{\epsilon_1}) - f(u_+)) \\ &+ (\rho_-(u_- - u_\delta^{\epsilon_1}) + \rho_+(u_\delta^{\epsilon_1} - u_+)) f'(u_\delta^{\epsilon_1}) + 2\epsilon_1 (u_+ - u_-) > 0, \end{aligned} \tag{3.16}$$

for

$$\epsilon_1 < \frac{\rho_-(f(u_-) - f(u_\delta^{\epsilon_1})) + \rho_+(f(u_\delta^{\epsilon_1}) - f(u_+)) + (\rho_-(u_- - u_\delta^{\epsilon_1}) + \rho_+(u_\delta^{\epsilon_1} - u_+))f'(u_\delta^{\epsilon_1})}{2(u_- - u_+)} := b.$$

Therefore, taking $\overline{\epsilon_1} = \min(a, b)$, one can get that when $0 < \epsilon_1 < \overline{\epsilon_1}$, there exists one and only one zero point of function $F(u_\delta^{\epsilon_1})$ in (u_+, u_-) according to the zero point theorem. This means that when $0 < \epsilon_1 < \overline{\epsilon_1}$, Eq. (3.11) has a unique solution denoted by $u_\delta^{\epsilon_1}$ under the entropy condition (3.7). Then we can return to (3.10) to solve the σ^{ϵ_1} and $w_0^{\epsilon_1}$ uniquely. Thus, we have the following theorem.

Theorem 3.1 *Let $u_- > u_+$. For $0 < \epsilon_1 < \overline{\epsilon_1}$, the Riemann problem (1.5) and (1.6) admits a unique entropy measure solution of the form*

$$(u, \rho)(t, x) = \begin{cases} (u_-, \rho_-)(t, x), & x < \sigma^{\epsilon_1} t, \\ \Big(u_\delta^{\epsilon_1}, w^{\epsilon_1}(t)\delta(x - \sigma^{\epsilon_1} t)\Big), & x = \sigma^{\epsilon_1} t, \\ (u_+, \rho_+)(t, x), & x > \sigma^{\epsilon_1} t, \end{cases}$$

where $w^{\epsilon_1}(t) = w_0^{\epsilon_1} t$ and all of the three constants σ^{ϵ_1}, $w_0^{\epsilon_1}$, and $u_\delta^{\epsilon_1}$ are determined uniquely by (3.10) under the entropy condition (3.7) .

3.2. Limit analysis of Riemann solutions of (1.5) as $\epsilon_1 \to 0$

Now the limit of Riemann solutions of the system (1.5) as $\epsilon_1 \to 0$ for $\rho_- \neq \rho_+$ can be discussed. We need to investigate two cases: $u_- > u_+$ and $u_- < u_+$.

In the case $u_- > u_+$, we can check that $u_\delta^{\epsilon_1} \to u_\delta$ when $\epsilon_1 \to 0$ from (3.11). Returning to (3.10), we immediately get that $w_0^{\epsilon_1} \to w_0$ and $\sigma^{\epsilon_1} \to \sigma$ as $\epsilon_1 \to 0$. Thus, the following theorem holds.

Theorem 3.2 *Let $u_- > u_+$. For $0 < \epsilon_1 < \overline{\epsilon_1}$, assume $(u^{\epsilon_1}, \rho^{\epsilon_1})$ is the delta-shock solution of (1.5) and (1.6). Then, as $\epsilon_1 \to 0$, the limit functions of ρ^{ϵ_1} and $\rho^{\epsilon_1} u^{\epsilon_1}$ are the sums of a step function and a δ-function with the weights $\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho f(u)])$ and $\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u] - [\rho u f(u)])$, respectively, which is the delta-shock solution of (1.1) and (1.6).*

Then we consider the case $u_- < u_+$. In this case, the limit of solution of (1.5) is obvious. We can directly get from (3.3) that, as $\epsilon_1 \to 0$, the limit of solution is just the vacuum solution (2.1) of the system (1.1).

4. Riemann solutions and limit analysis of (1.3) as $\epsilon_1, \epsilon_2 \to 0$

In this section, we solve the Riemann problem (1.3) and (1.6), then discuss the limit of Riemann solutions as $\epsilon_1, \epsilon_2 \to 0$.

4.1. Riemann solutions of (1.3)

For small $\epsilon_1, \epsilon_2 > 0$, the two eigenvalues of the system (1.3) are

$$\lambda_1 = f(u) - \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}\,, \qquad \lambda_2 = f(u) + \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}, \tag{4.1}$$

and the corresponding right eigenvectors are

$$r_1 = \left(1, -\sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho f'(u) - 2\epsilon_1}}\right)^\top, \qquad r_2 = \left(1, \sqrt{\frac{\epsilon_2 \rho^{\gamma-2}}{\rho f'(u) - 2\epsilon_1}}\right)^\top.$$

Then we have $\nabla \lambda_i \cdot r_i \neq 0$ $(i = 1, 2)$ for $\sqrt{\epsilon_2} f''(u)\rho^\gamma - \sqrt{\rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}\big((\gamma+1)(\rho f'(u) - 2\epsilon_1) + 6\epsilon_1\big) \neq 0$, and the system (1.3) is thus strictly hyperbolic and genuinely nonlinear.

By seeking the self-similar solution, we can get

$$\begin{cases} -\xi \rho_\xi + (\rho f(u) - 2\epsilon_1 u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u f(u) - \epsilon_1 u^2 + \dfrac{\epsilon_2 \rho^\gamma}{\gamma})_\xi = 0, \end{cases} \tag{4.2}$$

and

$$(u, \rho)(\pm\infty) = (u_\pm, \rho_\pm), \tag{4.3}$$

which, for smooth solutions, provides either the backward rarefaction wave

$$\overleftarrow{R}(u_-, \rho_-): \begin{cases} \xi = \lambda_1 = f(u) - \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}, \\ u - u_- = -\displaystyle\int_{\rho_-}^{\rho} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s f'(u) - 2\epsilon_1}} ds, \end{cases} \tag{4.4}$$

or the forward rarefaction wave

$$\overrightarrow{R}(u_-, \rho_-): \begin{cases} \xi = \lambda_2 = f(u) + \sqrt{\epsilon_2 \rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}, \\ u - u_- = \displaystyle\int_{\rho_-}^{\rho} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s f'(u) - 2\epsilon_1}} ds. \end{cases} \tag{4.5}$$

Through differentiating ξ with respect to ρ and u in the first equation of (4.4) and noticing $u_\rho = \frac{u_\xi}{\rho_\xi}$, we have

$$1 = \left(\frac{\gamma+1}{2} f'(u) - \frac{\sqrt{\epsilon_2}\rho^{\gamma-1} f''(u)}{2\sqrt{\rho^{\gamma-2}(\rho f'(u) - 2\epsilon_1)}} - \frac{\epsilon_1(\gamma-2)}{\rho}\right) u_\xi. \tag{4.6}$$

Thus, we can get $u_\xi > 0$ from (4.6) for ϵ_1, ϵ_2 sufficiently small, which means that the set (u, ρ) joining to (u_-, ρ_-) by the backward rarefaction wave is made up of the half-branch of $\overleftarrow{R}(u_-, \rho_-)$ with $u \geq u_-$. In the same way, for the forward rarefaction wave, we have $u_\xi > 0$ for ϵ_1, ϵ_2 sufficiently small, which implies that the set (u, ρ) joining to (u_-, ρ_-) by the forward rarefaction wave is made up of the half-branch of $\overrightarrow{R}(u_-, \rho_-)$ with $u \geq u_-$.

On the backward rarefaction wave curve, taking $\rho = \frac{2\epsilon_1}{f'(u)}$ in the second equation of (4.4) leads to

$$u = u_- + \int_{\frac{2\epsilon_1}{f'(u)}}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{s f'(u) - 2\epsilon_1}} ds. \tag{4.7}$$

Set

$$G(u)=u-u_{-}-\int_{\frac{2\epsilon_1}{f'(u)}}^{\rho_-}\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u)-2\epsilon_1}}ds. \tag{4.8}$$

For every fixed $\widehat{u}>u_-$, since the integral $\int_{\frac{2\epsilon_1}{f'(\widehat{u})}}^{\rho_-}\sqrt{\frac{s^{\gamma-2}}{sf'(\widehat{u})-2\epsilon_1}}ds$ is convergent according to the Cauchy criterion, we have $G(\widehat{u})>0$ for ϵ_1,ϵ_2 sufficiently small. Thus, $G(u_-)G(\widehat{u})<0$. In addition, the function $G(u)$ is continuous with respect to $u\in[u_-,\widehat{u}]$. Therefore, there exists $u_1\in[u_-,\widehat{u}]$ such that $G(u_1)=0$, which means that the backward rarefaction wave curve intersects with the curve $\rho=\frac{2\epsilon_1}{f'(u)}$ at a point denoted by (u_1,ρ_1).

For the forward rarefaction wave, passing to the limit $\rho\to+\infty$ in the second equation of (4.5) yields

$$\lim_{\rho\to+\infty}u=u_-+\int_{\rho_-}^{+\infty}\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u)-2\epsilon_1}}ds, \tag{4.9}$$

which gives $\lim\limits_{\rho\to+\infty}u=+\infty$. In fact, if $\lim\limits_{\rho\to+\infty}u=c\in(u_-,+\infty)$, then there exists $N>0$, and when $\rho>N$, one can get that

$$\mid u-c\mid<1,$$

which means that u is bounded, so $f'(u)$ is bounded. Set $\mid f'(u)\mid\leq M$ for some $M>0$. Since

$$\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u)-2\epsilon_1}}>\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sM}},$$

we have

$$\int_{\rho_-}^{+\infty}\sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u)-2\epsilon_1}}ds>\sqrt{\frac{\epsilon_2}{M}}\int_{\rho_-}^{+\infty}s^{\frac{\gamma-3}{2}}ds=+\infty, \tag{4.10}$$

which indicates a contradiction. Then we get $\lim\limits_{\rho\to+\infty}u=+\infty$ from (4.9).

For a bounded discontinuity at $\xi=\sigma^{\epsilon_1\epsilon_2}$, the Rankine–Hugoniot relation is

$$\begin{cases}-\sigma^{\epsilon_1\epsilon_2}[\rho]+[\rho f(u)-2\epsilon_1 u]=0,\\ -\sigma^{\epsilon_1\epsilon_2}[\rho u]+[\rho u f(u)-\epsilon_1 u^2+\dfrac{\epsilon_2\rho^\gamma}{\gamma}]=0,\end{cases} \tag{4.11}$$

where $[h]=h_r-h_l$ with $h_l=h(t,x(t)-0)$ and $h_r=h(t,x(t)+0)$. Eliminating $\sigma^{\epsilon_1\epsilon_2}$ from (4.11), together with the Lax entropy inequalities, we obtain the backward shock wave curve

$$\overleftarrow{S}(u_-,\rho_-):\quad u-u_-=-\sqrt{\frac{\rho_-\rho(u-u_-)\big(f(u)-f(u_-)\big)-\dfrac{\epsilon_2}{\gamma}(\rho-\rho_-)(\rho^\gamma-\rho_-^\gamma)}{\epsilon_1(\rho_-+\rho)}},\quad \rho>\rho_-, \tag{4.12}$$

and the forward shock wave curve

$$\overrightarrow{S}(u_-,\rho_-): \quad u-u_- = -\sqrt{\frac{\rho_-\rho(u-u_-)\big(f(u)-f(u_-)\big)-\dfrac{\epsilon_2}{\gamma}(\rho-\rho_-)(\rho^\gamma-\rho_-^\gamma)}{\epsilon_1(\rho_-+\rho)}}, \quad \rho<\rho_-. \tag{4.13}$$

Furthermore, it is easy to check that $du/d\rho<0$ for the backward shock wave curve and $du/d\rho>0$ for the forward shock wave. When $\rho\to+\infty$ in (4.12), we find $\lim\limits_{\rho\to+\infty}u=-\infty$. When $\rho\to 0$ in (4.13), we get $\lim\limits_{\rho\to 0}u=-\infty$, which indicates that the forward shock wave curve intersects with the curve $\rho=\frac{2\epsilon_1}{f'(u)}$ at a point denoted by (u_2,ρ_2).

Through the analysis above, for small ϵ_1,ϵ_2, given a left state (u_-,ρ_-), the phase plane can be divided into five regions by the wave curves (see Figure 3):

① $(u_+,\rho_+)\in I(u_-,\rho_-): \overleftarrow{R}+\overrightarrow{R}$; ② $(u_+,\rho_+)\in II(u_-,\rho_-): \overleftarrow{R}+\overrightarrow{S}$;

③ $(u_+,\rho_+)\in III(u_-,\rho_-): \overleftarrow{S}+\overrightarrow{R}$; ④ $(u_+,\rho_+)\in IV(u_-,\rho_-): \overleftarrow{S}+\overrightarrow{S}$;

⑤ $(u_+,\rho_+)\in V(u_-,\rho_-): \overleftarrow{R}+$ generalized constant density state $(\rho=\frac{2\epsilon_1}{f'(u)})+\overrightarrow{R}$.

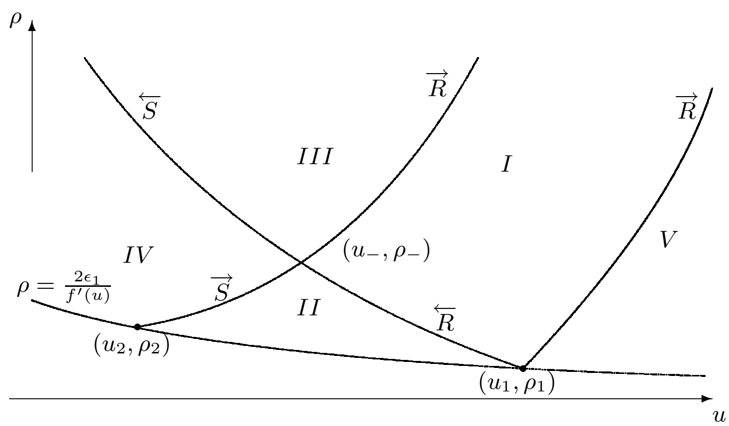

Figure 3. Curves of elementary waves.

4.2. Limit analysis of Riemann solutions of (1.3) as $\epsilon_1,\epsilon_2\to 0$

As $\epsilon_1,\epsilon_2\to 0$, the two regions $II(u_-,\rho_-)$ and $III(u_-,\rho_-)$ have empty interiors. Thus, we only need to consider the limit process for the two cases $(u_+,\rho_+)\in IV(u_-,\rho_-)$ and $(u_+,\rho_+)\in I(u_-,\rho_-)\cup V(u_-,\rho_-)$.

4.2.1. Formation of delta-shocks

In the case $(u_+,\rho_+)\in IV(u_-,\rho_-)$ with $u_->u_+$, the Riemann solution contains a two-shock wave and a nonvacuum intermediate constant state. Let $(u_*^{\epsilon_1\epsilon_2},\rho_*^{\epsilon_1\epsilon_2})$ be the intermediate state. We suppose that (u_-,ρ_-) and $(u_*^{\epsilon_1\epsilon_2},\rho_*^{\epsilon_1\epsilon_2})$ are connected by backward shock wave $\overleftarrow{S}$ with speed $\sigma_1^{\epsilon_1\epsilon_2}$, and that $(u_*^{\epsilon_1\epsilon_2},\rho_*^{\epsilon_1\epsilon_2})$ and (u_+,ρ_+) are connected by forward shock wave $\overrightarrow{S}$ with speed $\sigma_2^{\epsilon_1\epsilon_2}$. We thus obtain

$$u_*^{\epsilon_1\epsilon_2} - u_- = -\sqrt{\frac{\rho_-\rho_*^{\epsilon_1\epsilon_2}(u_*^{\epsilon_1\epsilon_2} - u_-)\big(f(u_*^{\epsilon_1\epsilon_2}) - f(u_-)\big) - \frac{\epsilon_2}{\gamma}(\rho_*^{\epsilon_1\epsilon_2} - \rho_-)((\rho_*^{\epsilon_1\epsilon_2})^\gamma - \rho_-^\gamma)}{\epsilon_1(\rho_- + \rho_*^{\epsilon_1\epsilon_2})}}, \quad \rho_*^{\epsilon_1\epsilon_2} > \rho_-, \tag{4.14}$$

on $\overleftarrow{S}$, and

$$u_+ - u_*^{\epsilon_1\epsilon_2} = -\sqrt{\frac{\rho_*^{\epsilon_1\epsilon_2}\rho_+(u_+ - u_*^{\epsilon_1\epsilon_2})\big(f(u_+) - f(u_*^{\epsilon_1\epsilon_2})\big) - \frac{\epsilon_2}{\gamma}(\rho_+ - \rho_*^{\epsilon_1\epsilon_2})(\rho_+^\gamma - (\rho_*^{\epsilon_1\epsilon_2})^\gamma)}{\epsilon_1(\rho_*^{\epsilon_1\epsilon_2} + \rho_+)}}, \quad \rho_*^{\epsilon_1\epsilon_2} > \rho_+, \tag{4.15}$$

on $\overrightarrow{S}$.

Then we can give three lemmas as follows.

Lemma 4.1 $\lim_{\epsilon_1,\epsilon_2\to 0} \rho_*^{\epsilon_1\epsilon_2} = +\infty$.

Proof If $\lim_{\epsilon_1,\epsilon_2\to 0} \rho_*^{\epsilon_1\epsilon_2} = d \in (\max(\rho_-, \rho_+), +\infty)$, and noting $u_+ < u_*^{\epsilon_1\epsilon_2} < u_-$, we can deduce from (4.14) and (4.15) that $u_+ - u_- = -\infty$ as $\epsilon_1, \epsilon_2 \to 0$. Therefore, we must have $\lim_{\epsilon_1,\epsilon_2\to 0} \rho_*^{\epsilon_1\epsilon_2} = +\infty$. □

Lemma 4.2 *Set* $\lim_{\epsilon_1,\epsilon_2\to 0} u_*^{\epsilon_1\epsilon_2} = u_\delta \in (u_+, u_-)$ *and* $\sigma = f(u_\delta)$. *Then*

$$\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_2(\rho_*^{\epsilon_1\epsilon_2})^\gamma = \gamma\rho_-(u_\delta - u_-)(f(u_\delta) - f(u_-)) = \gamma\rho_+(u_+ - u_\delta)(f(u_+) - f(u_\delta)), \tag{4.16}$$

$$\lim_{\epsilon_1,\epsilon_2\to 0} \sigma_1^{\epsilon_1\epsilon_2} = \lim_{\epsilon_1,\epsilon_2\to 0} \sigma_2^{\epsilon_1\epsilon_2} = \sigma, \tag{4.17}$$

$$\lim_{\epsilon_1,\epsilon_2\to 0} \int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}} \rho_*^{\epsilon_1\epsilon_2} d\xi = \sigma[\rho] - [\rho f(u)], \tag{4.18}$$

$$\lim_{\epsilon_1,\epsilon_2\to 0} \int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}} \rho_*^{\epsilon_1\epsilon_2} u_*^{\epsilon_1\epsilon_2} d\xi = \sigma[\rho u] - [\rho u f(u)]. \tag{4.19}$$

Proof Letting $\epsilon_1, \epsilon_2 \to 0$ in (4.14) and (4.15), respectively, and noting Lemma 4.1, we can obtain (4.16).

From the first equation of (4.11), it follows that

$$\sigma_1^{\epsilon_1\epsilon_2} = \frac{\rho_*^{\epsilon_1\epsilon_2} f(u_*^{\epsilon_1\epsilon_2}) - \rho_- f(u_-) + 2\epsilon_1(u_- - u_*^{\epsilon_1\epsilon_2})}{\rho_*^{\epsilon_1\epsilon_2} - \rho_-}, \tag{4.20}$$

$$\sigma_2^{\epsilon_1\epsilon_2} = \frac{\rho_+ f(u_+) - \rho_*^{\epsilon_1\epsilon_2} f(u_*^{\epsilon_1\epsilon_2}) + 2\epsilon_1(u_*^{\epsilon_1\epsilon_2} - u_+)}{\rho_+ - \rho_*^{\epsilon_1\epsilon_2}}. \tag{4.21}$$

Then the result of (4.17) is easily reached, and we have

$$\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})=\sigma[\rho]-[\rho f(u)]. \tag{4.22}$$

Similarly, using the second equation of (4.11), one can deduce that

$$\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}u_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})=\sigma[\rho u]-[\rho u f(u)]. \tag{4.23}$$

Thus, (4.18) and (4.19) hold. □

It is shown from Lemma 4.1 and Lemma 4.2 that the density between $\overleftarrow{S}$ and $\overrightarrow{S}$ becomes singular when $\epsilon_1,\epsilon_2\to 0$.

Lemma 4.3 *The quantity u_δ, i.e. the limit of $u_*^{\epsilon_1\epsilon_2}$ in Lemma 4.2, is just the propagation speed of the delta-shock of (1.1) and (1.6).*

Proof Noticing

$$\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}u_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})=\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})\cdot\lim_{\epsilon_1,\epsilon_2\to 0}u_*^{\epsilon_1\epsilon_2} \tag{4.24}$$

and (4.22) and (4.23), we get that u_δ is uniquely determined by

$$f(u_\delta)[\rho u]-[\rho u f(u)]=(f(u_\delta)[\rho]-[\rho f(u)])u_\delta \tag{4.25}$$

under the entropy condition (2.10). Thus, the lemma is true. □

Now a theorem that can characterize the limit of solutions of (1.3) and (1.6) as $\epsilon_1,\epsilon_2\to 0$ can be given as follows.

Theorem 4.4 *Let $u_->u_+$. Assume $(u^{\epsilon_1\epsilon_2},\rho^{\epsilon_1\epsilon_2})$ to be the Riemann solutions containing two shock waves of (1.3) and (1.6) constructed in Subsection 4.1. Then, as $\epsilon_1,\epsilon_2\to 0$, $\rho^{\epsilon_1\epsilon_2}$ and $\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}$ converge in the sense of distributions, and their limit functions are the sum of a step function and a δ-function with the weights*

$$\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho]-[\rho f(u)]) \quad and \quad \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u]-[\rho u f(u)]),$$

respectively, which is just the delta-shock solution of (1.1) with the same Riemann data (1.6).

Proof (i). The two-shock wave solution of (1.3) can be given as

$$(u^{\epsilon_1\epsilon_2},\rho^{\epsilon_1\epsilon_2})(\xi)=\begin{cases}(u_-,\rho_-), & \xi<\sigma_1^{\epsilon_1\epsilon_2},\\ (u_*^{\epsilon_1\epsilon_2},\rho_*^{\epsilon_1\epsilon_2}), & \sigma_1^{\epsilon_1\epsilon_2}<\xi<\sigma_2^{\epsilon_1\epsilon_2},\\ (u_+,\rho_+), & \xi>\sigma_2^{\epsilon_1\epsilon_2}.\end{cases} \tag{4.26}$$

For any $\phi\in C_0^1(-\infty,+\infty)$, (4.26) satisfies weak formulations

$$\int_{-\infty}^{+\infty}(-\rho^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-2\epsilon_1u^{\epsilon_1\epsilon_2})\phi'd\xi-\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}\phi d\xi=0, \tag{4.27}$$

and

$$\int_{-\infty}^{+\infty}\Big(-\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-\epsilon_1(u^{\epsilon_1\epsilon_2})^2+\frac{\epsilon_2}{\gamma}(\rho^{\epsilon_1\epsilon_2})^{\gamma}\Big)\phi' d\xi \\ -\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\phi d\xi=0. \tag{4.28}$$

(ii). Decomposing the first integral in (4.27) into

$$\Big(\int_{-\infty}^{\sigma_1^{\epsilon_1\epsilon_2}}+\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}+\int_{\sigma_2^{\epsilon_1\epsilon_2}}^{+\infty}\Big)(-\rho^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-2\epsilon_1u^{\epsilon_1\epsilon_2})\phi' d\xi, \tag{4.29}$$

and computing the limit of the sum of the first and last term of (4.29), we have

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0}\Big(\int_{-\infty}^{\sigma_1^{\epsilon_1\epsilon_2}}+\int_{\sigma_2^{\epsilon_1\epsilon_2}}^{+\infty}\Big)(-\rho^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-2\epsilon_1u^{\epsilon_1\epsilon_2})\phi' d\xi\\
&=\lim_{\epsilon_1,\epsilon_2\to 0}\int_{-\infty}^{\sigma_1^{\epsilon_1\epsilon_2}}(-\rho_-\xi+\rho_-f(u_-)-2\epsilon_1u_-)\phi' d\xi\\
&\qquad\qquad+\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\sigma_2^{\epsilon_1\epsilon_2}}^{+\infty}(-\rho_+\xi+\rho_+f(u_+)-2\epsilon_1u_+)\phi' d\xi\\
&=(\sigma[\rho]-[\rho f(u)])\phi(\sigma)+\int_{-\infty}^{+\infty}Q(\xi-\sigma)\phi d\xi
\end{aligned} \tag{4.30}$$

with

$$Q(\xi-\sigma)=\begin{cases}\rho_-, & \xi<\sigma,\\ \rho_+, & \xi>\sigma.\end{cases}$$

For the limit of the second term of (4.29), one can get

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}(-\rho^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-2\epsilon_1u^{\epsilon_1\epsilon_2})\phi' d\xi\\
&=\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}(-\rho_*^{\epsilon_1\epsilon_2}\xi+\rho_*^{\epsilon_1\epsilon_2}f(u_*^{\epsilon_1\epsilon_2})-2\epsilon_1u_*^{\epsilon_1\epsilon_2})\phi' d\xi\\
&=\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})\Big(\tfrac{\phi(\sigma_2^{\epsilon_1\epsilon_2})-\phi(\sigma_1^{\epsilon_1\epsilon_2})}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}f(u_*^{\epsilon_1\epsilon_2})-\tfrac{\sigma_2^{\epsilon_1\epsilon_2}\phi(\sigma_2^{\epsilon_1\epsilon_2})-\sigma_1^{\epsilon_1\epsilon_2}\phi(\sigma_1^{\epsilon_1\epsilon_2})}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}\\
&\quad+\tfrac{1}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}\phi d\xi\Big)-\lim_{\epsilon_1,\epsilon_2\to 0}2\epsilon_1u_*^{\epsilon_1\epsilon_2}\Big(\phi(\sigma_2^{\epsilon_1\epsilon_2})-\phi(\sigma_1^{\epsilon_1\epsilon_2})\Big)\\
&=(\sigma[\rho]-[\rho f(u)])\Big(\sigma\phi'(\sigma)-\sigma\phi'(\sigma)-\phi(\sigma)+\phi(\sigma)\Big)\\
&=0.
\end{aligned} \tag{4.31}$$

Then combining (4.30) with (4.31) yields

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}\phi d\xi=(\sigma[\rho]-[\rho f(u)])\phi(\sigma)+\int_{-\infty}^{+\infty}Q(\xi-\sigma)\phi d\xi. \tag{4.32}$$

(iii) We employ the weak formulation (4.28) to consider the limit of $\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}$. Similarly, from the first integral of (4.28), we have

$$\Big(\int_{-\infty}^{\sigma_1^{\epsilon_1\epsilon_2}}+\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}+\int_{\sigma_2^{\epsilon_1\epsilon_2}}^{+\infty}\Big)\Big(-\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-\epsilon_1(u^{\epsilon_1\epsilon_2})^2+\frac{\epsilon_2}{\gamma}(\rho^{\epsilon_1\epsilon_2})^{\gamma}\Big)\phi' d\xi. \tag{4.33}$$

Taking the limit $\epsilon_1,\epsilon_2\to 0$ in the sum of the first and last term of (4.33) leads to

$$\begin{aligned}&\lim_{\epsilon_1,\epsilon_2\to 0}\Big(\int_{-\infty}^{\sigma_1^{\epsilon_1\epsilon_2}}+\int_{\sigma_2^{\epsilon_1\epsilon_2}}^{+\infty}\Big)\Big(-\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-\epsilon_1(u^{\epsilon_1\epsilon_2})^2+\frac{\epsilon_2}{\gamma}(\rho^{\epsilon_1\epsilon_2})^{\gamma}\Big)\phi' \xi\\&=(\sigma[\rho u]-[\rho u f(u)])\phi(\sigma)+\int_{-\infty}^{+\infty}\widetilde{Q}(\xi-\sigma)\phi d\xi\end{aligned} \tag{4.34}$$

with

$$\widetilde{Q}(\xi-\sigma)=\begin{cases}\rho_-u_-, & \xi<\sigma,\\ \rho_+u_+, & \xi>\sigma.\end{cases}$$

For the limit of the second term of (4.33), using Lemmas 4.1–4.2, we can obtain

$$\begin{aligned}&\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}\Big(-\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\xi+\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}f(u^{\epsilon_1\epsilon_2})-\epsilon_1(u^{\epsilon_1\epsilon_2})^2+\frac{\epsilon_2}{\gamma}(\rho^{\epsilon_1\epsilon_2})^{\gamma}\Big)\phi' d\xi\\&=\lim_{\epsilon_1,\epsilon_2\to 0}\rho_*^{\epsilon_1\epsilon_2}(\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2})\Bigg(\frac{\phi(\sigma_2^{\epsilon_1\epsilon_2})-\phi(\sigma_1^{\epsilon_1\epsilon_2})}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}u_*^{\epsilon_1\epsilon_2}f(u_*^{\epsilon_1\epsilon_2})+\frac{\epsilon_2(\rho_*^{\epsilon_1\epsilon_2})^{\gamma-1}}{\gamma}\frac{\phi(\sigma_2^{\epsilon_1\epsilon_2})-\phi(\sigma_1^{\epsilon_1\epsilon_2})}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}\\&\quad-\frac{\sigma_2^{\epsilon_1\epsilon_2}\phi(\sigma_2^{\epsilon_1\epsilon_2})-\sigma_1^{\epsilon_1\epsilon_2}\phi(\sigma_1^{\epsilon_1\epsilon_2})}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}u_*^{\epsilon_1\epsilon_2}+\frac{u_*^{\epsilon_1\epsilon_2}}{\sigma_2^{\epsilon_1\epsilon_2}-\sigma_1^{\epsilon_1\epsilon_2}}\int_{\sigma_1^{\epsilon_1\epsilon_2}}^{\sigma_2^{\epsilon_1\epsilon_2}}\phi d\xi\Bigg)\\&\quad-\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1(u_*^{\epsilon_1\epsilon_2})^2\Big(\phi(\sigma_2^{\epsilon_1\epsilon_2})-\phi(\sigma_1^{\epsilon_1\epsilon_2})\Big)\\&=0.\end{aligned} \tag{4.35}$$

Returning to (4.28), we get

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}\phi d\xi=(\sigma[\rho u]-[\rho u f(u)])\phi(\sigma)+\int_{-\infty}^{+\infty}\widetilde{Q}(\xi-\sigma)\phi d\xi. \tag{4.36}$$

(iiii). Finally, as $\epsilon_1,\epsilon_2\to 0$, we investigate the limit of $\rho^{\epsilon_1\epsilon_2}$ and $\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}$ depending on t. For any test function $\psi(x,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, we have

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}(x/t)\psi(x,t)dxdt\\
&=\lim_{\epsilon_1,\epsilon_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}(\xi)\psi(\xi t,t)d(\xi t)dt\\
&=\lim_{\epsilon_1,\epsilon_2\to 0}\int_0^{+\infty} t\Big(\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}(\xi)\psi(\xi t,t)d\xi\Big)dt\\
&=\int_0^{+\infty} t\Big((\sigma[\rho]-[\rho f(u)])\psi(\sigma t,t)+\int_{-\infty}^{+\infty}Q(\xi-\sigma)\psi(\xi t,t)d\xi\Big)dt\\
&=\int_0^{+\infty}(\sigma[\rho]-[\rho f(u)])t\psi(\sigma t,t)dt+\int_0^{+\infty} t\Big(\int_{-\infty}^{+\infty}Q(\xi-\sigma)\psi(\xi t,t)d\xi\Big)dt\\
&=\int_0^{+\infty}(\sigma[\rho]-[\rho f(u)])t\psi(\sigma t,t)dt+\int_0^{+\infty}\int_{-\infty}^{+\infty}Q(x-\sigma t)\psi(x,t)dxdt.
\end{aligned}\tag{4.37}$$

Thus, by the definition (2.2), we get

$$\int_0^{+\infty}(\sigma[\rho]-[\rho f(u)])t\psi(\sigma t,t)dt=\Big\langle w_1(\cdot)\delta_S,\psi(\cdot,\cdot)\Big\rangle$$

with $w_1(t)=\dfrac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho]-[\rho f(u)])$.

Similarly, it can be shown that

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho^{\epsilon_1\epsilon_2}u^{\epsilon_1\epsilon_2}(x/t)\psi(x,t)dxdt\\
&=\Big\langle w_2(\cdot)\delta_S,\psi(\cdot,\cdot)\Big\rangle+\int_0^{+\infty}\int_{-\infty}^{+\infty}\widetilde{Q}(x-\sigma t)\psi(x,t)dxdt,
\end{aligned}\tag{4.38}$$

with $w_2(t)=\dfrac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u]-[\rho u f(u)])$.

This completes the proof of Theorem 4.1. □

4.2.2. Formation of vacuum states

In the case $(u_+,\rho_+)\in I(u_-,\rho_-)\cup V(u_-,\rho_-)$ with $u_-<u_+$, we get that, on the backward rarefaction wave, the solution $(u^{\epsilon_1\epsilon_2},\rho^{\epsilon_1\epsilon_2})$ satisfies

$$\begin{cases}\xi=f(u^{\epsilon_1\epsilon_2})-\sqrt{\epsilon_2(\rho^{\epsilon_1\epsilon_2})^{\gamma-2}(\rho^{\epsilon_1\epsilon_2}f'(u^{\epsilon_1\epsilon_2})-2\epsilon_1)}, & \rho_*^{\epsilon_1\epsilon_2}<\rho_-,\\ f(u_-)-\sqrt{\epsilon_2\rho_-^{\gamma-2}(\rho_-f'(u_-)-2\epsilon_1)}<\xi<f(u_*^{\epsilon_1\epsilon_2})-\sqrt{\epsilon_2(\rho_*^{\epsilon_1\epsilon_2})^{\gamma-2}(\rho_*^{\epsilon_1\epsilon_2}f'(u_*^{\epsilon_1\epsilon_2})-2\epsilon_1)},\end{cases}\tag{4.39}$$

and on the forward rarefaction wave

$$\begin{cases}\xi=f(u^{\epsilon_1\epsilon_2})+\sqrt{\epsilon_2(\rho^{\epsilon_1\epsilon_2})^{\gamma-2}(\rho^{\epsilon_1\epsilon_2}f'(u^{\epsilon_1\epsilon_2})-2\epsilon_1)}, & \rho_*^{\epsilon_1\epsilon_2}<\rho_+,\\ f(u_*^{\epsilon_1\epsilon_2})+\sqrt{\epsilon_2(\rho_*^{\epsilon_1\epsilon_2})^{\gamma-2}(\rho_*^{\epsilon_1\epsilon_2}f'(u_*^{\epsilon_1\epsilon_2})-2\epsilon_1)}<\xi<f(u_+)+\sqrt{\epsilon_2(\rho_+)^{\gamma-2}(\rho_+f'(u_+)-2\epsilon_1)}.\end{cases}\tag{4.40}$$

Now we analyze the formation of the vacuum state in the limit of solutions of (1.3) and (1.6).

Note that $(u_*^{\epsilon_1\epsilon_2}, \rho_*^{\epsilon_1\epsilon_2})$ is on the curve of the backward rarefaction wave $\overleftarrow{R}(u_-, \rho_-)$. Thus,

$$u_*^{\epsilon_1\epsilon_2} = u_- - \int_{\rho_-}^{\rho_*^{\epsilon_1\epsilon_2}} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u_*^{\epsilon_1\epsilon_2}) - 2\epsilon_1}} ds \leq u_- + \int_{\frac{2\epsilon_1}{f'(u_1)}}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u_1) - 2\epsilon_1}} ds := B^{\epsilon_1\epsilon_2}. \tag{4.41}$$

If $u_- < u_+ < B^{\epsilon_1\epsilon_2}$, the solution does not contain a generalized constant density. However, if $B^{\epsilon_1\epsilon_2} < u_+$, the intermediate state becomes a generalized constant density.

We claim that there exists a $\epsilon_0 > 0$ such that when $0 < \epsilon_1 < \epsilon_0$ and $0 < \epsilon_2 < \epsilon_0$, the intermediate state is a generalized constant density. In fact, for any ϵ_1, ϵ_2, setting $\epsilon_1 = \epsilon_2 = \epsilon$, we obtain from (4.41) that

$$u_*^{\epsilon_1\epsilon_2} = u_- - \int_{\rho_-}^{\rho_*^{\epsilon_1\epsilon_2}} \sqrt{\frac{\epsilon s^{\gamma-2}}{sf'(u_*^{\epsilon_1\epsilon_2}) - 2\epsilon}} ds \leq u_- + \int_{\frac{2\epsilon}{f'(u_1)}}^{\rho_-} \sqrt{\frac{\epsilon s^{\gamma-2}}{sf'(u_1) - 2\epsilon}} ds := B^{\epsilon}. \tag{4.42}$$

Thus, if $u_- < u_+ < B^\epsilon$, there exists ϵ_* such that $(u_+, \rho_+) \in I(u_-, \rho_-)$ when $u_- < u_+ < B^{\epsilon_*}$. However, if $B^\epsilon < u_+$, there exists ϵ_{**} such that $(u_+, \rho_+) \in V(u_-, \rho_-)$ when $B^{\epsilon_{**}} < u_+$.

Let $E(\epsilon) = \int_{\frac{2\epsilon}{f'(u_1)}}^{\rho_-} \sqrt{\frac{\epsilon s^{\gamma-2}}{sf'(u_1) - 2\epsilon}} ds - u_+ + u_-$. Since the integral $\int_0^{f'(u_1)\rho_-} \sqrt{\frac{(s+2\epsilon_*)^{\gamma-2}}{s}} ds$ is convergent, we can deduce that the integral $\int_{\frac{2\epsilon}{f'(u_1)}}^{\rho_-} \sqrt{\frac{\epsilon s^{\gamma-2}}{sf'(u_1) - 2\epsilon}} ds$ is uniformly convergent in $\epsilon \leq \epsilon_*$ by using the M-criterion, and then $E(\epsilon)$ is continuous with respect to ϵ and $E(\epsilon_*)E(\epsilon_{**}) < 0$. Thus, there exists $\epsilon_0 \in [\epsilon_{**}, \epsilon_*]$ such that $E(\epsilon_0) = 0$.

Therefore, if $0 < \epsilon_1 < \epsilon_0$ and $0 < \epsilon_2 < \epsilon_0$, the intermediate state is the generalized constant density

$$(u_*^{\epsilon_1\epsilon_2}, \rho_*^{\epsilon_1\epsilon_2})(\xi) = \left(u^{\epsilon_1\epsilon_2}, \frac{2\epsilon_1}{f'(u^{\epsilon_1\epsilon_2})} \right), \qquad u_{01}^{\epsilon_1\epsilon_2} \leq u^{\epsilon_1\epsilon_2}(\xi) \leq u_{02}^{\epsilon_1\epsilon_2}, \tag{4.43}$$

where

$$u_{01}^{\epsilon_1\epsilon_2} = u_- + \int_{\frac{2\epsilon_1}{f'(u_{01}^{\epsilon_1\epsilon_2})}}^{\rho_-} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u_{01}^{\epsilon_1\epsilon_2}) - 2\epsilon_1}} ds, \quad u_{02}^{\epsilon_1\epsilon_2} = u_+ - \int_{\frac{2\epsilon_1}{f'(u_{02}^{\epsilon_1\epsilon_2})}}^{\rho_+} \sqrt{\frac{\epsilon_2 s^{\gamma-2}}{sf'(u_{02}^{\epsilon_1\epsilon_2}) - 2\epsilon_1}} ds.$$

Then, when $0 < \epsilon_1 < \epsilon_0$ and $0 < \epsilon_2 < \epsilon_0$, taking $\epsilon_1, \epsilon_2 \to 0$, we have

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \rho_*^{\epsilon_1\epsilon_2} = 0,$$

which means that the vacuum occurs. Moreover, noting that $\rho^{\epsilon_1\epsilon_2}$ and $f'(u^{\epsilon_1\epsilon_2})$ are uniform boundedness with respect to ϵ_1, ϵ_2, we find

$$\lim_{\epsilon_1, \epsilon_2 \to 0} u_{01}^{\epsilon_1\epsilon_2} = u_-, \qquad \lim_{\epsilon_1, \epsilon_2 \to 0} u_{02}^{\epsilon_1\epsilon_2} = u_+,$$

and

$$\lim_{\epsilon_1, \epsilon_2 \to 0} f(u^{\epsilon_1\epsilon_2}) = \xi \quad \text{for} \quad \xi \in (f(u_-), f(u_+)).$$

In conclusion, from the analysis above, it is clear that the limit of the solution is the solution of the system (1.1), which contains two contact discontinuities $\xi = x/t = f(u_\pm)$ and a vacuum state in between.

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