Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2018) 42: 2752 – 2762
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doi:10.3906/mat-1711-15

$\mathbb{Q}$-Korselt numbers

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Received: 03.11.2017 • **Accepted/Published Online:** 21.08.2018 • **Final Version:** 27.09.2018

Abstract: Let $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0\}$; a positive integer N is said to be an α*-Korselt number* (K_α*-number*, for short) if $N \neq \alpha$ and $\alpha_2 p - \alpha_1$ divides $\alpha_2 N - \alpha_1$ for every prime divisor p of N. In this paper we prove that for each squarefree composite number N there exist finitely many rational numbers α such that N is a K_α-number and if $\alpha \leq 1$ then N has at least three prime factors. Moreover, we prove that for each $\alpha \in \mathbb{Q} \setminus \{0\}$ there exist only finitely many squarefree composite numbers N with two prime factors such that N is a K_α-number.

Key words: Prime number, Carmichael number, Korselt number, squarefree composite number, Korselt set, Korselt weight

1. Introduction

A Carmichael number is a composite number N that divides $a^N - a$ for all integers a [2, 4]. In 1899, Korselt gave a complete characterization of Carmichael numbers.

Theorem 1.1 (Korselt criterion [8]) *A composite integer $N > 1$ is a Carmichael number if and only if $p-1$ divides $N-1$ for all prime factors p of N.*

This criterion helped in the discovery of the existence of infinitely many Carmichael numbers in 1994 by Alford et al. (see [1] for details). In the proof of the infinitude of Carmichael numbers the authors asked if this proof can be generalized to produce other kinds of pseudoprimes by writing the following:

"*One can modify our proof to show that for any fixed nonzero integer a, there are many squarefree, composite integers n such that $p-a$ divides $n-1$ for all primes p dividing n. However, we have been unable to prove this for $p-a$ dividing $n-b$, for b other than 0 or 1.*"

The query of Alford et al. inspired Bouallegue et al. to state in a recent paper a new kind of pseudoprimes called Korselt numbers (see [3] for details). For $\alpha \in \mathbb{Z} \setminus \{0\}$, a number N is called an α-Korselt number if $p - \alpha \mid N - \alpha$ for each prime divisor p of N. By this definition, Carmichael numbers are exactly the squarefree composite 1-Korselt numbers. In this paper, we extend the definition of α-Korselt numbers given in [3] by allowing α to be a rational number. We state the following definition.

Definition 1.2 *Let $N \in \mathbb{N} \setminus \{0,1\}$ and $\alpha = \frac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \{0\}$. N is said to be an α-*Korselt number *(K_α-*number*, for short) if $N \neq \alpha$ and $\alpha_2 p - \alpha_1$ divides $\alpha_2 N - \alpha_1$ for every prime divisor p of N.*

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2010 *AMS Mathematics Subject Classification:* Primary 11Y16; Secondary 11Y11, 11A51

The set of all K_α-numbers, where $\alpha \in \mathbb{Q}$, is called the set of $\mathbb{Q}$-Korselt numbers.

For a fixed $N \in \mathbb{N} \setminus \{0,1\}$, we need to determine the set of all $\alpha \in \mathbb{Q} \setminus \{0\}$ such that N is a K_α-number. This leads to the following definition.

Definition 1.3 *Let N be a positive integer and $\mathbb{A}$ be a nonempty subset of $\mathbb{Q}$.*

1. *By the* $\mathbb{A}$-Korselt set *of N, we mean the set $\mathbb{A}$-$\mathcal{KS}(N)$ of all $\alpha \in \mathbb{A} \setminus \{0, N\}$ such that N is a K_α-number.*
2. *The cardinality of $\mathbb{A}$-$\mathcal{KS}(N)$ will be called the* $\mathbb{A}$-Korselt weight *of N; we denote it by $\mathbb{A}$-$\mathcal{KW}(N)$.*

By this definition, the notion of $\mathbb{Q}$-Korselt numbers generalizes that given by Bouallegue et al. and thus Carmichael numbers. Among the most recent works in this area are the papers [3, 5–7], where the notion of Korselt numbers over $\mathbb{Z}$ was studied and several related results were obtained. In this paper, our aim is to introduce the notion of $\mathbb{Q}$-Korselt numbers and to discuss generalizations of properties holding when $\alpha \in \mathbb{Z}$. Therefore, we proceed as follows:

- In Section 2, after giving some general results about $\mathbb{Q}$-Korselt numbers, we prove that for each squarefree composite number N, there exist only finitely many rational numbers α such that N is a K_α-number.

- In section 3, we prove that for every rational number $\alpha \leq 1$, if a squarefree composite number N is a K_α-number then N must have at least three prime factors. Furthermore, we show that for each rational number $\alpha > 1$, there exist only finitely many K_α-numbers with two prime factors.

Throughout this paper and for $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}$, we will suppose without loss of generality that $\alpha_2 > 0$, $\alpha_1 \in \mathbb{Z}$, and $\gcd(\alpha_1, \alpha_2) = 1$. Moreover, in this work we are concerned only with squarefree composite numbers N.

2. $\mathbb{Q}$-Korselt set properties

Proposition 2.1 *Let $\alpha \in \mathbb{Q} \setminus \{0\}$ and $N = p_1 p_2 \ldots p_m$ be a K_α-number such that $p_1 < p_2 < \ldots < p_m$ and $m \geq 2$. Then the following inequalities hold:*

$$\frac{(m+2)p_1 - N}{m+1} \leq \alpha \leq \frac{N + m p_m}{m+1}.$$

Proof $\alpha \in \mathbb{Q}$-$\mathcal{KS}(N)$ implies that $N - \alpha = k_i(p_i - \alpha)$ with $k_i \in \mathbb{Z}$ for each $i = 1 \ldots m$. We consider two cases:

Case 1: Assume that $\alpha < 0$. First, let us show that $k_m \geq 3$.

Since $N - \alpha > p_m - \alpha > 0$, then $k_m = \dfrac{N - \alpha}{p_m - \alpha} > 1$.

Next, we show that $k_m \neq 2$. Suppose by contradiction that $k_m = 2$.

Then $\alpha = 2p_m - N \in \mathbb{Z}$, but as $\alpha \neq p_m$ and $\alpha \neq 0$, we get $N \neq p_m$ and $N \neq 2p_m$. Thus, there exists an integer $N_1 \geq 3$ such that $N = N_1 p_m$. Let p_s be a prime factor of N_1. Then

$$p_s - \alpha = p_s + (N_1 - 2)p_m \mid N - \alpha = 2p_m(N_1 - 1).$$

However, as $\gcd(p_m, p_s - \alpha) = 1$, it follows that

$$p_s - \alpha = p_s + (N_1 - 2)p_m \mid 2(N_1 - 1),$$

and hence

$$p_s + (N_1 - 2)p_m \leq 2(N_1 - 1).$$

Since $4 \leq p_s + 2 \leq p_m$, we get

$$2 + 4(N_1 - 2) \leq p_s + (N_1 - 2)p_m \leq 2(N_1 - 1).$$

Therefore, $N_1 \leq 2$, which contradicts $N_1 \geq 3$, so $k_m \geq 3$.

Now, as $(p_i - \alpha)_{1\leq i\leq m}$ is increasing and positive, then $\left(k_i = \dfrac{N-\alpha}{p_i - \alpha}\right)_{1\leq i\leq m}$ is decreasing. Hence, as $k_m \geq 3$, $\dfrac{N-\alpha}{p_1-\alpha} = k_1 \geq m+2$. Thus,

$$\frac{(m+2)p_1 - N}{m+1} \leq \alpha.$$

Case 2: Suppose that $\alpha > 0$. We claim that $\alpha < N$. If not, then (as $\alpha \neq N$) we get $p_m < N < \alpha$. This implies that $0 < \alpha - N < \alpha - p_m$, and hence $0 < \dfrac{\alpha - N}{\alpha - p_m} = k_m < 1$, contradicting the fact that $k_m \in \mathbb{Z}$.

Now let us prove that $\alpha \leq \dfrac{N + mp_m}{m+1}$.

- If $\alpha \leq p_m$, it is immediate.
- Now suppose that $p_m < \alpha < N$. Since $(\alpha - p_i)_{1\leq i\leq m}$ is decreasing and positive, then $\left(\mid k_i \mid = \dfrac{N-\alpha}{\alpha - p_i}\right)_{1\leq i\leq m}$ is increasing. Hence, $\mid k_m \mid \geq m$ and consequently $N - \alpha = \mid k_m \mid (\alpha - p_m) \geq m(\alpha - p_m)$. Thus,

$$\alpha \leq \frac{N + mp_m}{m+1}.$$

Finally, combining the two cases, we get

$$\frac{(m+2)p_1 - N}{m+1} \leq \alpha \leq \frac{N + mp_m}{m+1}.$$

□

By the following result, we provide a characterization of the $\mathbb{Q}$-Korselt set of a squarefree composite number N.

Proposition 2.2 *Let N be a squarefree composite number with prime divisors p_i, $1 \leq i \leq m$. If we let*

$$A_{ij} = \left\{ \frac{dp_j - \delta p_i}{d - \delta};\ d \neq \delta,\ \delta \mid (N - p_i),\ d \mid (N - p_j), \quad \text{and} \quad (p_i - p_j) \mid (d - \delta) \right\},$$

for $1 \leq i < j \leq m$, then

$$\mathbb{Q}\text{-}\mathcal{KS}(N) = \bigcap_{1\leq i<j\leq m} A_{ij}.$$

Proof First note that for each $1 \leq i \leq m$, N is a K_α-number if and only if $\alpha_2 p_i - \alpha_1 \mid \alpha_2 N - \alpha_1$ or equivalently $\alpha_2 p_i - \alpha_1 \mid N - p_i$.

Now let $\alpha \in \mathbb{Q}\text{-}\mathcal{KS}(N)$. Then for each (i,j) with $1 \leq i < j \leq m$, we have

$$\begin{cases} \alpha_2 p_i - \alpha_1 \mid N - p_i \\ \alpha_2 p_j - \alpha_1 \mid N - p_j. \end{cases}$$

This implies that there are two distinct divisors d and δ of $N - p_i$ and $N - p_j$, respectively, such that

$$\begin{cases} \alpha_2 p_i - \alpha_1 = d \\ \alpha_2 p_j - \alpha_1 = \delta. \end{cases}$$

Solving the system we get

$$\alpha_1 = \frac{dp_j - \delta p_i}{p_i - p_j}, \ \alpha_2 = \frac{d - \delta}{p_i - p_j},$$

and so $\alpha = \dfrac{dp_j - \delta p_i}{d - \delta}$. Since α_1 and α_2 are integers we conclude that $\alpha \in A_{ij}$ and hence

$$\mathbb{Q}\text{-}\mathcal{KS}(N) \subseteq \bigcap_{1 \leq i < j \leq m} A_{ij}.$$

Next let $\alpha \in \bigcap_{1 \leq i < j \leq m} A_{ij}$. Then $\alpha \in A_{ij}$, for each pair (i,j) such that $1 \leq i < j \leq m$. This implies that $\alpha = \dfrac{dp_j - \delta p_i}{d - \delta}$, for some divisors d and δ of $N - p_i$ and $N - p_j$, respectively, with $(p_i - p_j) \mid (d - \delta)$.

Setting $\alpha_1 = \dfrac{dp_j - \delta p_i}{p_i - p_j}$ and $\alpha_2 = \dfrac{d - \delta}{p_i - p_j}$, then $\alpha_1, \alpha_2 \in \mathbb{Z}$ and

$$\alpha_2 p_i - \alpha_1 = d \mid N - p_i \quad \text{for} \quad i = 1 \ldots m.$$

Therefore, $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}\text{-}\mathcal{KS}(N)$. □

By the previous proposition, we immediately get the following result.

Theorem 2.3 *For any given squarefree composite number N, there are only finitely many rational numbers α for which N is a K_α-number.*

By the characterization of the $\mathbb{Q}$-Korselt set of a squarefree composite number N, given in Proposition 2.2, and with a simple Maple program, we provide in Table 1 and Table 2 data representing some squarefree composite numbers and their $\mathbb{Q}$-Korselt sets as follows:

- Table 1 gives for each integer $2 \leq d \leq 8$ the $\mathbb{Q}$-Korselt set of the smallest $\mathbb{Q}$-Korselt number N_d with d prime factors.
- Table 2 gives for each integer $0 \leq k \leq 10$ the smallest squarefree composite number N_k such that $\mathbb{Q}\text{-}\mathcal{KW}(N_k) = k$.

Table 1. $\mathbb{Q}$-$\mathcal{KS}(N_d)$ where N_d is the smallest $\mathbb{Q}$-Korselt number with d prime factors.

d	N_d	$\mathbb{Q}$-$\mathcal{KS}(N_d)$
2	$6 = 2 \cdot 3$	$\left\{4, \frac{3}{2}, \frac{10}{3}, \frac{14}{5}, \frac{8}{3}, \frac{5}{2}, \frac{18}{7}, \frac{12}{5}, \frac{9}{4}\right\}$
3	$30 = 2 \cdot 3 \cdot 5$	$\left\{4, 6, \frac{15}{8}, \frac{40}{13}, \frac{5}{2}, \frac{10}{3}, \frac{15}{4}, \frac{24}{5}\right\}$
4	$210 = 2 \cdot 3 \cdot 5 \cdot 7$	$\left\{6, \frac{21}{4}\right\}$
5	$2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	$\left\{\frac{15}{2}\right\}$
6	$255255 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	$\{15\}$
7	$8580495 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23$	$\{15\}$
8	$294076965 = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	$\{21\}$

Table 2. The smallest squarefree composite number N_k such that $\mathbb{Q}$-$\mathcal{KW}(N_k) = k$.

k	N_k	$\mathbb{Q}$-$\mathcal{KS}(N_k)$
0	$138 = 2 \cdot 3 \cdot 23$	$\emptyset$
1	$22 = 2 \cdot 11$	$\{12\}$
2	$102 = 2 \cdot 3 \cdot 17$	$\left\{12, \frac{17}{5}\right\}$
3	$14 = 2 \cdot 7$	$\left\{8, 6, \frac{7}{2}\right\}$
4	$42 = 2 \cdot 3 \cdot 7$	$\left\{6, \frac{21}{8}, \frac{28}{9}, \frac{9}{2}\right\}$
5	$10 = 2 \cdot 5$	$\left\{4, 6, \frac{10}{3}, \frac{5}{2}, \frac{14}{3}\right\}$
6	$273 = 3 \cdot 7 \cdot 13$	$\left\{-7, 8, 9, \frac{78}{11}, \frac{19}{31}, \frac{21}{2}\right\}$
7	$70 = 2 \cdot 5 \cdot 7$	$\left\{4, 6, \frac{5}{2}, \frac{7}{4}, \frac{56}{11}, \frac{25}{4}, \frac{48}{7}\right\}$
8	$30 = 2 \cdot 3 \cdot 5$	$\left\{4, 6, \frac{15}{8}, \frac{40}{13}, \frac{5}{2}, \frac{10}{3}, \frac{15}{4}, \frac{24}{5}\right\}$
9	$6 = 2 \cdot 3$	$\left\{4, \frac{3}{2}, \frac{10}{3}, \frac{14}{5}, \frac{8}{3}, \frac{5}{2}, \frac{18}{7}, \frac{12}{5}, \frac{9}{4}\right\}$
10	$110 = 2 \cdot 5 \cdot 11$	$\left\{8, 20, \frac{44}{13}, \frac{55}{14}, \frac{88}{17}, \frac{22}{5}, \frac{31}{2}, \frac{13}{2}, \frac{35}{4}, \frac{46}{5}\right\}$

3. $\mathbb{Q}$-Korselt numbers with two prime factors

In this section, we shall discus the case where N is a squarefree composite number with two prime factors. Let p and q be two prime numbers such that $p < q$, $N = pq$ and $\alpha = \dfrac{\alpha_1}{\alpha_2}$ be a rational number.

Proposition 3.1 *If N is a K_α-number such that* $\gcd(\alpha_1, N) = 1$, *then*

$$q - p + 1 \leq \alpha \leq q + p - 1.$$

Proof Since N is a K_α-number, then

$$(S_1) \qquad \begin{cases} \alpha_2 p - \alpha_1 \mid p(q-1) \\ \alpha_2 q - \alpha_1 \mid q(p-1). \end{cases}$$

As, in addition, $\gcd(\alpha_1, p) = \gcd(\alpha_1, q) = 1$, it follows that

$$(S_2) \qquad \begin{cases} \alpha_2 p - \alpha_1 \mid q-1 & (3.1) \\ \alpha_2 q - \alpha_1 \mid p-1. & (3.2) \end{cases}$$

Hence, by (3.2), we get

$$-p+1 \leq \alpha_1 - \alpha_2 q \leq p-1.$$

Knowing that $\alpha_2 \geq 1$, we deduce that

$$q-p+1 \leq q - \frac{p-1}{\alpha_2} \leq \alpha = \frac{\alpha_1}{\alpha_2} \leq q + \frac{p-1}{\alpha_2} \leq q+p-1.$$

□

In order to establish the set of $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q}$ with $\gcd(\alpha_1, N) \neq 1$ and for which N is a K_α-number, we need the next two results.

Proposition 3.2 *Let N be a K_α-number such that $\alpha < q-p+1$. Then the following assertions hold:*

1) q divides α_1.

2) If p divides α_1 (i.e. N divides α_1 and so $\gcd(\alpha_1, N) = N$), then $\alpha_1 = N$ and $\alpha_2 = 2p-1$.

Proof

1) Since $\alpha = \dfrac{\alpha_1}{\alpha_2} < q-p+1$, we have $\alpha_2(p-1) < \alpha_2 q - \alpha_1$.

   If $\gcd(q, \alpha_1) = 1$, then by (3.2) it follows that

$$\alpha_2(p-1) < \alpha_2 q - \alpha_1 \leq p-1.$$

   Hence, $\alpha_2 < 1$, which contradicts $\alpha_2 \in \mathbb{N} \setminus \{0\}$. Thus, $q \mid \alpha_1$.

2) Let $\alpha_1 = \alpha_1^{''} pq$ with $\alpha_1^{''} \in \mathbb{N} \setminus \{0\}$. Then (S_1) gives

$$(S_3) \qquad \begin{cases} \alpha_2 - \alpha_1^{''} q \mid q-1 & (3.3) \\ \alpha_2 - \alpha_1^{''} p \mid p-1. & (3.4) \end{cases}$$

   Let us show that $\alpha_1 = N$ and $\alpha_2 = 2p-1$.

   As $\alpha = \dfrac{\alpha_1}{\alpha_2} < q-p+1$, then

$$\alpha_2(p-1) < \alpha_2 q - \alpha_1 = (\alpha_2 - \alpha_1^{''} p)q.$$

It follows by (3.4), that

$$\alpha_2(p-1) < q(\alpha_2 - \alpha_1^{''}p) \leq q(p-1).$$

Hence, $\alpha_2 < q$. Furthermore, since by (3.3), $\alpha_1^{''}q - \alpha_2 < q-1$, it follows that $\alpha_1^{''}q < \alpha_2 + q - 1 < 2q-1$, and this forces $\alpha_1^{''} = 1$. Therefore, $\alpha_1 = pq = N$.

Now let us prove that $\alpha_2 = 2p-1$. First, as $\frac{pq}{\alpha_2} = \alpha < q-p+1$, then $p < \alpha_2(\frac{q-p+1}{q}) < \alpha_2$. Consequently, as $\alpha_1^{''} = 1$ and $\alpha_2 - p > 0$, it follows by (3.4) that $\alpha_2 - p = \frac{p-1}{k}$ with $k \in \mathbb{N} \setminus \{0\}$. We claim that $k=1$. Indeed, suppose by contradiction that $k \neq 1$; then $\alpha_2 - p \leq \frac{p-1}{2}$ and hence

$$\alpha_2 \leq \frac{3p-1}{2}. \tag{3.5}$$

Furthermore, since by hypothesis $\frac{pq}{\alpha_2} = \alpha < q-p+1$, it follows by (3.5) that $pq < \alpha_2(q-p+1) \leq \frac{3p-1}{2}(q-p+1)$. This is equivalent to $q-3p+1 < p(q-3p+1)$ and hence

$$3p-1 < q. \tag{3.6}$$

However, as in addition $\alpha \neq N$, i.e. $\alpha_2 \neq 1$ and $\alpha_1^{''} = 1$, we get by (3.3) $q - \alpha_2 \leq \frac{q-1}{2}$. This yields by (3.5) $q \leq 2\alpha_2 - 1 \leq 3p-2$, a contradiction with (3.6). Thus, $k=1$ and so $\alpha_2 = 2p-1$.

□

Lemma 3.3 *If N is a K_α-number such that $\gcd(\alpha_1, N) \neq 1$ and $q+p-1 < \alpha$, then $\alpha_1 = pq = N$.*

Proof As $q+p-1 < \alpha = \frac{\alpha_1}{\alpha_2}$, then we have

$$0 < \alpha_2(q-1) < \alpha_1 - \alpha_2 p \tag{3.7}$$

and

$$0 < \alpha_2(p-1) < \alpha_1 - \alpha_2 q. \tag{3.8}$$

First we claim that $\gcd(p, \alpha_1) \neq 1$. Indeed, if not, then by combining (3.1) and (3.7), we get

$$0 < \alpha_2(q-1) < \alpha_1 - \alpha_2 p \leq q-1.$$

This implies that $\alpha_2 < 1$, which contradicts $\alpha_2 \in \mathbb{N} \setminus \{0\}$. Thus, $p \mid \alpha_1$.

Similarly, by (3.2) and (3.8) we get $q \mid \alpha_1$. Hence, $\alpha_1 = \alpha_1^{''}pq$ with $\alpha_1^{''} \in \mathbb{N}$. Let us show that $\alpha_1^{''} = 1$. By (3.3) and (3.4), we get respectively

$$\alpha_1^{''}q - \alpha_2 \leq q-1 \tag{3.9}$$

and

$$\alpha_1^{''}p - \alpha_2 \leq p-1. \tag{3.10}$$

Multiplying (3.9) by p and combining it with (3.7), we obtain

$$\alpha_2(q-1) < \alpha_1 - \alpha_2 p = p(\alpha_1^{''} q - \alpha_2) \leq p(q-1),$$

and hence

$$\alpha_2 < p. \tag{3.11}$$

Now, combining (3.10) and (3.11), we get

$$(\alpha_1^{''} - 1)p < \alpha_1^{''} p - \alpha_2 \leq p - 1.$$

This implies that $\alpha_1^{''} = 1$, so $\alpha_1 = pq = N$. □

Proposition 3.4 *Suppose that N is a K_α-number with $\gcd(\alpha_1, N) \neq 1$. Then the following assertions hold:*

1) If $\alpha \in \mathbb{Z}$ (i.e. $\alpha_2 = 1; \alpha = \alpha_1$), then $q \nmid \alpha$, $p \mid \alpha$ and

$$\alpha \in \left\{ \left\lfloor \frac{q}{p} \right\rfloor p, \left\lceil \frac{q}{p} \right\rceil p \right\}.$$

2) If $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, then $\frac{q}{p} \leq \alpha \leq q + p - 1$.

Proof

1) See [7, Corollary 3.6].

2) Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ be such that $\gcd(\alpha_1, N) \neq 1$. Let us show that $\alpha \leq q + p - 1$.

Assume that $q + p - 1 < \alpha$. Then, by Lemma 3.3, (S_1), and (3.11), we have $0 < q - \alpha_2 = \frac{q-1}{k}$ with $k \in \mathbb{N}$. Since $\alpha \neq N$ (i.e. $\alpha_2 \neq 1$) and hence $k \geq 2$, it follows that $q - \alpha_2 \leq \frac{q-1}{2}$; therefore, $\alpha_2 \geq \frac{q+1}{2} > \frac{q}{2}$. As by Lemma 3.3, $\alpha_1 = pq = N$, it yields that $\alpha = \frac{pq}{\alpha_2} < \frac{2pq}{q} = 2p < p + q - 1$, which contradicts the assumption $\alpha > q + p - 1$.

It remains to prove that $\frac{q}{p} \leq \alpha$. First, since $\frac{q}{p} < q - p + 1$, we may suppose that $\alpha < q - p + 1$.

By Proposition 3.2, $\alpha_1 = \alpha_1^{'} q$ with $\alpha_1^{'} \in \mathbb{Z}$. Let us prove that $\alpha_1^{'} > 0$. The result is immediate by Proposition 3.2 when $p \mid \alpha_1$. Now, if $\gcd(p, \alpha_1) = 1$ and by (3.1) we have

$$\alpha_2 p - \alpha_1^{'} q = \alpha_2 p - \alpha_1 \leq q - 1,$$

this implies that $p < \alpha_2 p + 1 \leq q(1 + \alpha_1^{'})$, which forces $\alpha_1^{'} > 0$.

On the other hand, we have by (S_1)

$$(\alpha_2 - \alpha_1^{'})q = \alpha_2 q - \alpha_1 \leq q(p-1).$$

Hence, $\alpha_2 \leq \alpha_1^{'} + p - 1$, so

$$\alpha = \frac{\alpha_1}{\alpha_2} = \frac{\alpha_1^{'} q}{\alpha_2} \geq \frac{\alpha_1^{'} q}{\alpha_1^{'} + p - 1}.$$

Since, in addition, $\dfrac{\alpha_1^{'}}{\alpha_1^{'} + p - 1}$ is minimum when $\alpha_1^{'} = 1$, it follows that $\alpha \geq \dfrac{q}{p}$.

□

By Propositions 3.4 and 3.1, the next two results follow immediately.

Corollary 3.5 *Let* $\alpha \in \mathbb{Q} \setminus \{0\}$.

If N *is a* K_α*-number, then* $\dfrac{q}{p} \leq \alpha \leq q + p - 1$.

Theorem 3.6 *Let* $\alpha \in \mathbb{Q} \setminus \{0\}$. *If* $\alpha \leq 1$, *then each* K_α*-number has at least three prime factors.*

The next result shows that an $\alpha > 1$ can belong to only finitely many $\mathbb{Q}$-$\mathcal{KS}(pq)$.

Theorem 3.7 *Let* $\alpha \in \mathbb{Q} \setminus \{0\}$ *with* $\alpha > 1$, *and suppose that* N *is a* K_α*-number. Then the following assertions hold:*

(a) *If* $\alpha \in \mathbb{Z}$, *then* $p < q \leq 4\alpha - 3$.

(b) *If* $\alpha = \dfrac{\alpha_1}{\alpha_2} \in \mathbb{Q} \setminus \mathbb{Z}$, *then* $p < q \leq \alpha_1$.

Proof

(a) See [3, Theorem 1.10].

(b) First, if q divides α_1, then the result is immediate.

Now assume that $\gcd(q, \alpha_1) = 1$. As $N = pq$ is a K_α-number, it follows by (S_2) that $\alpha_2 q - \alpha_1$ divides $p - 1$. This implies that $\alpha_2 q - \alpha_1 \leq p - 1 < q - 1$. Thus, $q < \dfrac{\alpha_1 - 1}{\alpha_2 - 1} < \alpha_1$.

□

Remark 3.8 In case (b) of Theorem 3.7, the upper bound can be reached when $q = 3, p = 2$, and $\alpha = \dfrac{3}{2}$.

We obtain immediately from Theorem 3.7 the following result.

Theorem 3.9 *Let* $\alpha \in \mathbb{Q} \setminus \{0\}$. *Then there are only finitely many* K_α*-numbers with exactly two prime factors.*

Now we ask: do there exist (and how many) rationals $1 < \alpha < C$, where C is a fixed rational number, for which there are no K_α-numbers with two prime factors? Computationally, this problem can be solved by running a computer program with exhaustive research (see [3, Example 1.11]). However, for the case $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, it seems to be more difficult computationally and theoretically to find such a solution. This does not prevent us from providing, by the next proposition, all rationals $1 < \alpha < 2$ for which there are no K_α-numbers with two prime factors.

Proposition 3.10 *Let $\alpha \in \mathbb{Q}$ be such that $1 < \alpha < 2$. $N = pq$ is a K_α-number if and only if $\alpha = \frac{q}{p}$ with $(p,q) \in \{(2,3),(3,5)\}$.*

Proof Suppose that $\alpha \in \mathbb{Q}\text{-}\mathcal{KS}(N)$. Since $\alpha < 2 \leq q - p + 1$, then by Proposition 3.2, q divides α_1. Hence, $\alpha_1 = \alpha_1^{'} q$ with $\alpha_1^{'} \in \mathbb{N}$.

First we claim that $\gcd(p, \alpha_1) = 1$. Suppose by contradiction that p divides $\alpha_1^{'}$ (i.e. $N \mid \alpha_1$). Then, by Proposition 3.2, $\alpha = \frac{pq}{2p-1}$, but, as by hypothesis $\frac{pq}{2p-1} = \alpha < 2$, we obtain $p(q-4) < -2$. Hence, $q = 3$ and $p = 2$, and so $\alpha_1 = pq = 6$ and $\alpha_2 = 2p - 1 = 3$, which contradicts the fact that $\gcd(\alpha_1, \alpha_2) = 1$.

Now, as $\gcd(p, \alpha_1) = 1$, then (S_1) gives

$$(S_4) \qquad \begin{cases} \alpha_2 p - \alpha_1^{'} q \mid q - 1 & (3.12) \\ \alpha_2 - \alpha_1^{'} \quad \mid p - 1. & (3.13) \end{cases}$$

Since $\alpha = \frac{\alpha_1}{\alpha_2} < 2$, i.e. $\frac{\alpha_1^{'} q}{2} = \frac{\alpha_1}{2} < \alpha_2$, we get by (3.12)

$\frac{\alpha_1^{'}}{2} qp - \alpha_1^{'} q \leq \alpha_2 p - \alpha_1^{'} q \leq q - 1$. Hence, $\alpha_1^{'} q(\frac{p}{2} - 1) < q$, so $p = 2$ or ($\alpha_1^{'} = 1$ and $p = 3$).

- If $p = 2$, then by (3.13), we get $\frac{\alpha_1^{'} q}{2} - \alpha_1^{'} < \alpha_2 - \alpha_1^{'} \leq p - 1 = 1$. Hence, $\alpha_1^{'}(q-2) < 2$, and consequently $\alpha_1^{'} = 1$, $q = 3$, and $\alpha = \frac{3}{2}$.
- Now assume that $p = 3$ and $\alpha_1^{'} = 1$. As $\alpha_1 = q$ and $\alpha_2 > \frac{q}{2}$, then by (3.13), we get $\frac{q}{2} - 1 < \alpha_2 - \alpha_1^{'} = \alpha_2 - 1 \leq p - 1 = 2$. Therefore, $q < 6$. However, as in addition $q > p = 3$, necessarily $q = 5$, and so $\alpha_2 = 3$ and $\alpha = \frac{5}{3}$.

Conversely, we verify easily that $2 * 3 = 6$ is a $K_{\frac{3}{2}}$-number and $3 * 5 = 15$ is a $K_{\frac{5}{3}}$-number.

□

By Proposition 3.10, we may say that for each $1 < \alpha < 2$ with $\alpha \neq \frac{3}{2}$ and $\alpha \neq \frac{5}{3}$, there is no squarefree composite number N with two prime factors such that N is a K_α-number. The question about the infinitude of the K_α-numbers for a given $\alpha \in \mathbb{Q}$ remains posed. This can not be easily solved with an idea inspired by the proof of the case $\alpha = 1$ given by Alford et al. in [1]. However, following the heuristic ideas of Erdos, we believe the following:

Conjecture 3.11 *For any given $\alpha \in \mathbb{Q} \setminus \{0\}$ there exist infinitely many K_α-numbers.*

Acknowledgment I thank the referee for his/her report improving both the presentation and the mathematical content of the paper.

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