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Bargraphs in bargraphs

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Abstract: Bargraphs are lattice paths in $\mathbb{N}_0^2$ that start at the origin and end upon their first return to the x-axis. Each bargraph is represented by a sequence of column heights $\pi_1\pi_2\cdots\pi_m$ such that column j contains π_j cells. In this paper, we study the number of bargraphs with n cells and m columns according to the distribution for the statistic that records the number of times a given shape lies entirely within a bargraph for various small shapes.

Key words: Bargraphs, generating functions, C-vertices, combinatorial statistic

1. Introduction

Bargraphs (or *barcharts*) are closed lattice paths in $\mathbb{N}_0^2$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, that start with $(0, 0)$ and upon their first return to the x-axis end their progression to the right and return to $(0, 0)$. Each step is an up step $(0, 1)$, a down step $(0, -1)$, a right horizontal step $(1, 0)$, or a left horizontal step $(-1, 0)$. The first step has to be an up step, the right horizontal steps must all lie above the x-axis, and the left horizontal steps must all lie on the x-axis. An up step cannot directly follow a down step and vice versa. Clearly, the number of down steps must equal the number of up steps.

While similar lattice paths, such as Dyck and Motzkin paths, have been extensively studied (see [6, 11]), bargraphs have not attracted the same amount of combinatorial interest. Bargraphs have been studied previously from the standpoint of statistical physics (see, e.g., [12, 16–18]), and a connection between bargraphs and probability theory can be found in [8]. Enumeration of bargraphs was undertaken in [5, 8, 9], where in [8, 9] they are referred to as *wall polyominoes*. Bargraphs are also closely related to compositions of integers [10, 15].

Other results pertaining to bargraphs appear in [1–4], where the authors study the distribution of several statistics such as number of peaks, number of levels, height, and width. More recent results appear in [14], where the authors study corners in compositions and set partitions of a fixed size represented geometrically as bargraphs.

In [7], the authors found a simple bijection between bargraphs and Motzkin paths without peaks or valleys. Based on it, they use the recursive structure of Motzkin paths to enumerate bargraphs with respect to several parameters, find simpler derivations of known results, and obtain many new results.

Before we state the goal of the paper, we provide a few definitions. Let B be a bargraph. Four points (x, y), $(x+1, y)$, $(x+1, y+1)$, $(x, y+1)$ that lie on or within B define what we call a cell of B. Each bargraph

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will be identified as a sequence of column heights $\pi = \pi_1\pi_2\cdots\pi_m$ such that column j contains exactly π_j cells. We denote the set of bargraphs with n cells and m columns by $\mathcal{B}_{n,m}$. Bargraphs with n cells and m columns are clearly in a one-to-one correspondence with compositions of n with exactly m parts, whence $|\mathcal{B}_{n,m}| = \binom{n-1}{m-1}$ (see, e.g., [10, p. 2]). For instance, Figure 1 presents the bargraph 2331422.

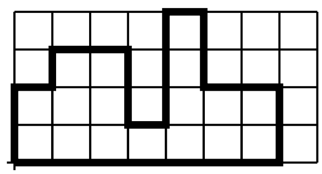

Figure 1. The bargraph 2331422.

Let B and C be bargraphs. We say that a vertex (x, y) of B is a C*-vertex* if C lies entirely in B when positioned starting at (x, y). We denote the number of C-vertices of B by $C(B)$. In other words, $C(B)$ denotes the number of ways C can be positioned within B so that its vertices coincide with those contained on or within B. For instance, if B is the bargraph given in Figure 1 and C is the bargraph 12, then the C-vertices of B are $(0, 0)$, $(1, 0)$, $(3, 0)$, $(4, 0)$, $(5, 0)$, $(0, 1)$, and $(1, 1)$, which implies $C(B) = 7$.

Remark 1.1 Our definition of C-vertices is related to *the generalized factor order in words* (see, e.g., [13]). We say that the word $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ is a generalized factor order of the word $\pi = \pi_1\pi_2\cdots\pi_n$ at position j if $\sigma_s \le \pi_{j-1+s}$ for all $s = 1, 2, \ldots, m$. Therefore, our $C = C_1C_2\cdots C_m$-vertex at (a, b) in bargraph $B = B_1B_2\cdots B_n$ corresponds to the word $(C_1 + b)(C_2 + b)\cdots(C_m + b)$, which is a generalized factor order of the word $B_{a+1}B_{a+2}\cdots B_{a+m}$. For instance, if $C = 12$, then the bargraph 232 contains C three times, namely, the C-vertices are $(0, 0)$, $(0, 1)$, and $(1, 0)$, which correspond to the factor 12 at position 1, the factor 23 at position 1, and the factor 12 at position 2, respectively.

In this paper, we are interested in studying the generating function $F_C(x, y, q)$ for the number of bargraphs with n cells and m columns according to the number of C-vertices tracked by q, where C is a fixed bargraph, namely,

$$F_C(x, y, q) = \sum_{n\ge 0}\sum_{m=0}^{n}\sum_{B\in\mathcal{B}_{n,m}} x^n y^m q^{C(B)}.$$

In order to do so, we extend our notation as follows. We denote the generating function for the number of bargraphs $\pi = a_1a_2\cdots a_s\pi'$ with n cells and m columns that begin with $a_1a_2\cdots a_s$ according to the number of C-vertices by $F_C(x, y, q|a_1a_2\cdots a_s)$.

2. Counting C-vertices

2.1. C has one column

First, we consider the case when C is a bargraph having c cells lying in a single column. By the definitions, we have

$$F_C(x, y, q|a) = x^a y F_C(x, y, q), \quad 1 \le a \le c - 1,$$

and

$$F_C(x, y, q|a) = x^a y q^{a+1-c} F_C(x, y, q), \quad a \ge c.$$

Thus,

$$F_C(x,y,q)-1=\sum_{a\geq 1}F_C(x,y,q|a)=\sum_{a=1}^{c-1}x^ayF_C(x,y,q)+\sum_{a\geq c}x^ayq^{a+1-c}F_C(x,y,q)$$
$$=\frac{(x-x^c)y}{1-x}F_C(x,y,q)+\frac{x^cyq}{1-qx}F_C(x,y,q),$$

which implies the following result.

Theorem 2.1 *Let C be a bargraph with c cells and one column. Then*

$$F_C(x,y,q)=\frac{1}{1-\frac{x-x^c}{1-x}y-\frac{x^cyq}{1-xq}}.$$

Table 1. The triangular distribution of $F_C(x,1,q)$ for $c=3$.

n/k	0	1	2	3	4	5	...
1	1						
2	2						
3	3	1					
4	5	2	1				
5	8	5	2	1			
6	13	10	6	2	1		
7	21	20	13	7	2	1	

Theorem 2.1 gives $\frac{d}{dq}F_C(x,1,q)\mid_{q=1}=\frac{x^c}{(1-2x)^2}$. Thus, the number of C-vertices in all bargraphs with n cells is given by $(n-c+1)2^{n-c}$ for all $n\geq c$. Note that since the last expression depends only on the difference $n-c$, for each c, we have basically the same sequence, which corresponds to A001787 in [19]. In the following table, we show the triangular distribution of $F_C(x,1,q)$ for $c=3$, which corresponds to A076791 in [19].

2.2. Counting $11\cdots 1$-vertices

Let $C=1^c=11\cdots 1$, where 1 is repeated $c\geq 1$ times. Note that the bargraph B contains 1^c if and only if there exist x and y such that the vertices $(x,y),(x+1,y),\ldots,(x+c,y)$ lie on or within B. In order to study the generating function $F_C(x,y,q)$ in this case, we need the following further definitions.

A *forest bargraph* is a sequence of bargraphs including the empty bargraph separated by empty columns. In other words, a *forest bargraph* of n with m columns is a sequence $\sigma_1\sigma_2\cdots\sigma_m$ such that $\sum_{i=1}^m\sigma_i=n$ with $\sigma_j\geq 0$ for $1\leq j\leq m$ (where 0 presents an empty column). For example, Figure 2 presents the forest bargraph 1300102.

Let $G(x,y,q)$ be the generating function for the number of forest bargraphs of n with m columns according to the number of C-vertices. Note that each forest bargraph can be written as either π or $\pi 0\pi'$, where π is a bargraph (each letter of π is at least 1) and π' is a forest bargraph. For instance, for the forest 1300102, we have $\pi=13$ and $\pi'=0102$ (see Figure 2), and for the forest 0102, we have π is the empty word (bargraph) and $\pi'=102$. Thus, the generating function $G(x,y,q)$ is given by $G(x,y,q)=F_C(x,y,q)+yF_C(x,y,q)G(x,y,q)$,

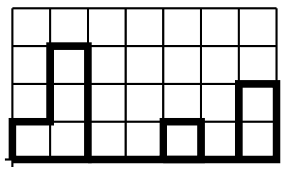

Figure 2. The forest bargraph 1300102 of 7 with 7 columns.

which leads to

$$G(x,y,q)=\frac{F_C(x,y,q)}{1-yF_C(x,y,q)}. \tag{1}$$

Now let us write an equation for the generating function $F_C(x,y,q)$. A bargraph π is either empty, of the form $\pi=\pi_1\pi_2\cdots\pi_j$ with $1\le j\le c-1$, or contains at least c columns. The contributions from the first two cases are given by 1 and $\frac{x^jy^j}{(1-x)^j}$ for $1\le j\le c-1$, respectively. For the third case, we consider the forest bargraph $\pi'=(\pi_1-1)\cdots(\pi_m-1)$ with $m\ge c$, which gives the contribution $\frac{1}{q^{c-1}}\left(G(x,qxy,q)-\sum_{j=0}^{c-1}\frac{q^jx^jy^j}{(1-x)^j}\right)$. By adding all the contributions, we obtain

$$F_C(x,y,q)=\sum_{j=0}^{c-1}\frac{x^jy^j}{(1-x)^j}+\frac{1}{q^{c-1}}\left(G(x,qxy,q)-\sum_{j=0}^{c-1}\frac{q^jx^jy^j}{(1-x)^j}\right).$$

Hence, by (1), the generating function $F_C(x,y,q)$ satisfies

$$F_C(x,y,q)=\sum_{j=0}^{c-1}\frac{x^jy^j(1-q^{j-c+1})}{(1-x)^j}+\frac{1}{q^{c-1}}\frac{F_C(x,qxy,q)}{1-qxyF_C(x,qxy,q)}. \tag{2}$$

Define $\alpha(x,y,q)=\sum_{j=0}^{c-1}\frac{x^jy^j(1-q^{j-c+1})}{(1-x)^j}$. Then

$$F_C(x,y,q)=\alpha(x,y,q)-\frac{1/q^{c-1}}{qxy-\frac{1}{F_C(x,qxy,q)}},$$

which leads to the following result (see Table 2).

Theorem 2.2 *Let $C=1^c$ with $c\ge 1$. Then*

$$F_C(x,y,q)=\alpha(x,y,q)-\cfrac{1/q^{c-1}}{qxy-\cfrac{1}{\alpha(x,qxy,q)-\cfrac{1/q^{c-1}}{(qx)^2y-\cfrac{1}{\alpha(x,(qx)^2y,q)-\cfrac{1/q^{c-1}}{(qx)^3y-\ddots}}}}}.$$

Note that (2) gives $F_C(x,y,1)=\frac{F_C(x,xy,1)}{1-xyF_C(x,xy,1)}$. Clearly, $F_C(x,y,1)=\frac{1}{1-\frac{xy}{1-x}}$. Let $A(x,y)=\frac{d}{dq}F_C(x,y,q)\mid_{q=1}$. By the fact that

$$\frac{\partial}{\partial q}F_C(x,qxy,q)\mid_{q=1}=xy\frac{\partial}{\partial u}F_C(x,u,1)\mid_{u=xy}+A(x,xy)$$

and $F_C(x,y,1) = \frac{1}{1-\frac{xy}{1-x}}$, (2) gives

$$
\begin{aligned}
A(x,y) &= \sum_{j=0}^{c-1} \frac{(c-1-j)x^j y^j}{(1-x)^j} - (c-1)\frac{F_C(x,xy,1)}{1-xyF_C(x,xy,1)} \\
&+ \frac{xyF_C^2(x,xy,1) + xy\frac{d}{du}F_C(x,u,1)\mid_{u=xy} + A(x,xy)}{(1-xyF_C(x,xy,1))^2} \\
&= \sum_{j=0}^{c-1} \frac{(c-1-j)x^j y^j}{(1-x)^j} - (c-1)F_C(x,y,1) + xyF_C^2(x,y,1) + \frac{xy\frac{d}{du}F_C(x,u,1)\mid_{u=xy}}{(1-xyF_C(x,xy,1))^2} \\
&+ \frac{A(x,xy)}{(1-xyF_C(x,xy,1))^2} \\
&= \frac{x^c y^c}{(1-x)^{c-2}(xy+x-1)^2} + \frac{(x^2y+x-1)^2}{(xy+x-1)^2}A(x,xy).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
A(x,y) &= \frac{x^c y^c}{(xy+x-1)^2(1-x)^{c-2}} + \frac{(x^2y+x-1)^2}{(xy+x-1)^2}A(x,xy) \\
&= \frac{(x^c + x^{2c})y^c}{(xy+x-1)^2(1-x)^{c-2}} + \frac{(x^3y+x-1)^2}{(xy+x-1)^2}A(x,x^2y) \\
&= \frac{(x^c + x^{2c} + x^{3c})y^c}{(xy+x-1)^2(1-x)^{c-2}} + \frac{(x^4y+x-1)^2}{(xy+x-1)^2}A(x,x^3y) \\
&= \cdots .
\end{aligned}
$$

Assume $|x|, |y| < 1$. Iterating infinitely many times and noting $A(x,0) = 0$, we obtain

$$
A(x,y) = \sum_{j\geq 0} \frac{x^{c+cj}y^c}{(1-x)^{c-2}(xy+x-1)^2},
$$

which leads to $A(x,y) = \frac{d}{dq}F_C(x,y,q)\mid_{q=1} = \frac{x^c y^c}{(1-x)^{c-2}(1-x^c)(1-x-xy)^2}$. Hence, we can state the following result.

Corollary 2.3 *The generating function for the number of* 1^c*-vertices in all bargraphs with* n *cells is given by*

$$
\frac{x^c}{(1-x)^{c-2}(1-x^c)(1-2x)^2}.
$$

For example, the generating functions for the number of 1-vertices, 11-vertices, and 111-vertices in all bargraphs with n cells are given by $\frac{x}{(1-2x)^2}$, $\frac{x^2}{(1-x^2)(1-2x)^2}$, and $\frac{x^3}{(1-x)(1-x^3)(1-2x)^2}$, respectively.

In the following table, we show the triangular distribution of $F_C(x,1,q)$ for $c=2$, which corresponds to A110971 in [19].

Table 2. The triangular distribution of $F_C(x,1,q)$ for $c=2$.

n/k	0	1	2	3	4	...
1	1					
2	1	1				
3	1	2	1			
4	1	2	4	1		
5	1	2	6	6	1	

2.3. Counting $c1$-vertices

Let $C=c1$. By the definitions, we have

$$
\begin{aligned}
F_C(x,y,q|i) &= x^i y F_C(x,y,q), \quad i=1,2,\dots,c-1,\\
F_C(x,y,q|a) &= x^a y + \sum_{b=1}^{a-c+1} F_C(x,y,q|ab) + \sum_{b\geq a-c+2} F_C(x,y,q|ab)\\
&= x^a y + x^a y \sum_{b=1}^{a-c+1} q^b F_C(x,y,q|b) + x^a y q^{a-c+1} \sum_{b\geq a-c+2} F_C(x,y,q|b)\\
&= x^a y(1-q^{a-c+1}) + x^a y q^{a-c+1} F_C(x,y,q) + x^a y \sum_{b=1}^{a-c+1} (q^b - q^{a-c+1}) F_C(x,y,q|b),
\end{aligned}
$$

where $a\geq c$. In order to solve this recurrence relation, we define $F_C(u)=F_C(x,y,q;u)=1+\sum_{a\geq 1} F_C(x,y,q|a)u^a$. Multiplying the recurrence by u^a and summing over $a\geq c$, we find

$$
\begin{aligned}
\sum_{a\geq c} F_C(x,y,q|a)u^a &= \sum_{a\geq c} x^a u^a y(1-q^{a-c+1}) + \sum_{a\geq c} x^a y q^{a-c+1} u^a F_C(x,y,q)\\
&\quad + \sum_{a\geq c}\left(x^a y \sum_{b=1}^{a-c+1} (q^b - q^{a-c+1}) F_C(x,y,q|b)\right) u^a,
\end{aligned}
$$

which implies

$$\begin{aligned}
F_C(u) &= 1 + \sum_{a=1}^{c-1} F_C(x,y,q|a)u^a + \frac{x^c u^c y}{1-xu} - \frac{x^c u^c yq}{1-qxu} + \frac{x^c u^c yq}{1-qxu} F_C(x,y,q) \\
&+ \sum_{a\geq 1}\sum_{j\geq a} x^{c+j-1}u^{c+j-1}y(q^a - q^j)F_C(x,y,q|a) \\
&= 1 + \sum_{a=1}^{c-1} F_C(x,y,q|a)u^a + \frac{x^c u^c y}{1-xu} - \frac{x^c u^c yq}{1-qxu} + \frac{x^c u^c yq}{1-qxu} F_C(x,y,q) \\
&+ \sum_{a\geq 1}\left(\frac{x^{c+a-1}u^{c+a-1}yq^a}{1-xu} - \frac{x^{c+a-1}u^{c+a-1}yq^a}{1-qxu}\right) F_C(x,y,q|a) \\
&= 1 + \sum_{a=1}^{c-1} F_C(x,y,q|a)u^a + \frac{x^c u^c y}{1-xu} - \frac{x^c u^c yq}{1-qxu} + \frac{x^c u^c yq}{1-qxu} F_C(x,y,q) \\
&+ \frac{x^{c-1}u^{c-1}y}{1-xu}(F_C(qxu)-1) - \frac{x^{c-1}u^{c-1}y}{1-qxu}(F_C(qxu)-1).
\end{aligned}$$

By using the initial conditions $F_C(x,y,q|i) = x^i y F_C(x,y,q) = x^i y F_C(1)$ for $i = 1, 2, \ldots, c-1$, we have

$$F_C(u) = 1 + \left(xyu + \cdots + x^{c-1}u^{c-1}y + \frac{qx^c yu^c}{1-qxu}\right) F_C(1) + \frac{(1-q)x^c yu^c}{(1-xu)(1-qxu)} F_C(xqu).$$

Assume that $|x|, |q| < 1$ and $|u| \leq 1$. Iterating the above equation infinitely many times yields

$$F_C(u) = \sum_{j\geq 0}\left(1 + \left(\sum_{s=1}^{c-1} x^{s(j+1)}q^{sj}yu^s + \frac{q^{1+cj}x^{c(1+j)}yu^c}{1-(qx)^{j+1}u}\right)F_C(1)\right)\prod_{i=0}^{j-1}\frac{x^{ci+c}q^{ci}y(1-q)u^c}{(1-x^{i+1}q^i u)(1-(xq)^{i+1}u)}.$$

By substituting $u = 1$, and solving for $F_C(1)$, we obtain the following result.

Theorem 2.4 *The generating function for the number of bargraphs with n cells and m columns according to the number of $c1$-vertices is given by*

$$F_C(x,y,q) = \frac{\sum_{j\geq 0}\frac{x^{c\binom{j+1}{2}}q^{c\binom{j}{2}}y^j(1-q)^j}{\prod_{i=0}^{j-1}(1-x^{i+1}q^i)(1-(xq)^{i+1})}}{1 - \sum_{j\geq 0}\frac{x^{c\binom{j+1}{2}}q^{c\binom{j}{2}}(1-q)^j y^j\left(\frac{y((x^{j+1}q^j)^c - x^{j+1}q^j)}{x^{j+1}q^j - 1} + \frac{q^{1+cj}x^{c(1+j)}y}{1-(qx)^{j+1}}\right)}{\prod_{i=0}^{j-1}(1-x^{i+1}q^i)(1-(xq)^{i+1})}}.$$

Since $\frac{d}{dq}(1-q)^m\mid_{q=1} = 0$ for all $m \geq 2$, Theorem 2.4 leads to

$$\begin{aligned}
\frac{d}{dq}F_C(x,1,q)\mid_{q=1} &= \frac{d}{dq}\left.\frac{1 + \frac{x^c(1-q)}{(1-x)(1-xq)}}{1 - \sum_{s=1}^{c-1}x^s - \frac{qx^c}{1-qx} - \frac{x^c(1-q)\left(\sum_{s=1}^{c-1}x^{2s}q^s + \frac{q^{1+c}x^{2c}}{1-(qx)^2}\right)}{(1-x)(1-xq)}}\right|_{q=1} \\
&= \frac{x^{c+1}}{(1-2x)^2(1-x^2)} \\
&= x^c\left(\frac{1}{2(1-x)} - \frac{1}{18(1+x)} + \frac{2}{3(1-2x)^2} - \frac{10}{9(1-2x)}\right).
\end{aligned}$$

Corollary 2.5 *The number of $c1$-vertices in all bargraphs with n cells is given by* $\frac{9-(-1)^{n-c}}{18}+\frac{3(n-c)-2}{9}2^{n-c+1}$ *for all $n \geq c$.*

For instance, the number of $c1$-vertices in all bargraphs with $n = c, c+1, c+2$ cells is given by 0, 1, 4, respectively. Note that since the expression in the prior corollary depends only on the difference $n - c$, for each c, we have essentially the same sequence, which corresponds to A102301 in [19]. In Table 3, we give the triangular distribution of $F_C(x, 1, q)$ for $c = 2$, which corresponds to A298637 in [19].

Table 3. The triangular distribution of $F_C(x, 1, q)$ for $c = 2$.

n/k	0	1	2	...
1	1			
2	2			
3	3	1		
4	4	4		
5	5	9	2	

2.4. Counting cd-vertices

Theorem 2.4 can be extended to the case of cd-vertices. Let $C = cd$. By symmetry, namely, applying the reversal operation, we may assume $c \geq d$. By the definitions, we have

$$
\begin{aligned}
&F_C(x, y, q|i) = x^i y F_C(x, y, q), \quad i = 1, 2, \ldots, c-1,\\
&F_C(x, y, q|a) = x^a y + \sum_{b=1}^{d-1} F_C(x, y, q|ab) + \sum_{b=d}^{a-c+d} F_C(x, y, q|ab) + \sum_{b \geq a-c+d+1} F_C(x, y, q|ab)\\
&\qquad = x^a y + x^a y \sum_{b=1}^{d-1} x^b y F_C(x, y, q) + x^a y \sum_{b=d}^{a-c+d} q^{b-d+1} F_C(x, y, q|b)\\
&\qquad + x^a y q^{a-c+1} \sum_{b \geq a-c+d+1} F_C(x, y, q|b)\\
&\qquad = x^a y(1 - q^{a-c+1}) \left(1 + F_C(x, y, q) \sum_{b=1}^{d-1} x^b y\right) + x^a y q^{a-c+1} F_C(x, y, q)\\
&\qquad + x^a y \sum_{b=d}^{a-c+d} (q^{b-d+1} - q^{a-c+1}) F_C(x, y, q|b),
\end{aligned}
$$

where $a \geq c$. In the last equality, note that we used the relation $F_C(x, y, q) = 1 + \sum_{j \geq 1} F_C(x, y, q|j)$. Similar to the previous subsection, in order to solve the recurrence, we define

$$F_C(u) = F_C(x, y, q; u) = 1 + \sum_{a \geq 1} F_C(x, y, q|a) u^a.$$

By multiplying both sides of the recurrence by u^a, summing over $a \geq c$, and proceeding as before, we obtain

$$\begin{aligned}
F_C(u) &= 1 + \frac{(1-q)x^c y u^c}{(1-xu)(1-qxu)} \\
&+ \left(\sum_{j=1}^{c-1} x^j y u^j + \frac{x^c y u^c}{1-xu}\sum_{j=1}^{d-1} x^j y - \frac{x^c y u^c q}{1-qxu}\sum_{j=1}^{d-1} x^j y + \frac{q x^c y u^c}{1-qxu}\right) F_C(1) \\
&+ \left(\frac{x^{c-d} y u^{c-d}}{q^{d-1}(1-xu)} - \frac{x^{c-d} y u^{c-d}}{q^{d-1}(1-qxu)}\right)\left(F_C(qxu) - 1 - F_C(1)\sum_{j=1}^{d-1} q^j x^{2j} y u^j\right).
\end{aligned}$$

This is equivalent to

$$F_C(u) = \alpha(u) + \alpha'(u)F_C(1) + \beta(u)F_C(qxu), \tag{3}$$

where

$$\begin{aligned}
\alpha(u) &= 1 + \frac{(1-q)x^c y u^c}{(1-xu)(1-qxu)} - \beta(u), \\
\alpha'(u) &= \sum_{j=1}^{c-1} x^j y u^j + \frac{(1-q)x^c y u^c}{(1-xu)(1-qxu)}\sum_{j=1}^{d-1} x^j y + \frac{q x^c y u^c}{1-qxu} - \beta(u)\sum_{j=1}^{d-1} q^j x^{2j} y u^j, \\
\beta(u) &= \frac{x^{c-d} y u^{c-d}}{q^{d-1}(1-xu)} - \frac{x^{c-d} y u^{c-d}}{q^{d-1}(1-qxu)} = \frac{(1-q)x^{c-d+1} y u^{c-d+1}}{q^{d-1}(1-xu)(1-qxu)}.
\end{aligned}$$

Assume $|x|, |y| < 1$. Iterating (3) infinitely many times yields

$$F_C(u) = \sum_{j\geq 0}\left(\alpha(q^j x^j u) + \alpha'(q^j x^j u)F_C(1)\right)\prod_{i=0}^{j-1}\beta(q^i x^i u),$$

which gives

$$F_C(1) = \frac{\sum_{j\geq 0}\alpha(q^j x^j)\prod_{i=0}^{j-1}\beta(q^i x^i)}{1-\sum_{j\geq 0}\alpha'(q^j x^j)\prod_{i=0}^{j-1}\beta(q^i x^i)}.$$

Thus, by substituting the expressions of $\alpha(u), \alpha'(u), \beta(u)$ into $F_C(1)$, we obtain the following result.

Theorem 2.6 *Let $c \geq d \geq 1$ and $C = cd$. Then the generating function for the number of bargraphs with n cells and m columns according to the number of cd-vertices is given by*

$$F_C(x,y,q) = \frac{\sum_{j\geq 0}\alpha(q^j x^j)\prod_{i=0}^{j-1}\beta(q^i x^i)}{1-\sum_{j\geq 0}\alpha'(q^j x^j)\prod_{i=0}^{j-1}\beta(q^i x^i)},$$

where

$$\alpha(u) = 1 - (1 - y(qxu)^{d-1})\frac{(1-q)x^{c-d+1}yu^{c-d+1}}{q^{d-1}(1-xu)(1-qxu)},$$

$$\alpha'(u) = \sum_{j=1}^{c-1} x^j yu^j + \frac{(1-q)x^c yu^c}{(1-xu)(1-qxu)}\sum_{j=1}^{d-1} x^j y + \frac{qx^c yu^c}{1-qxu} - \beta(u)\sum_{j=1}^{d-1} q^j x^{2j} yu^j,$$

$$\beta(u) = \frac{(1-q)x^{c-d+1}yu^{c-d+1}}{q^{d-1}(1-xu)(1-qxu)}.$$

Theorem 2.6 with $q=1$ (here $\beta(u)=0$) gives $F_C(x,y,1) = \frac{\alpha(1)}{1-\alpha'(1)} = \frac{1}{1-\frac{xy}{1-x}}$, as expected. Note that

$$\alpha(q^j x^j)\mid_{q=1} = 1,\ \alpha'(q^j x^j)\mid_{q=1} = \frac{x^{j+1}y}{1-x^{j+1}},\ \beta(q^j x^j)\mid_{q=1} = 0$$

and

$$\frac{d}{dq}\alpha(q^j x^j)\mid_{q=1} = -\frac{x^{c(j+1)}y}{(1-x^{j+1})^2} - \frac{d}{dq}\beta(q^j x^j)\mid_{q=1},$$

$$\frac{d}{dq}\alpha'(q^j x^j)\mid_{q=1} = \frac{y(jx^{j+1}+x^{c(j+1)})}{(1-x^{j+1})^2} - \frac{x^{c(j+1)}y}{(1-x^{j+1})^2}\sum_{s=1}^{d-1} x^s y - \frac{d}{dq}\beta(q^j x^j)\mid_{q=1}\sum_{s=1}^{d-1} x^{s(j+2)}y,$$

$$\frac{d}{dq}\beta(q^j x^j)\mid_{q=1} = -\frac{yx^{(j+1)(c-d+1)}}{(1-x^{j+1})^2}.$$

Therefore, after several algebraic steps,

$$\frac{d}{dq}F_C(x,y,q)\mid_{q=1} = \frac{d}{dq}\left.\frac{\alpha(1)+\alpha(qx)\beta(1)}{1-\alpha'(1)-\alpha'(qx)\beta(1)}\right|_{q=1}$$

$$= \frac{x^{c+d}y^2}{(1-x^2)(1-x-xy)^2}.$$

Let $K = \frac{1}{(1-x^2)(1-2x)^2} = \frac{1}{2(1-x)} + \frac{1}{18(1+x)} + \frac{4}{3(1-2x)^2} - \frac{8}{9(1-2x)}$. Then the coefficient of x^n in $\frac{d}{dq}F_C(x,1,q)\mid_{q=1}$ is given by $[x^{n-c-d}]K$, where $[x^n]K = \frac{9+(-1)^n}{18} + \frac{3n+1}{9}2^{n+2}$, which occurs as [19, Sequence A102301]. Hence, we have the following result.

Corollary 2.7 *Let $c \geq d \geq 1$. The number of cd-vertices in all bargraphs with n cells is given by*

$$\frac{9+(-1)^{n-c-d}}{18} + \frac{(3(n-c-d)+1)2^{n-c-d+2}}{9}$$

for all $n \geq c+d$.

For instance, for $d=1$ it yields Corollary 2.5. Note that the case $c=3, d=2$ corresponds to A118869 in [19].

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