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A note on the associated primes of local cohomology modules for regular local rings

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Abstract: Let R be a regular local ring. In this note, we prove that $Ass_R H^2_I(R)$ is finite for any ideal I of R. We also give a sufficient condition for $Ass_R H^3_{(x,y,z)}(R)$ to be finite for x, y an R-regular sequence and $z \in R$, which would imply that Lyubeznik's conjecture is true in the regular local rings case.

Key words: Local cohomology modules, associated primes, regular local rings

1. Introduction

Throughout this note, R denotes a commutative Noetherian ring with identity, I an ideal of R and M a finite R-module. The i-th local cohomology module of M with support in I is defined as

$$H^i_I(M) = \varinjlim_n Ext^i_R(R/I^n, M).$$

In general, the i-th local cohomology module $H^i_I(M)$ is not finitely generated. For example, if $t \geq 1$ is the largest integer such that $H^t_I(M)$ is nonzero, then $H^t_I(M)$ is not finitely generated. Grothendieck [3] conjectured that $Hom_R(R/I, H^i_I(M))$ is a finite R-module for all $i \geqslant 0$. Later, Hartshorne [4] gave a counterexample to this conjecture. In [6], Huneke conjectured that $Ass_R H^i_I(M)$ is always a finite set. However, Singh [13] and Katzman [8] gave examples to show that $H^i_I(M)$ may have infinitely many associated primes. Huneke and Sharp [7] (in the case of positive characteristic) and Lyubeznik [9] (in characteristic zero) showed that if R is a regular local ring containing a field, then $H^i_I(R)$ has only finitely many associated primes for all ideals I and $i \geqslant 0$. Lyubeznik [10] also showed this result for an unramified regular local ring of mixed characteristic R.

In [9], Lyubeznik conjectured that if R is a regular ring and I an ideal of R, then $H^i_I(R)$ has finitely many associated primes for any $i \geqslant 0$. In [2], Bhatt et al. proved that for a smooth $\mathbb{Z}$-algebra R, the conjecture is true.

For a Cohen-Macaulay local ring R, Hellus [5] showed that if $H^2_{(x,y)}(R)$ and $H^3_{(x,y,z)}(R)$ have finitely many associated primes for some kind of x, y, and z in R, then $Ass_R H^i_I(R)$ is finite for all ideals I and $i \geqslant 0$. In this note, for a regular local ring R, we prove that $Ass_R H^2_I(R)$ is finite for any ideal I of R and give a sufficient condition for $Ass_R H^3_{(x,y,z)}(R)$ to be a finite set.

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2. Main Results

Proposition 2.1 *Let $\phi : R \to S$ be a faithfully flat homomorphism of Noetherian rings and $\phi^a : Spec\ S \to Spec\ R$ the map induced by ϕ. Let M be an R-module. Then $Ass_R M = \phi^a(Ass_S M \otimes_R S)$.*

Proof By Theorem 23.2 in [12], $Ass_S M \otimes_R S = \cup_{p\in Ass_R M} Ass_S(S/pS)$. Since S is a faithfully flat R-algebra, S/pS is nonzero and $\phi^a(Ass_S S/pS) = \{p\}$. Then $\phi^a(Ass_S M \otimes_R S) = \cup_{p\in Ass_R M}\phi^a(Ass_S S/pS) = Ass_R M$.
□

Let (R, m) be a regular local ring and $\widehat{R}$ the m-adic completion of R. Since $\widehat{R}$ is a faithfully flat R-algebra and $H^i_I(R) \otimes_R \widehat{R} \cong H^i_{I\widehat{R}}(\widehat{R})$, by the proposition above, $Ass_R H^i_I(R) = \phi^a(Ass_{\widehat{R}} H^i_{I\widehat{R}}(\widehat{R}))$. Hence, when we consider the problem of whether $Ass_R H^i_I(R)$ is finite, we can assume that R is a complete regular local ring.

Proposition 2.2 *Let x, y be an R-sequence in a Noetherian ring. Then $Ass_R H^2_{(x,y)}(R) = Ass_R R/(x, y)$.*

Proof Since x, y is an R-sequence, $ht(x, y) = grade(x, y) = 2$. By Theorem 1.1($b$) in [10], $Ass_R H^2_{(x,y)}(R) = Ass_R Ext^2_R(R/(x, y), R)$. The Koszul complex $K_.(x, y)$ gives a free resolution of $R/(x, y)$,

$$0 \to R \to R \oplus R \to R \to R/(x, y) \to 0.$$

From the above resolution, we see that $Ext^2_R(R/(x, y), R) \cong R/(x, y)$. Hence, $Ass_R H^2_{(x,y)}(R) = Ass_R R/(x, y)$.
□

The following theorem is proved in Theorem 2.4 in [1]. Using a different method, we give a much simpler proof here.

Theorem 2.3 *Let R be a regular local ring and I an ideal of R. Then $H^2_I(R)$ has only a finite number of associated prime ideals.*

Proof If $dim(R) \leqslant 1$, then $H^2_I(R) = 0$; If $ht(I) > 2$, since $ht(I) = grade(I, R)$ and $grade(I, R)$ is the least integer i such that $H^i_I(R)$ is nonzero; hence, $H^2_I(R) = 0$; if $ht(I) = 0$, then $I = (0)$ and also $H^2_I(R) = 0$. Therefore, we can assume that $dim(R) \geqslant 2$ and consider the cases $ht(I) = 1$ and 2, respectively.

If $ht(I) = 1$ and $I = q_1 \cap \cdots \cap q_s \cap q_{s+1} \cap \cdots \cap q_m$ is an irredundant primary decomposition of I with $\sqrt{q_i} = p_i$. Suppose that $\{p_i | 1 \leqslant i \leqslant s\}$ is the set of minimal primes containing I, let $J = p_1 \cap \cdots \cap p_s$. By Lemma 3 in [5], the finiteness of $Ass_R H^2_J(R)$ will imply that of $Ass_R H^2_I(R)$. Since R is a unique factorization domain, $p_i = (u_i)$ for some irreducible element of R, then $J = (u_1 \cdots u_s)$ and $H^2_{(u_1\cdots u_s)}(R) = 0$; hence, $Ass_R H^2_I(R)$ is a finite set when $ht(I) = 1$.

If $ht(I) = 2$, then $Ass_R H^2_I(R) = Ass_R Ext^2_R(R/I, R)$ from Proposition 1.1(b) in [10]. Since $Ext^2_R(R/I, R)$ is a finite R-module, $Ass_R Ext^2_R(R/I, R)$ is a finite set; hence, $Ass_R H^2_I(R)$ is finite. □

Theorem 2.4 *Let R be a regular local ring, $x, y \in R$ an R-regular sequence with $Ass_R R/(x, y) = \{p_i \mid 1 \leq i \leq t\}$, and $(x^n, y^n) = q_{n1} \cap \cdots \cap q_{nt}$ the irredundant primary decomposition of (x^n, y^n) such that $\sqrt{q_{ni}} = p_i$. Let $z \in R$ and the prime ideals to which z belongs are $\{p_i | s + 1 \leqslant i \leqslant t\}$ for some positive integer s. If $\bigcup_{n=1}^{\infty} Ass_R R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$ is finite, then $Ass_R H^3_{(x,y,z)}(R)$ is a finite set.*

Proof From the spectral sequence $E_2^{p,q} = H_I^p(H_J^q(R)) \Rightarrow H_{I+J}^{p+q}(R)$, we see that $H^3_{(x,y,z)}(R) \cong H^1_{(z)}(H^2_{(x,y)}(R))$. Let $M = H^2_{(x,y)}(R)$, we have $H^1_{(z)}(M) \cong H^1_{(z)}(M/\Gamma_{(z)}(M))$. From the spectral sequence $Ext^p_R(R/(z), H^q_{(z)}(\bar{M})) \Rightarrow Ext^{p+q}_R(R/(z), \bar{M})$ associated to the R-module $\bar{M} = M/\Gamma_{(z)}(M)$ and the fact that $H^0_{(z)}(\bar{M}) = 0$, we have $Hom(R/(z), H^1_{(z)}(\bar{M})) \cong Ext^1_R(R/(z), \bar{M})$. Since R is a domain, z is R-regular, and hence $Ext^1_R(R/(z), \bar{M}) \cong \bar{M}/(z)\bar{M}$. Now we have

$$Ass_R H^3_{(x,y,z)}(R) = Ass_R \bar{M}/(z)\bar{M} = Ass_R H^2_{(x,y)}(R)/(\Gamma_{(z)}(H^2_{(x,y)}(R)) + (z)H^2_{(x,y)}(R)).$$

Since R is a Cohen-Macaulay local ring and x, y an R-sequence, $Ass_R R/(x,y) = Ass_R R/(x^n, y^n) = \{p_i | 1 \leqslant i \leqslant t\}$ are just the minimal prime ideals containing (x,y). Assume now that the prime ideals to which z belongs are $\{p_i | s+1 \leqslant i \leqslant t\}$ for some positive integer s. Since $H^2_{(x,y)}(R) \cong \varinjlim R/(x^n, y^n)$, where the map $R/(x^n, y^n) \to R/(x^{n+1}, y^{n+1})$ is multiplication by xy, then $\Gamma_{(z)}(H^2_{(x,y)}(R)) = H^0_{(z)}(\varinjlim R/(x^n, y^n)) \cong \varinjlim \Gamma_{(z)}(R/(x^n, y^n))$. Let $\bar{R} = R/(x^n, y^n)$. Then

$$\bar{M} = H^2_{(x,y)}(R)/\Gamma_{(z)}(H^2_{(x,y)}(R)) \cong \varinjlim \bar{R}/\varinjlim \Gamma_{(z)}(\bar{R}) \cong \varinjlim \bar{R}/\Gamma_{(z)}(\bar{R}),$$

$$\bar{M}/(z)\bar{M} \cong (\varinjlim \bar{R}/\Gamma_{(z)}(\bar{R})) \otimes_R R/(z) \cong \varinjlim (\bar{R}/\Gamma_{(z)}(\bar{R}) \otimes_R R/(z)) \cong \varinjlim \bar{R}/(\Gamma_{(z)}(\bar{R}) + (z)\bar{R}).$$

We have $\bar{R}/\Gamma_{(z)}(\bar{R}) \cong R/\cup_{l \geqslant 0}((x^n, y^n) :_R (z^l))$. Let $(x^n, y^n) = q_{n1} \cap \cdots \cap q_{nt}$ be the irredundant primary decomposition of (x^n, y^n) with $\sqrt{q_{ni}} = p_i$. Since $((x^n, y^n) :_R (z)) \subseteq \cdots \subseteq ((x^n, y^n) :_R (z^l)) \subseteq ((x^n, y^n) :_R (z^{l+1})) \subseteq \cdots$ is an ascending chain of ideals, there is a positive integer l_0 such that $\cup_{l \geqslant 0}((x^n, y^n) :_R (z^l)) = ((x^n, y^n) :_R (z^{l_0}))$. Note that we assume the prime ideals to which z belongs are $\{p_i | s+1 \leqslant i \leqslant t\}$, we can choose l_0 large enough such that $z^{l_0} \in q_{n(s+1)} \cap \cdots \cap q_{nt}$. Then we can see that $((x^n, y^n) :_R (z^{l_0})) = q_{n1} \cap \cdots \cap q_{ns}$ and we have $\bar{R}/(\Gamma_{(z)}(\bar{R}) + (z)\bar{R}) \cong R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$. Altogether we have $Ass_R H^3_{(x,y,z)}(R) = Ass_R \varinjlim R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$. Therefore, if $\bigcup_{n=1}^{\infty} Ass_R R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$ is finite, then $Ass_R H^3_{(x,y,z)}(R)$ is a finite set. □

Theorem 2.5 *Let R be a regular local ring. If for all x, y, and z as in Theorem 2.4, $\bigcup_{n=1}^{\infty} Ass_R R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$ is finite, then $Ass_R H^i_I(R)$ is finite for all ideals I of R and $i \geq 0$.*

Proof By Theorem 4 in [5], to prove that $Ass_R H^i_I(R)$ is finite for all ideals I of R and $i \geq 0$, we only need to check if the following two conditions are satisfied:

(1) $Ass_R H^2_{(x,y)}(R)$ is finite for all $x, y \in R$.

(2) $Ass_R H^3_{(x,y,z)}(R)$ is finite whenever $x, y \in R$ is an R-sequence and $z \in R$.

By Theorem 2.3, the statement (1) is true. As for the statement (2), we consider the following three cases.

If z is $R/(x,y)$-regular, then $grade(x,y,z) = 3$. From Proposition 1.1(b) in [13], $Ass_R H^3_{(x,y,z)}(R)$ is a finite set.

If $z \in \sqrt{(x,y)}$, since the R-module $H^2_{(x,y)}(R)$ is (x,y)-torsion, $H^3_{(x,y,z)}(R) \cong H^1_{(z)}(M/\Gamma_{(z)}(M)) = 0$ and $Ass_R H^3_{(x,y,z)}(R) = \emptyset$

If z is not $R/(x,y)$-regular and $z \notin \sqrt{(x,y)}$, then z satisfies the conditions in Theorem 2.4, an hence if $\bigcup_{n=1}^{\infty} Ass_R R/(q_{n1} \cap \cdots \cap q_{ns} + (z))$ is finite, then $Ass_R H^3_{(x,y,z)}(R)$ is a finite set. Now the theorem is proved. □

Remark. In the notation of Theorem 2.5, if we delete z, we have $\bigcup_{n=1}^{\infty} Ass_R R/q_{n1} \cap \cdots \cap q_{ns} = Ass_R R/q_{n1} \cap \cdots \cap q_{ns} = \{p_i | 1 \leqslant i \leqslant s\}$ for every $n \geqslant 1$. Let $mAss_R M$ denote the minimal associated primes of M. Since for a finite R-module M and an M-regular element $z \in R$, the minimal primes over $p + (z)$ for every $p \in Ass_R M$ belong to $Ass_R M/(z)M$, we have $mAss_R R/(q_{n1} \cap \cdots \cap q_{ns} + (z)) = \cup_{i=1}^{s} mAss_R R/(p_i + (z))$ for all $n \geqslant 1$.

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