On 3-dimensional $\widetilde{J}$-tangent centro-affine hypersurfaces and $\widetilde{J}$-tangent affine hyperspheres with some null-directions

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Abstract: Let $\widetilde{J}$ be the canonical para-complex structure on $\mathbb{R}^4$. In this paper we study 3-dimensional centro-affine hypersurfaces with a $\widetilde{J}$-tangent centro-affine vector field (sometimes called $\widetilde{J}$-tangent centro-affine hypersurfaces) as well as 3-dimensional $\widetilde{J}$-tangent affine hyperspheres with the property that at least one null-direction of the second fundamental form coincides with either $\mathcal{D}^+$ or $\mathcal{D}^-$. The main purpose of this paper is to give a full local classification of the above-mentioned hypersurfaces. In particular, we prove that every nondegenerate centro-affine hypersurface of dimension 3 with a $\widetilde{J}$-tangent centro-affine vector field that has two null-directions $\mathcal{D}^+$ and $\mathcal{D}^-$ must be both an affine hypersphere and a hyperquadric. Some examples of these hypersurfaces are also given.

Key words: Centro-affine hypersurface, almost paracontact structure, affine hypersphere, null-direction

1. Introduction

Paracomplex and paracontact geometry plays an important role in mathematical physics (see [1, 6, 12]). On the other hand, affine differential geometry and in particular affine hyperspheres have been extensively studied over the past decades. Some relations between paracomplex and affine differential geometry can be found in [2, 7, 9].

Let us denote by $\widetilde{\mathbb{C}}$ the real algebra of paracomplex numbers (for details we refer to [3, 4]) and let $\widetilde{J}$ be the canonical paracomplex structure on $\mathbb{R}^{2n+2} \cong \widetilde{\mathbb{C}}^{n+1}$. A transversal vector field for an affine hypersurface $f \colon M \to \mathbb{R}^{2n+2}$ is called $\widetilde{J}$-tangent if $\widetilde{J}$ maps it into a tangent space. Such a vector field induces in a natural manner an almost paracontact structure (φ, ξ, η). We also have the biggest $\widetilde{J}$-invariant distribution in TM, which we denote by $\mathcal{D}$. $\mathcal{D}$ splits into $\mathcal{D}^+$ and $\mathcal{D}^-$ – eigenspaces related to eigenvalues $+1$ and -1 respectively (for details we refer to Section 3 of this paper).

In [11] the author studied affine hypersurfaces with a $\widetilde{J}$-tangent transversal vector field and gave a local classification of $\widetilde{J}$-tangent affine hyperspheres with an involutive distribution $\mathcal{D}$.

In this paper we study real affine hypersurfaces $f \colon M^3 \to \mathbb{R}^4 \cong \widetilde{\mathbb{C}}^2$ of the paracomplex space $\widetilde{\mathbb{C}}^2$ with a $\widetilde{J}$-tangent transversal vector field C and an induced almost paracontact structure (φ, ξ, η). In [11] the author

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showed that when C is centro-affine (not necessarily Blaschke) then f can be locally expressed in the form:

$$f(x_1, \ldots, x_{2n}, z) = \widetilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z, \tag{1.1}$$

where g is some smooth immersion defined on an open subset of $\mathbb{R}^{2n}$. Based on the above result we provide a local classification of all 3-dimensional nondegenerate centro-affine hypersurfaces with a $\widetilde{J}$-tangent transversal centro-affine vector field as well as $\widetilde{J}$-tangent affine hyperspheres with the null-direction $\mathcal{D}^+$ or $\mathcal{D}^-$. Moreover, in this case distribution $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ is not involutive, so affine hyperspheres are completely different from those studied in [11]. In particular, we give explicit examples of such hyperspheres.

In Section 2 we briefly recall the basic formulas of affine differential geometry. We recall the notion of a Blaschke field and an affine hypersphere.

In Section 3 we recall the definition of an almost paracontact structure introduced for the first time in [5]. We recall the notion of $\widetilde{J}$-tangent transversal vector field and the $\widetilde{J}$-invariant distribution. We also recall some elementary results for induced almost paracontact structures that will be used later in this paper (for details we refer to [10]).

In Section 4 we introduce the definition of the null-direction for a nondegenerate affine hypersurface. In this section we find local representation for nondegenerate centro-affine hypersurfaces $f \colon M \to \mathbb{R}^4$ with a $\widetilde{J}$-tangent centro-affine transversal vector field with a property that either $\mathcal{D}^+$ or $\mathcal{D}^-$ is the null-direction for f. To illustrate this situation we give some explicit examples of such hypersurfaces. Moreover, we prove that every centro-affine nondegenerate hypersurface with a $\widetilde{J}$-tangent centro-affine vector field and two null-directions $\mathcal{D}^+$ and $\mathcal{D}^-$ is both the affine hypersphere and the hyperquadric.

Section 5 concerns the case of $\widetilde{J}$-tangent affine hyperspheres. In this section we recall the notion of a $\widetilde{J}$-tangent affine hypersphere and prove classification theorems. We show that every 3-dimensional $\widetilde{J}$-tangent affine hypersphere with the null-direction $\mathcal{D}^+$ or $\mathcal{D}^-$ can be locally constructed from two regular flat curves $\alpha, \beta \colon I \to \mathbb{R}^2$ with a property $\det[\alpha, \alpha'] \neq 0$, $\det[\beta, \beta'] \neq 0$. As an application we give examples of such hyperspheres.

2. Preliminaries
In this section we briefly recall the basic formulas of affine differential geometry. For more details, we refer to [8].

Let $f \colon M \to \mathbb{R}^{n+1}$ be an orientable, connected differentiable n-dimensional hypersurface immersed in affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then, for any transversal vector field C, we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C \tag{2.1}$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M. For any transversal vector field ∇ is a torsion-free connection, h is a symmetric bilinear form on M called the second fundamental form, S is a tensor of type $(1, 1)$ called the shape operator, and τ is a 1-form called the transversal connection form.

We shall now consider the change of a transversal vector field for a given immersion f.

Theorem 2.1 ([8]) *Suppose we change a transversal vector field C to*

$$\bar{C} = \Phi C + f_*(Z),$$

where Z is a tangent vector field on M and Φ is a nowhere vanishing function on M. Then the affine fundamental form, the induced connection, the transversal connection form, and the affine shape operator change as follows:

$$\bar{h} = \frac{1}{\Phi}h,$$

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{\Phi}h(X,Y)Z,$$

$$\bar{\tau} = \tau + \frac{1}{\Phi}h(Z,\cdot) + d\ln|\Phi|,$$

$$\bar{S} = \Phi S - \nabla.Z + \bar{\tau}(\cdot)Z.$$

We have the following:

Theorem 2.2 ([8], Fundamental equations) *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.6}$$

Equations (2.3), (2.4), (2.5), and (2.6) are called the equation of Gauss, Codazzi for h, Codazzi for S, and Ricci, respectively.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$). For an affine hypersurface $f\colon M \to \mathbb{R}^{n+1}$ with the transversal vector field C we consider the following volume element on M:

$$\Theta(X_1, \ldots, X_n) := \det[f_* X_1, \ldots, f_* X_n, C]$$

for all $X_1, \ldots, X_n \in \mathcal{X}(M)$. We call Θ *the induced volume element* on M.

Immersion $f\colon M \to \mathbb{R}^{n+1}$ is said to be a *centro-affine hypersurface* if the position vector x (from origin o) for each point $x \in M$ is transversal to the tangent plane of M at x. In this case $S = \mathrm{I}$ and $\tau = 0$.

If h is nondegenerate (that is, h defines a semi-Riemannian metric on M), we say that the hypersurface or the hypersurface immersion is *nondegenerate*. In this paper we always assume that f is nondegenerate.

When f is nondegenerate, there exists a canonical transversal vector field C called *the affine normal field* (or *the Blaschke field*). The affine normal field is uniquely determined up to sign by the following conditions:

1. the induced volume form Θ is ∇-parallel (i.e. $\tau = 0$),

2. the metric volume form ω_h of h coincides with the induced volume form Θ.

Recall that ω_h is defined by

$$\omega_h(X_1, \ldots, X_n) = |\det[h(X_i, X_j)]|^{1/2},$$

where $\{X_1, \ldots, X_n\}$ is any positively oriented basis relative to the induced volume form Θ. The affine immersion f with a Blaschke field C is called *a Blaschke hypersurface*. In this case the fundamental equations can be rewritten as follows:

Theorem 2.3 ([8], Fundamental equations) *For a Blaschke hypersurface f, we have the following fundamental equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z),$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X),$$

$$h(X, SY) = h(SX, Y).$$

A Blaschke hypersurface is called *an affine hypersphere* if $S = \lambda \mathrm{I}$, where $\lambda = \mathrm{const}$.

If $\lambda = 0$ f is called *an improper affine hypersphere*, if $\lambda \neq 0$ a hypersurface f is called *a proper affine hypersphere*.

3. Induced almost paracontact structures

A $(2n+1)$-dimensional manifold M is said to have an *almost paracontact structure* if there exist on M a tensor field φ of type (1,1), a vector field ξ, and a 1-form η that satisfy

$$\varphi^2(X) = X - \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$ and the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker \eta$. That is, the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues $1, -1$ of φ have equal dimension n.

Let $\dim M = 2n + 1$ and $f : M \to \mathbb{R}^{2n+2}$ be a nondegenerate (relative to the second fundamental form) affine hypersurface. We always assume that $\mathbb{R}^{2n+2}$ is endowed with the standard paracomplex structure $\widetilde{J}$ given by

$$\widetilde{J}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) := (y_1, \ldots, y_{n+1}, x_1, \ldots, x_{n+1}).$$

Let C be a transversal vector field on M. We say that C is $\widetilde{J}$-tangent if $\widetilde{J}C_x \in f_*(T_x M)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest $\widetilde{J}$-invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_x M) \cap \widetilde{J}(f_*(T_x M)))$$

for every $x \in M$. We have that $\dim \mathcal{D}_x \geq 2n$. If for some x the $\dim \mathcal{D}_x = 2n+1$ then $\mathcal{D}_x = T_x M$ and it is not possible to find a $\widetilde{J}$-tangent transversal vector field in a neighborhood of x. Since we only study hypersurfaces

with a $\widetilde{J}$-tangent transversal vector field, then we always have $\dim \mathcal{D} = 2n$. The distribution $\mathcal{D}$ is smooth as an intersection of two smooth distributions and because $\dim \mathcal{D}$ is constant. A vector field X is called a $\mathcal{D}$-*field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields.

We say that the distribution $\mathcal{D}$ is nondegenerate if h is nondegenerate on $\mathcal{D}$.

To simplify the writing, we will be omitting f_* in front of vector fields in most cases.

Let $f: M \to \mathbb{R}^{2n+2}$ be a nondegenerate affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η, and a tensor field φ of type (1,1) as follows:

$$\xi := \widetilde{J}C; \tag{3.3}$$

$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1; \tag{3.4}$$

$$\varphi|_{\mathcal{D}} = \widetilde{J}|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0. \tag{3.5}$$

It is easy to see that (φ, ξ, η) is an almost paracontact structure on M. This structure is called the *induced almost paracontact structure*.

Lemma 3.1 *Let C be a $\widetilde{J}$-tangent transversal vector field. Then any other $\widetilde{J}$-tangent transversal vector field $\bar{C}$ has the following form:*

$$\bar{C} = \phi C + f_* Z,$$

where $\phi \neq 0$ and $Z \in \mathcal{D}$. Moreover, if (φ, ξ, η) is an almost paracontact structure induced by C, then $\bar{C}$ induces an almost paracontact structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta})$, where

$$\begin{cases} \bar{\xi} = \phi \xi + \varphi Z, \\ \bar{\eta} = \frac{1}{\phi} \eta, \\ \bar{\varphi} = \varphi - \eta(\cdot)\frac{1}{\phi}Z. \end{cases} \tag{3.6}$$

Proof By Theorem 2.1 we have that $\bar{C} = \phi C + f_* Z$ where $\phi \neq 0$ and $Z \in \mathcal{X}(M)$. Since both C and $\bar{C}$ are $\widetilde{J}$-tangent then in particular $\widetilde{J}f_* Z \in f_* TM$, that is, $Z \in \mathcal{D}$. Formulas (3.6) are an immediate consequence of (3.3)–(3.5). $\qquad \square$

For an induced almost paracontact structure we have the following theorem:

Theorem 3.2 ([10]) *Let $f: M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C. If (φ, ξ, η) is an induced almost paracontact structure on M then the following equations hold:*

$$\eta(\nabla_X Y) = h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.7}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y - \eta(Y)SX - h(X,Y)\xi, \tag{3.8}$$

$$\eta([X,Y]) = h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \tag{3.9}$$
$$+ \eta(Y)\tau(X) - \eta(X)\tau(Y),$$

$$\varphi([X,Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \eta(X)SY - \eta(Y)SX, \tag{3.10}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.11}$$

$$\eta(SX) = -h(X, \xi) \tag{3.12}$$

for every $X, Y \in \mathcal{X}(M)$.

We conclude this section with the following useful lemma related to differential equations.

Lemma 3.3 ([11]) *Let* $F\colon I \to \mathbb{R}^{2n}$ *be a smooth function on an interval* I. *If* F *satisfies the differential equation*

$$F'(z) = -\widetilde{J}F(z), \tag{3.13}$$

then F *is of the form*

$$F(z) = \widetilde{J}v\cosh z - v\sinh z, \tag{3.14}$$

where $v \in \mathbb{R}^{2n}$.

4. Centro-affine hypersurfaces with a $\widetilde{J}$-tangent centro-affine vector field

Let $f\colon M \to \mathbb{R}^{n+1}$ be a nondegenerate affine hypersurface. A 1-dimensional, smooth distribution $\mathcal{A} \subset TM$ we call *a null-direction for* f if for every $x \in M$ and for every $v \in \mathcal{A}_x$ we have $h_x(v,v) = 0$.

Lemma 4.1 *Let* $f\colon M \mapsto \mathbb{R}^4$ *be a nondegenerate centro-affine hypersurface with a* $\widetilde{J}$*-tangent centro-affine vector field* C. *If* $\mathcal{D}^+$ *is the null-direction for* f, *then for every point* $p \in M$ *there exist an open neighborhood* U *of* p *and a vector field* $X \in \mathcal{D}^+$, $X \neq 0$ *defined on* U *such that*

$$h(X,X) = 0, \tag{4.1}$$

$$\nabla_X \xi = \nabla_\xi X = -X, \tag{4.2}$$

$$\nabla_X X = 0. \tag{4.3}$$

Proof Let $p \in M$ and let X'' be a vector field defined on some neighborhood U of p such that $X'' \in \mathcal{D}^+$, $X'' \neq 0$. Since $\mathcal{D}^+$ is the null-direction for f we have $h(X'', X'') = 0$. Now from formulas (3.7) and (3.12) and due to the fact that $S = \mathrm{I}$, we get $h(X'', \xi) = 0$, $\eta(\nabla_{X''}\xi) = 0$, $\eta(\nabla_\xi X'') = 0$. That is, $\nabla_{X''}\xi, \nabla_\xi X'' \in \mathcal{D}$. Additionally, from (3.8) we obtain $\varphi(\nabla_{X''}\xi) = -X''$ and $\varphi(\nabla_\xi X'') = \nabla_\xi \varphi X'' = \nabla_\xi X''$ since $\varphi X'' = X''$. In particular, we have $\nabla_\xi X'' \in \mathcal{D}^+$. The above implies that

$$\nabla_{X''}\xi = -X''$$

and

$$\nabla_\xi X'' = \alpha X'',$$

where α is some smooth function on U. We claim that in a neighborhood of p there exists $X' \in \mathcal{D}^+, X' \neq 0$ such that

$$h(X', X') = 0, \tag{4.4}$$

$$\nabla_{X'}\xi = \nabla_\xi X' = -X'. \tag{4.5}$$

Indeed, let $X' := \beta X''$, where $\beta \neq 0$, be a solution of the differential equation $\xi(\beta) = -(\alpha+1)\beta$. Shrinking eventually U if needed, we may assume that X' is defined on U. It is easy to verify that X' satisfies (4.4) and (4.5). By (3.7) and (3.12) we get $\nabla_{X'}X' = aX'$ for some smooth function a. Moreover, from the Gauss equation we have

$$R(\xi, X')X' = 0 = \nabla_\xi(aX') - \nabla_{X'}(-X') = \xi(a)X',$$

that is, $\xi(a) = 0$. Let $X := bX'$, where $b \neq 0$, be a solution of the differential equation $X'(b) = -ab$. Again shrinking U if needed we may assume that X is defined on U. Since $\xi(a) = 0$ we also have that $\xi(b) = 0$. Of course we have

$$X \neq 0, X \in \mathcal{D}^+,$$

$$h(X, X) = 0.$$

Now, by straightforward computation, we obtain

$$\nabla_X X = 0, \quad \nabla_\xi X = \nabla_X \xi = -X.$$

The proof is completed. $\qquad\square$

When $\mathcal{D}^-$ is the null-direction for f, one may easily obtain a result similar to Lemma 4.1. Namely, we have:

Lemma 4.2 *Let* $f \colon M \mapsto \mathbb{R}^4$ *be a nondegenerate centro-affine hypersurface with a* $\widetilde{J}$*-tangent centro-affine vector field* C*. If* $\mathcal{D}^-$ *is the null-direction for* f*, then for every point* $p \in M$ *there exist an open neighborhood* U *of* p *and a vector field* $Y \in \mathcal{D}^-$*,* $Y \neq 0$ *defined on* U *such that*

$$h(Y, Y) = 0, \tag{4.6}$$

$$\nabla_Y \xi = \nabla_\xi Y = Y, \tag{4.7}$$

$$\nabla_Y Y = 0. \tag{4.8}$$

In order to simplify further computation, we will need the following technical lemma.

Lemma 4.3 *Let* $g \colon V \to \mathbb{R}^4$ *be a smooth function defined on an open subset* V *of* $\mathbb{R}^2$*. Let* $I \subset \mathbb{R}$ *be an open interval in* $\mathbb{R}$ *and let function* $f \colon V \times I \to \mathbb{R}^4$ *be given by the formula*

$$f(x, y, z) = \widetilde{J}g(x, y) \cosh z - g(x, y) \sinh z.$$

Then

$$\det[f_x, f_y, f_z, f] = \det[g_x, g_y, g, \widetilde{J}g].$$

Proof We have

$$\det[f_x, f_y, f_z, f] = \det[f_x, f_y, \widetilde{J}g \sinh z - g \cosh z, \widetilde{J}g \cosh z - g \sinh z]$$

$$= (\cosh^2 z - \sinh^2 z) \det[f_x, f_y, \widetilde{J}g, g] = \det[f_x, f_y, \widetilde{J}g, g].$$

Now

$$\det[f_x, f_y, \widetilde{J}g, g] = \det[\widetilde{J}g_x \cosh z - g_x \sinh z, \widetilde{J}g_y \cosh z - g_y \sinh z, \widetilde{J}g, g]$$

$$= \cosh^2 z \det[\widetilde{J}g_x, \widetilde{J}g_y, \widetilde{J}g, g] + \sinh^2 z \det[g_x, g_y, \widetilde{J}g, g]$$

$$- \sinh z \cosh z (\det[\widetilde{J}g_x, g_y, \widetilde{J}g, g] + \det[g_x, \widetilde{J}g_y, \widetilde{J}g, g])$$

$$= (\cosh^2 z - \sinh^2 z) \det[g_x, g_y, g, \widetilde{J}g],$$

where the last equality is a consequence of the following straightforward observation:

$$\det[\widetilde{J}v_1, v_2, v_3, \widetilde{J}v_3] = -\det[v_1, \widetilde{J}v_2, v_3, \widetilde{J}v_3] \tag{4.9}$$

for every $v_1, v_2, v_3 \in \mathbb{R}^4$. Summarizing,

$$\det[f_x, f_y, f_z, f] = \det[f_x, f_y, \widetilde{J}g, g] = \det[g_x, g_y, g, \widetilde{J}g].$$

$\square$

Now we shall state the classification theorems for centro-affine hypersurfaces.

Theorem 4.4 *Let* $f\colon M \mapsto \mathbb{R}^4$ *be a nondegenerate centro-affine hypersurface with a* $\widetilde{J}$*-tangent centro-affine vector field* C*. If* $\mathcal{D}^+$ *is the null-direction for* f *then for every point* $p \in M$ *there exists a neighborhood* U *of* p *such that* $f|_U$ *can be expressed in the following form:*

$$f(x, y, z) = \widetilde{J}g(x, y)\cosh z - g(x, y)\sinh z, \tag{4.10}$$

where $g(x, y) = x \cdot \gamma_1(y) + \gamma_2(y)$ *and* γ_1, γ_2 *are some curves such that* $\widetilde{J}\gamma_1 = \gamma_1$ *and* $\det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0$*.*
Moreover, when γ_1, γ_2 *are smooth curves such that* $\widetilde{J}\gamma_1 = \gamma_1$ *and* $\det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0$ *then* f *given by* (4.10)
is the nondegenerate centro-affine hypersurface with a $\widetilde{J}$*-tangent centro-affine vector field with the null-direction*
$\mathcal{D}^+$*.*

Proof By Lemma 4.1 for every $p \in M$ there exist a neighborhood U and vector field $X \in \mathcal{D}^+, X \neq 0$ defined on this neighbourhood such that

$$h(X, X) = 0,$$

$$\nabla_X X = 0,$$

$$\nabla_X \xi = \nabla_\xi X = -X. \tag{4.11}$$

By (4.11) we have $[X, \xi] = 0$, so there exists a local coordinate system (x, y, z) around p such that

$$\frac{\partial}{\partial x} = X \quad \text{and} \quad \frac{\partial}{\partial z} = \xi$$

in some neighborhood of p. Without loss of generality we may assume that the system (x, y, z) is defined on U. Since $\widetilde{J}C = f_*(\xi)$ we have

$$f_z = f_*\left(\frac{\partial}{\partial z}\right) = f_*(\xi) = \widetilde{J}C = -\widetilde{J}f. \tag{4.12}$$

By the Gauss formula we also have

$$f_{xx} = 0, \tag{4.13}$$

$$f_{xz} = f_*\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}\right) = f_*\left(\nabla_{\frac{\partial}{\partial x}}\xi\right) = f_*\left(-\frac{\partial}{\partial x}\right) = -f_x. \tag{4.14}$$

Solving (4.12) (see Lemma 3.3) we get that f can be locally expressed in the following form:

$$f(x, y, z) = \widetilde{J}g(x, y) \cosh z - g(x, y) \sinh z$$

where $g \colon V \ni (x, y) \mapsto g(x, y) \in \mathbb{R}^4$ is some smooth function defined on an open subset V of $\mathbb{R}^2$. Moreover, by (4.13) and (4.14), we have

$$g_{xx} = 0, \tag{4.15}$$

$$\widetilde{J}g_x = g_x. \tag{4.16}$$

Solving (4.15) we obtain that

$$g(x, y) = x \cdot \gamma_1(y) + \gamma_2(y),$$

where γ_1, γ_2 are some smooth curves. From (4.16) we get that $\widetilde{J}\gamma_1 = \gamma_1$. Since f is a centro-affine immersion we have $\det[f_x, f_y, f_z, f] \neq 0$ and in consequence $\det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0$, because by (4.9) $\det[\gamma_1, \gamma_1', \gamma_2, \widetilde{J}\gamma_2] = 0$.

In order to prove the last part of the theorem, first note that $\det[\gamma_1, \gamma_2, \gamma_2', \widetilde{J}\gamma_2] \neq 0$ implies that f given by (4.10) is a centro-affine immersion. Indeed, by Lemma 4.3 and due to the fact that $\widetilde{J}\gamma_1 = \gamma_1$, we have

$$\det[f_x, f_y, f_z, f] = \det[\gamma_1, x\gamma_1' + \gamma_2', x\gamma_1 + \gamma_2, x\gamma_1 + \widetilde{J}\gamma_2]$$

$$= \det[\gamma_1, x\gamma_1' + \gamma_2', \gamma_2, \widetilde{J}\gamma_2]$$

$$= \det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0.$$

Moreover, since $\widetilde{J}f = -f_z$, the centro-affine transversal vector field is $\widetilde{J}$-tangent. Now it is enough to show that $\mathcal{D}^+$ is the null-direction for f. Since

$$f_x = \widetilde{J}\gamma_1(y) \cosh z - \gamma_1(y) \sinh z = \gamma_1(y)(\cosh z - \sinh z), \tag{4.17}$$

we have that $\widetilde{J}f_x = f_x$, that is, $\mathcal{D}^+ = \mathrm{span}\{\frac{\partial}{\partial x}\}$. From (4.17) we also have that $f_{xx} = 0$, so, in particular, $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0$ and $\mathcal{D}^+$ is the null-direction for f. □

When $\mathcal{D}^-$ is the null-direction for f, using Lemma 4.2, one may prove an analogous result to Theorem 4.4. That is, we have the following:

Theorem 4.5 *Let $f \colon M \mapsto \mathbb{R}^4$ be a nondegenerate centro-affine hypersurface with a $\widetilde{J}$-tangent centro-affine vector field C. If $\mathcal{D}^-$ is the null-direction for f then for every point $p \in M$ there exists a neighborhood U of p such that $f|_U$ can be expressed in the following form:*

$$f(x, y, z) = \widetilde{J}g(x, y) \cosh z - g(x, y) \sinh z, \tag{4.18}$$

where $g(x, y) = y \cdot \gamma_1(x) + \gamma_2(x)$ and γ_1, γ_2 are some curves such that $\widetilde{J}\gamma_1 = -\gamma_1$ and $\det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0$. Moreover, when γ_1, γ_2 are smooth curves such that $\widetilde{J}\gamma_1 = -\gamma_1$ and $\det[\gamma_1, \gamma_2', \gamma_2, \widetilde{J}\gamma_2] \neq 0$ then f given by (4.18) is the nondegenerate centro-affine hypersurface with the $\widetilde{J}$-tangent centro-affine vector field with the null-direction $\mathcal{D}^-$.

In order to illustrate the above theorems we give some explicit examples of centro-affine hypersurfaces with a $\widetilde{J}$-tangent centro-affine vector field with the null-direction $\mathcal{D}^+$ or $\mathcal{D}^-$.

Example 4.6 *Let us consider the affine immersion defined by*

$$f\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \widetilde{J}g(x,y)\cosh z - g(x,y)\sinh z \in \mathbb{R}^4$$

, where

$$g\colon \mathbb{R}^2 \ni (x,y) \mapsto \begin{bmatrix} xy+1 \\ -x+y \\ xy \\ -x \end{bmatrix} \in \mathbb{R}^4$$

with the transversal vector field $C = -f$. Of course C is $\widetilde{J}$-tangent, $\tau = 0$, and $S = \mathrm{I}$. By straightforward computations we obtain

$$h = \begin{bmatrix} 0 & -\frac{1}{y^2+1} & 0 \\ \frac{1}{y^2+1} & 0 & \frac{2x+y}{y^2+1} \\ 0 & \frac{2x+y}{y^2+1} & -1 \end{bmatrix}$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ of $\mathbb{R}^3$, so in particular f is nondegenerate. It is easy to see that $\frac{\partial}{\partial x} \in \mathcal{D}^+$, and since $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0$, distribution $\mathcal{D}^+$ is the null-direction for f. One may also compute

$$\Theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = y^2 + 1$$

$$\omega_h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{1}{y^2+1}$$

so f is not an affine hypersphere. Moreover, f is not a hyperquadric since $\nabla h \neq 0$.

One may construct a similar example when $\mathcal{D}^-$ is the null-direction. Namely, we have:

Example 4.7 *Let us consider the affine immersion defined by*

$$f\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \widetilde{J}g(x,y)\cosh z - g(x,y)\sinh z \in \mathbb{R}^4,$$

where

$$g\colon \mathbb{R}^2 \ni (x,y) \mapsto \begin{bmatrix} xy+1 \\ x-y \\ -xy \\ y \end{bmatrix} \in \mathbb{R}^4$$

with the transversal vector field $C = -f$. Of course C is $\widetilde{J}$-tangent, $\tau = 0$, and $S = \mathrm{I}$. By straightforward computations we obtain

$$h = \begin{bmatrix} 0 & -\frac{1}{x^2+1} & -\frac{x+2y}{x^2+1} \\ \frac{1}{x^2+1} & 0 & 0 \\ -\frac{x+2y}{x^2+1} & 0 & -1 \end{bmatrix}$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ of $\mathbb{R}^3$, so in particular f is nondegenerate. It is easy to see that $\frac{\partial}{\partial y} \in \mathcal{D}^-$, and since $h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = 0$, distribution $\mathcal{D}^-$ is the null-direction for f. One may also compute

$$\Theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = x^2 + 1$$

$$\omega_h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{1}{x^2 + 1}$$

so f is not an affine hypersphere. Moreover, f is not a hyperquadric since $\nabla h \neq 0$.

The next example has the property that both $\mathcal{D}^+$ and $\mathcal{D}^-$ are null-directions.

Example 4.8 *Let us consider a function f defined by*

$$f \colon \mathbb{R}^3 \ni (x, y, z) \mapsto \widetilde{J}g(x, y) \cosh z - g(x, y) \sinh z \in \mathbb{R}^4$$

, where

$$g \colon \mathbb{R}^2 \ni (x, y) \mapsto \begin{bmatrix} x + \frac{1}{2}y - xy + \frac{1}{4} \\ x + \frac{1}{2}y + xy - \frac{1}{4} \\ x - \frac{1}{2}y + xy + \frac{3}{4} \\ x - \frac{1}{2}y - xy - \frac{3}{4} \end{bmatrix} \in \mathbb{R}^4.$$

It is easy to verify that f is an immersion and $C = -f$ is a transversal vector field. Of course C is $\widetilde{J}$-tangent, $\tau = 0$, and $S = \mathrm{I}$. By straightforward computations we obtain

$$h = \begin{bmatrix} 0 & -2 & -4y \\ -2 & 0 & 0 \\ -4y & 0 & -1 \end{bmatrix}$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. We also compute

$$\Theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 2$$

$$\omega_h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 2$$

so f is nondegenerate and f is an affine hypersphere. It is easy to see that $\frac{\partial}{\partial y} \in \mathcal{D}^-$, and since $h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = 0$, the distribution $\mathcal{D}^-$ is the null-direction for f.

Let

$$X := \frac{1}{4} \cdot \frac{\partial}{\partial x} + y^2 \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial z}.$$

By straightforward computations it can be checked that $X \in \mathcal{D}^+$ and $h(X, X) = 0$, so $\mathcal{D}^+$ is also the null-direction for f.

Moreover, one may compute that $\nabla h = 0$, so f is a hyperquadric.

The first two examples are neither hyperquadrics nor affine hyperspheres; however, the last example is both the affine hypersphere and the hyperquadric. This is not a coincidence. Namely, we have the following theorem:

Theorem 4.9 *Let $f \colon M \mapsto \mathbb{R}^4$ be a centro-affine nondegenerate hypersurface with a $\widetilde{J}$-tangent centro-affine vector field C and let $\mathcal{D}^-$ and $\mathcal{D}^+$ be null-directions for f. Then f is both the affine hypersphere and the hyperquadric. Moreover, if $f \colon M \to \mathbb{R}^4$ is a centro-affine nondegenerate hyperquadric with a $\widetilde{J}$-tangent centro-affine vector field C then $\mathcal{D}^-$ and $\mathcal{D}^+$ are null-directions for f.*

In order to prove the above theorem we need the following lemma:

Lemma 4.10 *Let $f \colon M \mapsto \mathbb{R}^4$ be a nondegenerate centro-affine hypersurface with a $\widetilde{J}$-tangent centro-affine vector field C. If both $\mathcal{D}^+$ and $\mathcal{D}^-$ are null-directions for f, then for every point $p \in M$ there exist a neighborhood U of p and vector fields X, Y defined on U such that $X \neq 0, X \in \mathcal{D}^+$, $Y \neq 0, Y \in \mathcal{D}^-$ and the following conditions are satisfied:*

$$\nabla_X \xi = \nabla_\xi X = -X, \quad \nabla_Y \xi = \nabla_\xi Y = Y, \tag{4.19}$$

$$h(X, X) = 0, \quad h(Y, Y) = 0, \tag{4.20}$$

$$\nabla_X X = 0, \tag{4.21}$$

$$\nabla_Y Y = 0, \tag{4.22}$$

$$\nabla_X Y = aY - e^\beta \xi, \tag{4.23}$$

$$\nabla_Y X = bX + e^\beta \xi, \tag{4.24}$$

$$h(X, Y) = e^\beta, \tag{4.25}$$

$$\xi(\beta) = \xi(a) = \xi(b) = 0, \tag{4.26}$$

$$X(\beta) = a, Y(\beta) = b, \tag{4.27}$$

for some smooth functions a, b, β on U.

Proof　By Lemma 4.1 and Lemma 4.2 there exist $X \in \mathcal{D}^+$, $X \neq 0$, $Y \in \mathcal{D}^-$, $Y \neq 0$ defined on some neighborhood of p such that (4.19)–(4.22) hold. Since f is nondegenerate, $h(X, Y) \neq 0$. Without loss of generality (replacing Y by $-Y$ if needed) we may assume that $h(X, Y) = e^\beta$, where β is a smooth function on U. From formulas (3.7), (3.8), and (3.12) and due to the fact that $S = \mathrm{I}$, we get

$$\eta(\nabla_X Y) = -h(X, Y) = -e^\beta,$$

$$\eta(\nabla_Y X) = h(X, Y) = e^\beta,$$

$$\varphi(\nabla_X Y) = -\nabla_X Y - e^\beta \xi,$$

$$\varphi(\nabla_Y X) = \nabla_Y X - e^\beta \xi.$$

The above implies that there exist smooth functions a and b such that

$$\nabla_X Y = aY - e^\beta \xi,$$

$$\nabla_Y X = bX + e^\beta \xi.$$

The above and the Gauss equation imply

$$e^\beta \xi = R(\xi, X)Y = \xi(a)Y - e^\beta \xi(\beta)\xi + e^\beta \xi$$

and

$$e^\beta \xi = R(\xi, Y)X = \xi(b)X + e^\beta \xi(\beta)\xi + e^\beta \xi.$$

The above implies $\xi(\beta) = \xi(a) = \xi(b) = 0$. From the Codazzi equation for h we have

$$0 = (\nabla_Y h)(X, X) = (\nabla_X h)(X, Y) = X(e^\beta) - h(X, aY - e^\beta \xi)$$

$$= e^\beta X(\beta) - ae^\beta;$$

that is, $X(\beta) = a$. In a similar way from the Codazzi equation for h we get

$$0 = (\nabla_X h)(Y, Y) = (\nabla_Y h)(X, Y) = Y(e^\beta) - h(bX + e^\beta \xi, Y)$$

$$= e^\beta Y(\beta) - be^\beta;$$

that is, $Y(\beta) = b$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Proof [Proof of Theorem 4.9] Let X, Y, and ξ be as in Lemma 4.10. It is easy to see that

$$(\nabla_\xi h)(U, V) = \xi(h(U, V)) - h(\nabla_\xi U, V) - h(U, \nabla_\xi V) = 0$$

for $U, V \in \{X, Y, \xi\}$, because $h(U, V) \in \{0, -1, e^\beta\}$ and $\xi(\beta) = 0$. Thus, we have $\xi(h(U, V)) = 0$. Moreover, we have

$$(\nabla_U h)(V, W) = 0$$

for every $U, V, W \in \{X, Y\}$. The above implies that

$$(\nabla_{Z_1} h)(Z_2, Z_3) = 0$$

for every $Z_1, Z_2, Z_3 \in \mathcal{X}(M)$.

Since $\nabla h = 0$, then in particular $\nabla \omega_h = 0$, so f is an affine hypersphere.

In order to prove the second part of the theorem first note that since f is a centro-affine hyperquadric we have $S = \mathrm{I}$, $\tau = 0$, and $\nabla h = 0$. Letting $X \in \mathcal{D}$, then we have

$$0 = (\nabla_X h)(X, \xi) = -h(\nabla_X X, \xi) - h(X, \nabla_X \xi).$$

By Theorem 3.2 we also have

$$-h(\nabla_X X, \xi) = \eta(\nabla_X X) = h(X, \varphi X)$$

and

$$-h(X, \nabla_X \xi) = -h(X, -\varphi X) = h(X, \varphi X).$$

Summarizing,

$$h(X, \varphi X) = 0$$

for every $X \in \mathcal{D}$. The above immediately implies that $h(X, X) = 0$ if $X \in \mathcal{D}^+$ or $X \in \mathcal{D}^-$, so both $\mathcal{D}^+$ and $\mathcal{D}^-$ are null-directions for f. □

5. $\widetilde{J}$-tangent affine hyperspheres

It is well known that every proper affine hypersphere is in particular a centro-affine hypersurface. Of course, the converse is not true in general. The purpose of this section is to give a local classification of 3-dimensional $\widetilde{J}$-tangent affine hyperspheres f with the property that either $\mathcal{D}^+$ or $\mathcal{D}^-$ is the null-direction for f.

First recall from [11] that an affine hypersphere with a transversal $\widetilde{J}$-tangent Blaschke field is called *a $\widetilde{J}$-tangent affine hypersphere*. Moreover, in [11] we proved that every $\widetilde{J}$-tangent affine hypersphere is proper. Now we shall prove classification theorems for affine hyperspheres with null-directions.

Theorem 5.1 *Let $f \colon M \mapsto \mathbb{R}^4$ be an affine hypersphere with the null-direction $\mathcal{D}^+$ and the $\widetilde{J}$-tangent Blaschke field C. Then there exist smooth planar curves $\alpha \colon I \to \mathbb{R}^2$, $\beta \colon I \to \mathbb{R}^2$ with the property*

$$\det[\alpha, \alpha'] \cdot \det[\beta, \beta'] \neq 0 \tag{5.1}$$

and a smooth function $A \colon I \to \mathbb{R}$, where I is an open interval in $\mathbb{R}$, such that f can be locally expressed in the form

$$f(x, y, z) = \widetilde{J}g(x, y) \cosh z - g(x, y) \sinh z, \tag{5.2}$$

where $g(x, y) = (x + A(y)) \cdot (\alpha, \alpha)(y) + B(y)(\alpha', \alpha')(y) + (\beta, -\beta)(y)$ and

$$B(y) = \frac{E}{\sqrt{|\det[\alpha(y), \alpha'(y)] \cdot \det[\beta(y), \beta'(y)]|}}$$

for some nonzero constant E.

Moreover, for any smooth curves $\alpha, \beta \colon I \to \mathbb{R}^2$ such that (5.1) and for any smooth function $A \colon I \to \mathbb{R}$ and a nonzero constant E the function f given by (5.2) is the $\widetilde{J}$-tangent affine hypersphere with the null-direction $\mathcal{D}^+$.

Proof Since f is a $\widetilde{J}$-tangent affine hypersphere it is proper, so there exists $\lambda \neq 0$ such that $C = -\lambda f$. In particular, f is centro-affine. Now, by Theorem 4.4 for every point $p \in M$ there exists a neighborhood U of p such that $f|_U$ has the following form:

$$f(x, y, z) = \widetilde{J}g(x, y) \cosh z - g \sinh z,$$

where $g(x, y) = x \cdot \gamma_1(y) + \widetilde{\gamma_2}(y)$, $\widetilde{J}\gamma_1 = \gamma_1$, and $\det[\gamma_1, \widetilde{\gamma_2}', \widetilde{\gamma_2}, \widetilde{J}\widetilde{\gamma_2}] \neq 0$. Let us define $\gamma_2 := \frac{1}{2} \cdot (\widetilde{\gamma_2} - \widetilde{J}\widetilde{\gamma_2})$ and $\gamma_3 := \frac{1}{2} \cdot (\widetilde{\gamma_2} + \widetilde{J}\widetilde{\gamma_2})$. Of course we have

$$\widetilde{J}\gamma_2 = -\gamma_2, \quad \widetilde{J}\gamma_3 = \gamma_3 \quad \text{and} \quad \widetilde{\gamma_2} = \gamma_2 + \gamma_3.$$

Because $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0$, $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = 0$ and since h is nondegenerate, we have that $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \neq 0$. By the Gauss formula we have

$$f_{xy} = f_*(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}) - \lambda h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})f.$$

Since $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \neq 0$, the above implies that f_{xy} and f_x are linearly independent and as a consequence γ_1 and γ_1' are linearly independent, too. Moreover, $\widetilde{J}\gamma_1' = \gamma_1'$; that is, γ_1, γ_1' form a basis of the eigenspace of $\widetilde{J}$ related to the eigenvalue 1. This means that there exist smooth functions A, B such that

$$\gamma_3 = A\gamma_1 + B\gamma_1'.$$

Now g can be rewritten in the form

$$g(x, y) = (x + A(y))\gamma_1(y) + B(y)\gamma_1'(y) + \gamma_2(y).$$

Since $\det[\gamma_1, \widetilde{\gamma_2}, \widetilde{\gamma_2}', \widetilde{J}\widetilde{\gamma_2}] \neq 0$, we compute

$$0 \neq \det[\gamma_1, \gamma_2 + \gamma_3, \gamma_2' + \gamma_3', -\gamma_2 + \gamma_3]$$

$$= \det[\gamma_1, \gamma_2, \gamma_2' + \gamma_3', -\gamma_2 + \gamma_3]$$

$$+ \det[\gamma_1, \gamma_3, \gamma_2' + \gamma_3', -\gamma_2 + \gamma_3]$$

$$= \det[\gamma_1, \gamma_2, \gamma_2', \gamma_3] + \det[\gamma_1, \gamma_2, \gamma_3', \gamma_3]$$

$$- \det[\gamma_1, \gamma_3, \gamma_2', \gamma_2] - \det[\gamma_1, \gamma_3, \gamma_3', \gamma_2]$$

$$= 2\det[\gamma_1, \gamma_2, \gamma_2', \gamma_3] = 2B\det[\gamma_1, \gamma_2, \gamma_2', \gamma_1'],$$

so $\det[\gamma_1, \gamma_1', \gamma_2, \gamma_2'] \neq 0$. Now, since $\widetilde{J}\gamma_1 = \gamma_1$ and $\widetilde{J}\gamma_2 = -\gamma_2$, there exist smooth curves $\alpha = (\alpha_1, \alpha_2)\colon I \to \mathbb{R}^2$, $\beta = (\beta_1, \beta_2)\colon I \to \mathbb{R}^2$ such that

$$\gamma_1 = (\alpha, \alpha) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \text{and} \quad \gamma_2 = (\beta, -\beta) = \begin{bmatrix} \beta_1 \\ \beta_2 \\ -\beta_1 \\ -\beta_2 \end{bmatrix}.$$

By straightforward computation we get

$$\det[\gamma_1, \gamma_1', \gamma_2, \gamma_2'] = 4 \cdot (\alpha_1\alpha_2' - \alpha_1'\alpha_2) \cdot (\beta_1\beta_2' - \beta_1'\beta_2)$$

$$= 4\det[\alpha, \alpha'] \cdot \det[\beta, \beta'],$$

$$\Theta(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = 2\lambda B \cdot \det[\gamma_1, \gamma_1', \gamma_2, \gamma_2']$$

$$= 8\lambda \cdot B \cdot \det[\alpha, \alpha'] \cdot \det[\beta, \beta'],$$

$$\omega_h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{1}{2|\lambda|\sqrt{|\lambda|} \cdot |B|}.$$

Since f is an affine hypersphere we have $\omega_h = |\Theta|$; that is,

$$|B| = \frac{1}{4|\lambda| \sqrt[4]{|\lambda|} \sqrt{|\det[\alpha, \alpha'] \cdot \det[\beta, \beta']|}}.$$

Now we take $E := \pm \frac{1}{4\lambda \sqrt[4]{|\lambda|}}$ (depending on the sign of B).

In order to prove the last part of the theorem let us define $C := -\lambda f$, where

$$|\lambda| = \frac{1}{\sqrt[5]{4^4} \cdot \sqrt[5]{E^4}}.$$

We also denote $\gamma_1 := (\alpha, \alpha)$ and $\gamma_2 := (\beta, -\beta)$. Now, by Lemma 4.3 and using the fact that $\gamma_1, \gamma_1', \gamma_1''$ are linearly dependent (in consequence $\det[\gamma_1, \gamma_1'', \gamma_2, \gamma_1'] = 0$), we get

$$\det[f_x, f_y, f_z, f] = \det[g_x, g_y, g, \widetilde{J}g] = -8B\det[\alpha, \alpha'] \cdot \det[\beta, \beta'] \neq 0.$$

The above implies that f is an immersion and C is transversal. Of course C is $\widetilde{J}$-tangent as well. We also have

$$|\Theta(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})| = |\det[f_x, f_y, f_z, -\lambda f]|$$

$$= 8|\lambda| \cdot |B| \cdot |\det[\alpha, \alpha'] \cdot \det[\beta, \beta']|$$

$$= \frac{2\sqrt{|\det[\alpha, \alpha'] \cdot \det[\beta, \beta']|}}{\sqrt[4]{|\lambda|}}.$$

Directly from (5.2) we obtain

$$f_x = (\alpha, \alpha)(y)(\cosh z - \sinh z),$$

so in particular $\frac{\partial}{\partial x} \in \mathcal{D}^+$ because $\widetilde{J}f_x = f_x$. Moreover, $f_{xx} = 0$ implies $h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0$, so $\mathcal{D}^+$ is the null-direction for f. Finally, using the Gauss formula we compute

$$\omega_h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{1}{\sqrt{|\lambda|}} \cdot |h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})| = \frac{1}{\sqrt{|\lambda|}} \cdot \frac{1}{2|\lambda| \cdot |B|}$$

$$= \frac{4|\lambda| \sqrt[4]{|\lambda|} \cdot \sqrt{|\det[\alpha, \alpha'] \cdot \det[\beta, \beta']|}}{2|\lambda| \sqrt{|\lambda|}}.$$

That is, $\omega_h = |\Theta|$. $\qquad \square$

In a similar way one may prove the following.

Theorem 5.2 *Let $f \colon M \mapsto \mathbb{R}^4$ be an affine hypersphere with the null-direction $\mathcal{D}^-$ and the $\widetilde{J}$-tangent Blaschke field C. Then there exist smooth planar curves $\alpha \colon I \to \mathbb{R}^2$, $\beta \colon I \to \mathbb{R}^2$ with the property*

$$\det[\alpha, \alpha'] \cdot \det[\beta, \beta'] \neq 0 \tag{5.3}$$

and a smooth function $A \colon I \to \mathbb{R}$, where I is an open interval in $\mathbb{R}$, such that f can be locally expressed in the form

$$f(x, y, z) = \widetilde{J}g(x, y)\cosh z - g(x, y)\sinh z, \tag{5.4}$$

where $g(x,y) = (y + A(x)) \cdot (\alpha, -\alpha)(x) + B(x)(\alpha', -\alpha')(x) + (\beta, \beta)(x)$ *and*

$$B(x) = \frac{E}{\sqrt{|\det[\alpha(x), \alpha'(x)] \cdot \det[\beta(x), \beta'(x)]|}}$$

for some nonzero constant E.

Moreover, for any smooth curves $\alpha, \beta \colon I \to \mathbb{R}^2$ such that (5.3) and for any smooth function $A \colon I \to \mathbb{R}$ and a nonzero constant E the function f given by (5.4) is the $\widetilde{J}$-tangent affine hypersphere with the null-direction $\mathcal{D}^-$.

Example 5.3 *Let us consider two affine immersions ($i = 1, 2$) defined by*

$$f_i \colon \mathbb{R}^3 \ni (x, y, z) \mapsto \widetilde{J} g_i(x, y) \cosh z - g_i(x, y) \sinh z \in \mathbb{R}^4,$$

where

$$g_1 \colon \mathbb{R}^2 \ni (x, y) \mapsto \begin{bmatrix} x + \cos y \\ yx + \sin y + \frac{1}{4} \\ x - \cos y \\ yx - \sin y + \frac{1}{4} \end{bmatrix} \in \mathbb{R}^4$$

and

$$g_2 \colon \mathbb{R}^2 \ni (x, y) \mapsto \begin{bmatrix} y + \cos x \\ xy + \sin x + \frac{1}{4} \\ -y + \cos x \\ -xy + \sin x - \frac{1}{4} \end{bmatrix} \in \mathbb{R}^4$$

with the transversal vector fields $C_i = -f_i$ for $i = 1, 2$. Of course C_i is $\widetilde{J}$-tangent and $\tau_i = 0$ and $S_i = \mathrm{I}$ for $i = 1, 2$. By straightforward computations we obtain

$$h_1 = \begin{bmatrix} 0 & -2 & 0 \\ -2 & \frac{1}{2} & 4x \\ 0 & 4x & -1 \end{bmatrix}, \qquad h_2 = \begin{bmatrix} \frac{1}{2} & -2 & -4y \\ -2 & 0 & 0 \\ -4y & 0 & -1 \end{bmatrix}$$

in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ of $\mathbb{R}^3$. Since $\det h_i = 4$, f_i is nondegenerate. In particular, $\omega_{h_i}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = 2$. By straightforward computations we also get

$$\Theta_i(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \det[f_{i_x}, f_{i_y}, f_{i_z}, C_i] = 2,$$

so f_i is the affine hypersphere for $i = 1, 2$.

For f_1 we have $f_{1_x} = \widetilde{J} f_{1_x}$, so $\frac{\partial}{\partial x} \in \mathcal{D}^+$. Since $h_1(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 0$ we have that $\mathcal{D}^+$ is the null-direction for f_1. We also have

$$\widetilde{J}(x^2 f_{1_x} + \frac{1}{4} f_{1_y} + x f_{1_z}) = -(x^2 f_{1_x} + \frac{1}{4} f_{1_y} + x f_{1_z}),$$

so the vector field $Y := x^2 \frac{\partial}{\partial x} + \frac{1}{4} \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$ belongs to $\mathcal{D}^-$. Now we compute

$$h_1(Y,Y) = 2 \cdot \frac{1}{4} x^2 \cdot h_1(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) + \frac{1}{16} \cdot h_1(\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$$

$$+2 \cdot \frac{1}{4} x \cdot h_1(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) + x^2 \cdot h_1(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \frac{1}{32} \neq 0,$$

so $\mathcal{D}^-$ is not the null-direction for f_1.

On the other hand, for f_2 we have $f_{2y} = -\widetilde{J} f_{2y}$, so $\frac{\partial}{\partial y} \in \mathcal{D}^-$. Since $h_2(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = 0$ we have that $\mathcal{D}^-$ is the null-direction for f_2. We also have

$$\widetilde{J}(\frac{1}{4} f_{2x} + y^2 f_{2y} - y f_{2z}) = \frac{1}{4} f_{2x} + y^2 f_{2y} - y f_{2z},$$

so the vector field $X := \frac{1}{4} \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}$ belongs to $\mathcal{D}^+$. Now we compute

$$h_2(X,X) = \frac{1}{16} \cdot h_2(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + 2 \cdot \frac{1}{4} y^2 \cdot h_2(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

$$+2 \cdot \frac{1}{4} (-y) \cdot h_2(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) + y^2 \cdot h_2(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \frac{1}{32} \neq 0,$$

so $\mathcal{D}^+$ is not the null-direction for f_2.

Summarizing, f_1 is a $\widetilde{J}$-tangent affine hypersphere with the null-direction $\mathcal{D}^+$ (and not $\mathcal{D}^-$) while f_2 is a $\widetilde{J}$-tangent affine hypersphere with the null-direction $\mathcal{D}^-$ (and not $\mathcal{D}^+$). Both hypersurfaces are not hyperquadrics.

Acknowledgment

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