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Second Hankel determinant for a subclass of analytic bi-univalent functions defined by subordination

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Abstract: In this work with a different technique we obtain upper bounds of the functional $\left|a_2a_4 - a_3^2\right|$ for functions belonging to a comprehensive subclass of analytic bi-univalent functions, which is defined by subordinations in the open unit disk. Moreover, our results extend and improve some of the previously known ones.

Key words: Bi-univalent functions, Fekete–Szegő determinant, second Hankel determinant, differential subordination, Carathéodory functions

1. Introduction and preliminaries

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in \mathbb{D}, \tag{1.1}$$

and let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\left(f(z)\right) = z, \; z \in \mathbb{D}, \qquad \text{and} \qquad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\; r_0(f) \geq \frac{1}{4}\right),$$

with the power series expansion

$$g(w) := f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent in* $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, and let Σ denote the class of bi-univalent functions in $\mathbb{D}$. Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ [23].

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Lewin [18] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for all the functions belonging to Σ. Recently, many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found nonsharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (see, for example, [1, 9, 16, 17, 23, 25, 26, 29]).

The problem of estimating the coefficients $|a_n|$ with $n \geq 4$ is presumably still an open problem. Using the *Faber polynomial* expansions, several authors obtained coefficient estimates of $|a_n|$ for the functions belonging in different subclasses of bi-univalent functions (see, for example, [10–13, 30]). First, we will recall some definitions and lemmas that will be used in this work.

One of the important tools in the theory of univalent functions are the *Hankel determinants*, which are used, for example, in showing that a function of bounded characteristic in $\mathbb{U}$, that is, a function that is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [5].

In 1976, Noonan and Thomas [19] defined the *q th Hankel determinant* for integers $n \geq 1$ and $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \qquad \text{and} \qquad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

where the Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$ are well known as *Fekete–Szegő* and *second Hankel determinant* functionals, respectively. Furthermore, Fekete and Szegő [8] introduced the generalized functional $a_3 - \lambda a_2^2$, where λ is some real number, and recently, problems in this direction have been considered by several authors (see, for example, [2, 6, 15, 20–22, 27, 28]).

Definition 1.1 *[7] For two functions f and g, which are analytic in $\mathbb{D}$, we say that the function f is* subordinate *to g and write $f(z) \prec g(z)$ if there exists a* Schwarz function *w, that is, a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.*

Throughout this paper, we assume that the function φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, such that $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function has the power series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, \ z \in \mathbb{D} \quad (B_1 > 0). \tag{1.3}$$

Definition 1.2 *[3, 24] A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma^\mu(\lambda, \varphi)$ if*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \varphi(z) \quad (\lambda \geq 1, \ \mu \geq 0),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec\varphi(w)\quad(\lambda\geq 1,\ \mu\geq 0),$$

where the function $g=f^{-1}$ *is given by* (1.2) *(all powers are the principal ones).*

Lemma 1.3 [7, p. 190] *Let* u *be analytic function in the unit disk* $\mathbb{D}$, *with* $u(0)=0$, *and* $|u(z)|<1$ *for all* $z\in\mathbb{D}$, *with the power series expansion*

$$u(z)=\sum_{n=1}^{\infty}c_n z^n,\quad z\in\mathbb{D}.$$

Then $|c_n|\leq 1$ *for all* $n=1,2,3,\ldots$. *Furthermore,* $|c_n|=1$ *for some* n $(n=1,2,3,\ldots)$ *if and only if* $u(z)=e^{i\theta}z^n$, $\theta\in\mathbb{R}$.

Lemma 1.4 [14] *If* $\psi(z)=\sum\limits_{n=1}^{\infty}\psi_n z^n$, $z\in\mathbb{D}$, *is a Schwarz function, then*

$$\psi_2=x\left(1-\psi_1^2\right),$$
$$\psi_3=\left(1-\psi_1^2\right)\left(1-|x|^2\right)s-\psi_1\left(1-\psi_1^2\right)x^2,$$

for some x, s, *with* $|x|\leq 1$ *and* $|s|\leq 1$.

The object of the present paper is to determine the functional $|H_2(2)|=\left|a_2a_4-a_3^2\right|$ for functions belonging to a comprehensive subclass of analytic bi-univalent functions, which is defined by subordinations in the open unit disk. Furthermore, our results generalize and improve some of the previously known results.

2. The functional $\left|a_2a_4-a_3^2\right|$ for the class $\mathcal{H}_{\Sigma}^{\mu}(\lambda,\varphi)$

First we state our main results and two interesting special cases.

Theorem 2.1 *If the function* $f\in\mathcal{H}_{\Sigma}^{\mu}(\lambda,\varphi)$ *is given by* (1.1), *then*

$$\left|a_2a_4-a_3^2\right|\leq B_1(P+Q+R),$$

where

$$\begin{aligned}
P&=\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|+2\Bigg(\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}\\
&\quad+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\Bigg)+\frac{2B_1}{2(\mu+3\lambda)(\mu+\lambda)}+\frac{B_1}{(2\lambda+\mu)^2},\\
Q&=2\left(\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right)+\frac{2B_1}{2(\mu+3\lambda)(\mu+\lambda)}+\frac{2B_1}{(2\lambda+\mu)^2},\\
R&=\frac{B_1}{(2\lambda+\mu)^2},
\end{aligned}\tag{2.1}$$

and B_1, B_2, B_3 *are given by* (1.3).

If we take in Theorem 2.1 the function

$$\varphi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+2(1-\beta)z^3+\cdots,\ z\in\mathbb{D}\quad(0\le\beta<1),$$

then we have the following special case:

Corollary 2.2 *If the function* $f\in\mathcal{H}_{\Sigma}^{\mu}\left(\lambda,\dfrac{1+(1-2\beta)z}{1-z}\right)$*, with* $0\le\beta<1$*, is given by* (1.1)*, then*

$$\begin{aligned}\left|a_2a_4-a_3^2\right|\le&4(1-\beta)^2\left[\left|\frac{-2\left(\mu^2+3\mu+2\right)(1-\beta)^2}{3(\mu+\lambda)^4}+\frac{1}{(\mu+3\lambda)(\mu+\lambda)}\right|\right.\\&+\frac{2}{(\mu+3\lambda)(\mu+\lambda)}+4\left(\frac{1-\beta}{2(\mu+\lambda)^2(\mu+2\lambda)}+\frac{1}{(\mu+3\lambda)(\mu+\lambda)}\right)\\&\left.+\frac{4}{(2\lambda+\mu)^2}\right].\end{aligned}$$

Remark 2.3 *Previous researchers got wrong results by miscalculation. We corrected their mistakes and obtained the correct result.*

(i) Theorem 2.2 is a correction of the obtained estimates given in [20, Theorem 2.1];

(ii) Letting the value $\lambda=1$ *in Theorem 2.2, we get a correction of the obtained estimates of [2, Theorem 2.1];*

(iii) Setting the values $\lambda=1$*,* $\mu=0$ *in Theorem 2.2, then we gain a correction of the obtained estimates that were given in [6, Theorem 2.1];*

(iv) Taking the values $\lambda=1$*,* $\mu=0$*, and* $\beta=0$ *in Theorem 2.2, then we obtain a correction of the obtained estimates given in [6, Corollary 2.2];*

(v) Supposing the values $\lambda=1$*,* $\mu=1$ *in Theorem 2.2, then we get a correction of the obtained estimates from [4, Theorem 1].*

For the special case

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+\frac{8\alpha^3+4\alpha}{6}z^3+\cdots,\ z\in\mathbb{D}\quad(0<\alpha\le1),$$

where the power is the principal one, Theorem 2.1 reduces to the next result:

Corollary 2.4 *If the function* $f\in\mathcal{H}_{\Sigma}^{\mu}\left(\lambda,\left(\dfrac{1+z}{1-z}\right)^{\alpha}\right)$*, with* $0<\alpha\le1$*, is given by* (1.1)*, then*

$$\begin{aligned}\left|a_2a_4-a_3^2\right|\le&4\alpha^2\left[\left|\frac{-2\left(\mu^2+3\mu+2\right)\alpha^2}{3(\mu+\lambda)^4}+\frac{2\alpha^2+1}{3(\mu+3\lambda)(\mu+\lambda)}\right|+\frac{2}{(\mu+3\lambda)(\mu+\lambda)}\right.\\&\left.+4\left(\frac{\alpha}{2(\mu+\lambda)^2(\mu+2\lambda)}+\frac{\alpha}{(\mu+3\lambda)(\mu+\lambda)}\right)+\frac{4}{(2\lambda+\mu)^2}\right].\end{aligned}$$

Remark 2.5 *Taking the values* $\lambda=1$*,* $\mu=1$ *in Theorem 2.4, then we obtain a correction of the obtained estimates given in [4, Theorem 2].*

3. Proof of the results

Proof of Theorem 2.1. If $f \in \mathcal{H}_{\Sigma}^{\mu}(\lambda, \varphi)$ then, by Definition 1.2 and Lemma 1.3, there exist two Schwartz functions u and v, of the form $u(z) = \sum_{n=1}^{\infty} p_n z^n$ and $v(z) = \sum_{n=1}^{\infty} q_n z^n$, $z \in \mathbb{D}$ such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \varphi(u(z)), \ z \in \mathbb{D}, \tag{3.1}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = \varphi(v(w)), \ w \in \mathbb{D}. \tag{3.2}$$

Using (1.3) we have

$$\varphi(u(z)) = 1 + B_1c_1z + \left(B_1c_2 + B_2c_1^2\right)z^2 + \left(B_1c_3 + 2c_1c_2B_2 + B_3c_1^3\right)z^3 + \cdots, \tag{3.3}$$

and

$$\varphi(v(w)) = 1 + B_1d_1w + \left(B_1d_2 + B_2d_1^2\right)w^2 + \left(B_1d_3 + 2d_1d_2B_2 + B_3d_1^3\right)w^3 + \cdots. \tag{3.4}$$

From (3.1), (3.3) and (3.2), (3.4), we get that, respectively,

$$(\lambda+\mu)a_2 = B_1c_1, \tag{3.5}$$

$$(\mu+2\lambda)\left[a_3 + \frac{a_2^2}{2}(\mu-1)\right] = B_1c_2 + B_2c_1^2, \tag{3.6}$$

$$(\mu+3\lambda)\left[a_4 + (\mu-1)a_2a_3 + (\mu-1)(\mu-2)\frac{a_2^3}{6}\right] = B_1c_3 + 2c_1c_2B_2 + B_3c_1^3, \tag{3.7}$$

and

$$-(\lambda+\mu)a_2 = B_1d_1, \tag{3.8}$$

$$(\mu+2\lambda)\left[\frac{a_2^2}{2}(\mu+3) - a_3\right] = B_1d_2 + B_2d_1^2, \tag{3.9}$$

$$(\mu+3\lambda)\left[-a_4 + (4+\mu)a_2a_3 - (4+\mu)(5+\mu)\frac{a_2^3}{6}\right] = B_1d_3 + 2d_1d_2B_2 + B_3d_1^3. \tag{3.10}$$

Now, from (3.5) and (3.8), we obtain

$$c_1 = -d_1. \tag{3.11}$$

and

$$a_2 = \frac{B_1c_1}{\lambda+\mu}.$$

In addition, from (3.6) and (3.9), we have

$$a_3 = \frac{B_1^2c_1^2}{(\mu+\lambda)^2} + \frac{B_1(c_2-d_2)}{2(2\lambda+\mu)}.$$

From (3.7) and (3.10), we also get

$$a_4 = \frac{-\left(\mu^2+3\mu-4\right)B_1^3}{6(\mu+\lambda)^3}c_1^3 + \frac{5B_1^2}{4(\mu+\lambda)(\mu+2\lambda)}c_1(c_2-d_2) + \frac{B_1(c_3-d_3)}{2(\mu+3\lambda)} + \frac{2B_2c_1(c_2+d_2)}{2(\mu+3\lambda)} + \frac{2B_3c_1^3}{2(\mu+3\lambda)}.$$

Therefore, we establish that

$$\left|a_2a_4-a_3^2\right| = \Bigg|\left[\frac{-\left(\mu^2+3\mu+2\right)B_1^4}{6(\mu+\lambda)^4} + \frac{B_3B_1}{(\mu+3\lambda)(\mu+\lambda)}\right]c_1^4 + \frac{B_1^3c_1^2(c_2-d_2)}{4(\mu+\lambda)^2(\mu+2\lambda)} + \frac{B_2B_1c_1^2(c_2+d_2)}{(\mu+3\lambda)(\mu+\lambda)} + \frac{B_1^2c_1(c_3-d_3)}{2(\mu+3\lambda)(\mu+\lambda)} - \frac{B_1^2(c_2-d_2)^2}{4(\mu+2\lambda)^2}\Bigg|. \tag{3.12}$$

According to Lemma 1.4, we have

$$c_2 = x\left(1-c_1^2\right) \qquad \text{and} \qquad d_2 = y\left(1-d_1^2\right),$$

so, from (3.11), we find that

$$c_2-d_2 = \left(1-c_1^2\right)(x-y) \qquad \text{and} \qquad c_2+d_2 = \left(1-c_1^2\right)(x+y), \tag{3.13}$$

and further

$$c_3 = \left(1-c_1^2\right)\left(1-|x|^2\right)s - c_1\left(1-c_1^2\right)x^2,$$

and

$$d_3 = \left(1-d_1^2\right)\left(1-|y|^2\right)w - d_1\left(1-d_1^2\right)y^2,$$

where

$$c_3-d_3 = \left(1-c_1^2\right)\left[\left(1-|x|^2\right)s - \left(1-|y|^2\right)w\right] - c_1\left(1-c_1^2\right)\left(x^2+y^2\right), \tag{3.14}$$

for some x, y, s, and w, with $|x|\le 1$, $|y|\le 1$, $|s|\le 1$, and $|w|\le 1$. Using (3.13) and (3.14), in (3.12), we obtain

$$\begin{aligned}\left|a_2a_4-a_3^2\right| =& B_1\Bigg|\left[\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4} + \frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right]c_1^4 \\ &+ \left[\frac{B_1^2(x-y)}{4(\mu+\lambda)^2(\mu+2\lambda)} + \frac{B_2(x+y)}{(\mu+3\lambda)(\mu+\lambda)}\right]c_1^2\left(1-c_1^2\right) \\ &- \frac{B_1c_1^2\left(1-c_1^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left(x^2+y^2\right) - \frac{B_1\left(1-c_1^2\right)^2}{4(\mu+2\lambda)^2}(x-y)^2 \\ &+ \frac{B_1c_1\left(1-c_1^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left[\left(1-|x|^2\right)s - \left(1-|y|^2\right)w\right]\Bigg|.\end{aligned}$$

Setting $c=|c_1|$, since $|c_1|\le 1$, then $c\in[0,1]$, and so we deduce that

$$\begin{aligned}
\left|a_2a_4-a_3^2\right| \le & B_1\Bigg[\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|c^4 \\
&+\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right]c^2\left(1+c^2\right)(|x|+|y|) \\
&+\frac{B_1c^2\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left(|x|^2+|y|^2\right)+\frac{B_1\left(1+c^2\right)^2}{4(2\lambda+\mu)^2}(|x|+|y|)^2 \\
&+\frac{B_1c\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left[\left(1-|x|^2\right)|s|+\left(1-|y|^2\right)|w|\right]\Bigg] \\
\le & B_1\Bigg[\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|c^4 \\
&+\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right]c^2\left(1+c^2\right)(|x|+|y|) \\
&+\frac{B_1c^2\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left(|x|^2+|y|^2\right)+\frac{B_1\left(1+c^2\right)^2}{4(2\lambda+\mu)^2}(|x|+|y|)^2 \\
&+\frac{B_1c\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\left[\left(1-|x|^2\right)+\left(1-|y|^2\right)\right]\Bigg] \\
= & B_1\Bigg[\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|c^4+\frac{B_1c(1+c^2)}{(\mu+3\lambda)(\mu+\lambda)} \\
&+\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right]c^2\left(1+c^2\right)(|x|+|y|) \\
&+\left[\frac{B_1c^2\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}-\frac{B_1c\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\right]\left(|x|^2+|y|^2\right)+\frac{B_1(1+c^2)^2}{4(2\lambda+\mu)^2}(|x|+|y|)^2\Bigg].
\end{aligned}$$

Now, for $\theta=|x|\le 1$ and $\vartheta=|y|\le 1$, we obtain

$$\left|a_2a_4-a_3^2\right|\le B_1F(\theta,\vartheta),\quad F(\theta,\vartheta):=T_1+(\theta+\vartheta)T_2+\left(\theta^2+\vartheta^2\right)T_3+(\theta+\vartheta)^2T_4,$$

where

$$T_1=T_1(c)=\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|c^4+\frac{B_1c\left(1+c^2\right)}{(\mu+3\lambda)(\mu+\lambda)}\ge 0,$$

$$T_2=T_2(c)=\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right]c^2\left(1+c^2\right)\ge 0,$$

$$T_3=T_3(c)=\frac{B_1c(c-1)\left(1+c^2\right)}{2(\mu+3\lambda)(\mu+\lambda)}\le 0,$$

$$T_4=T_4(c)=\frac{B_1\left(1+c^2\right)^2}{4(2\lambda+\mu)^2}\ge 0.$$

We now need to determine the maximum of the function $F(\theta,\vartheta)$ on the closed square $[0,1]\times[0,1]$ for $c\in[0,1]$. For this work, we must investigate the maximum of $F(\theta,\vartheta)$ according to $c\in(0,1)$, $c=0$, and $c=1$, taking into the account the sign of $F_{\theta\theta}F_{\vartheta\vartheta}-(F_{\theta\vartheta})^2$.

First, if we let $c=1$, then we obtain

$$F(\theta,\vartheta)=\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|+\frac{2B_1}{(\mu+3\lambda)(\mu+\lambda)}$$
$$+2\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right](\theta+\vartheta)+\frac{B_1}{(2\lambda+\mu)^2}(\theta+\vartheta)^2,$$

and hence we can see easily that

$$\max\{F(\theta,\vartheta):(\theta,\vartheta)\in[0,1]\times[0,1]\}=F(1,1)$$
$$=\left|\frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4}+\frac{B_3}{(\mu+3\lambda)(\mu+\lambda)}\right|+\frac{2B_1}{(\mu+3\lambda)(\mu+\lambda)}$$
$$+4\left[\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)}+\frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)}\right]+\frac{4B_1}{(2\lambda+\mu)^2}.$$

Second, letting $c=0$, then we have

$$F(\theta,\vartheta)=\frac{B_1}{4(2\lambda+\mu)^2}(\theta+\vartheta)^2,$$

and we can see easily that

$$\max\{F(\theta,\vartheta):(\theta,\vartheta)\in[0,1]\times[0,1]\}=F(1,1)=\frac{B_1}{(2\lambda+\mu)^2}.$$

Finally, let us consider the case $c\in(0,1)$. Since $T_3+2T_4>0$ and $T_3<0$, we conclude that

$$F_{\theta\theta}F_{\vartheta\vartheta}-(F_{\theta\vartheta})^2<0,$$

and thus the function F cannot have a local maximum in the interior of the square $[0,1]\times[0,1]$.

For $\theta=0$ and $0\le\vartheta\le1$ (similarly for $\vartheta=0$ and $0\le\theta\le1$), we get

$$H(\vartheta):=F(0,\vartheta)=(T_3+T_4)\vartheta^2+T_2\vartheta+T_1.$$

(i) If $T_3+T_4\ge0$, it is clear that $H'(\vartheta)=2(T_3+T_4)\vartheta+T_2>0$ for $0<\vartheta<1$ and any fixed $c\in(0,1)$; that is, $H(\vartheta)$ is an increasing function. Hence, for fixed $c\in(0,1)$, the maximum of $H(\vartheta)$ occurs at $\vartheta=1$, and then

$$\max\{H(\vartheta):\vartheta\in[0,1]\}=H(1)=T_1+T_2+T_3+T_4.$$

(ii) If $T_3+T_4<0$, we consider for critical point $\vartheta=\dfrac{-T_2}{2(T_3+T_4)}=\dfrac{T_2}{2k}$ for fixed $c\in(0,1)$, where $k=-(T_3+T_4)>0$, the following two cases:

Case 1. For $\vartheta = \dfrac{T_2}{2k} > 1$, it follows that $k < \dfrac{T_2}{2} \leq T_2$, and so $T_2 + T_3 + T_4 \geq 0$. Therefore,

$$H(0) = T_1 \leq T_1 + T_2 + T_3 + T_4 = H(1).$$

Case 2. For $\vartheta = \dfrac{T_2}{2k} \leq 1$, since $\dfrac{T_2}{2} \geq 0$, we get $\dfrac{T_2^2}{4k} \leq \dfrac{T_2}{2} \leq T_2$. Also, we have $H(1) = T_1+T_2+T_3+T_4 \leq T_1 + T_2$, and hence

$$H(0) = T_1 \leq T_1 + \frac{T_2^2}{4k} = H\left(\frac{T_2}{2k}\right) \leq T_1 + T_2.$$

By considering cases **(i)** and **(ii)**, for $\theta = 0$, $0 \leq \vartheta \leq 1$ and for fixed $c \in (0,1)$, it follows that $H(\vartheta)$ gets its maximum when $T_3 + T_4 \geq 0$, which means

$$\max\{H(\vartheta) : \vartheta \in [0,1]\} = H(1) = T_1 + T_2 + T_3 + T_4.$$

For $\theta = 1$ and $0 \leq \vartheta \leq 1$ (similarly for $\vartheta = 1$ and $0 \leq \theta \leq 1$), we get

$$G(\vartheta) := F(1,\vartheta) = (T_3 + T_4)\vartheta^2 + (T_2 + 2T_4)\vartheta + T_1 + T_2 + T_3 + T_4.$$

(iii) If $T_3 + T_4 \geq 0$, it is clear that $G'(\vartheta) = 2(T_3 + T_4)\vartheta + T_2 + 2T_4 > 0$ for $0 < \vartheta < 1$ and any fixed $c \in (0,1)$; that is, $G(\vartheta)$ is an increasing function. Hence, for fixed $c \in (0,1)$, the maximum of $G(\vartheta)$ occurs at $\vartheta = 1$, and

$$\max\{G(\vartheta) : \vartheta \in [0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

(iv) If $T_3 + T_4 < 0$, then we consider for critical point $\vartheta = \dfrac{-(T_2 + 2T_4)}{2(T_3 + T_4)} = \dfrac{T_2 + 2T_4}{2k}$ for any fixed $c \in (0,1)$, where $k = -(T_3 + T_4) > 0$, the following two cases:

Case 1. For $\mu = \dfrac{T_2 + 2T_4}{2k} > 1$, it follows that $k < \dfrac{T_2 + 2T_4}{2} \leq T_2 + 2T_4$, so $T_2 + T_3 + 3T_4 \geq 0$. Therefore,

$$G(0) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2 + T_3 + T_4 + T_2 + T_3 + 3T_4 = T_1 + 2T_2 + 2T_3 + 4T_4 = G(1).$$

Case 2. For $\vartheta = \dfrac{T_2 + 2T_4}{2k} \leq 1$, since $\dfrac{T_2 + 2T_4}{2} \geq 0$, we get that

$$\frac{(T_2 + 2T_4)^2}{4k} \leq \frac{T_2 + 2T_4}{2} \leq T_2 + 2T_4.$$

Therefore,

$$\begin{aligned} G(0) = T_1 + T_2 + T_3 + T_4 &\leq T_1 + T_2 + T_3 + T_4 + \frac{(T_2 + 2T_4)^2}{4k} \\ &= G\left(\frac{T_2 + 2T_4}{2k}\right) \leq T_1 + 2T_2 + T_3 + 3T_4. \end{aligned}$$

By considering cases **(iii)** and **(iv)** for $\theta = 1$, $0 \leq \vartheta \leq 1$ and for fixed $c \in (0,1)$, it follows that $G(\vartheta)$ gets its maximum when $T_3 + T_4 \geq 0$, which means

$$\max\{G(\vartheta) : \vartheta \in [0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $H(1) \leq G(1)$ for $c \in [0,1]$, then $\max\{F(\theta,\vartheta) : (\theta,\vartheta) \in [0,1]\times[0,1]\} = F(1,1)$, and thus the maximum of F in the closed square $[0,1]\times[0,1]$ occurs at $\theta = 1$ and $\vartheta = 1$.

Let the function $K : [0,1] \to \mathbb{R}$ defined by

$$\begin{aligned} K(c) :&= B_1 \max\{F(\theta,\vartheta) : (\theta,\vartheta) \in [0,1]\times[0,1]\} = B_1 F(1,1) \\ &= B_1 \left(T_1 + 2T_2 + 2T_3 + 4T_4\right). \end{aligned}$$

Substituting the values of T_1, T_2, T_3, and T_4 in the above function K, we have

$$\begin{aligned} K(c) = B_1 \Bigg\{ \Bigg[&\left| \frac{-\left(\mu^2+3\mu+2\right)B_1^3}{6(\mu+\lambda)^4} + \frac{B_3}{(\mu+3\lambda)(\mu+\lambda)} \right| \\ &+ 2\left(\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)} + \frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)} \right) + \frac{2B_1}{2(\mu+3\lambda)(\mu+\lambda)} + \frac{B_1}{(2\lambda+\mu)^2} \Bigg] c^4 \\ &+ \left[2\left(\frac{B_1^2}{4(\mu+\lambda)^2(\mu+2\lambda)} + \frac{|B_2|}{(\mu+3\lambda)(\mu+\lambda)} \right) + \frac{2B_1}{2(\mu+3\lambda)(\mu+\lambda)} + \frac{2B_1}{(2\lambda+\mu)^2} \right] c^2 \\ &+ \frac{B_1}{(2\lambda+\mu)^2} \Bigg\}. \end{aligned}$$

Setting $c^2 = t$, and letting P, Q, R be given by (2.1), since $P \geq 0$, $Q \geq 0$, $R \geq 0$ it follows that

$$\max\left\{Pt^2 + Qt + R : t \in [0,1]\right\} = P + Q + R,$$

and consequently

$$\left|a_2 a_4 - a_3^2\right| \leq B_1(P + Q + R),$$

which completes our proof. □

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