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On the coefficient problem for close-to-convex functions

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Abstract: This paper is concerned with the problem of estimating $|a_4 - a_2a_3|$, where a_k are the coefficients of a given close-to-convex function. The bounds of this expression for various classes of analytic functions have been applied to estimate the third Hankel determinant $H_3(1)$. The results for two subclasses of the class $\mathcal{C}$ of all close-to-convex functions are sharp. This bound is equal to 2. It is conjectured that this number is also the exact bound of $|a_4 - a_2a_3|$ for the whole class $\mathcal{C}$.

Key words: Close-to-convex functions, coefficient problem

1. Introduction

Let $\mathcal{A}$ be the family of all functions analytic in the unit disk $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ given by the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}^*$ denote the class of starlike functions in $\mathcal{A}$ and let $\mathcal{P}$ denote the class of all analytic functions p with a positive real part in Δ satisfying the normalization condition $p(0) = 1$.

Given $\beta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function $f \in \mathcal{A}$ is called close-to-convex with argument β with respect to g if

$$\mathfrak{Re}\left\{\frac{e^{i\beta} z f'(z)}{g(z)}\right\} > 0, \quad z \in \Delta. \tag{1.2}$$

Let $\mathcal{C}_\beta(g)$ be the class of all such functions. Moreover, let

$$\mathcal{C}(g) = \bigcup_{\beta\in(-\pi/2,\pi/2)} \mathcal{C}_\beta(g) \quad \text{and} \quad \mathcal{C}_\beta = \bigcup_{g\in\mathcal{S}^*} \mathcal{C}_\beta(g).$$

Let $\mathcal{C}$ denote the family of all close-to-convex functions (see [3, 5]). It is obvious that

$$\mathcal{C} = \bigcup_{\beta\in(-\pi/2,\pi/2)} \mathcal{C}_\beta = \bigcup_{g\in\mathcal{S}^*} \mathcal{C}(g).$$

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The number $e^{i\beta}$ is necessary in (1.2) for the definition of close-to-convex function. In addition, this factor significantly complicates the task of estimating some coefficient functionals. Therefore, to simplify the calculation, many authors take $\beta = 0$ or use a specific starlike function, for example the Koebe function

$$k(z) = \frac{z}{(1-z)^2}, \quad z \in \Delta. \tag{1.3}$$

Then inequality (1.2) becomes:

$$\mathfrak{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \Delta \tag{1.4}$$

or

$$\mathfrak{Re}\left\{e^{i\beta}(1-z)^2 f'(z)\right\} > 0, \quad z \in \Delta, \tag{1.5}$$

respectively, and defines the related subclass of close-to-convex functions $\mathcal{C}_0$ and $\mathcal{C}(k)$, respectively. Let us cite the most important results concerning the estimates of some coefficient functionals within the class $\mathcal{C}$. Keogh and Merkes in [6] solved the Fekete-Szegö problem in the class $\mathcal{C}_0$. Koepf in [7] extended this result for the class $\mathcal{C}$. Kowalczyk and Lecko, in [9], studied the Fekete-Szegö problem in the class $\mathcal{C}(k)$ of all close-to-convex functions with respect to the Koebe function (and in [8], in the subclass of close-to-convex with respect to other starlike functions). Recently, several authors have extensively investigated the Hankel determinant for close-to-convex functions (see for example [13, 15, 16, 18]) and the logarithmic coefficients of close-to-convex functions (see for example [20]).

The main aim of this paper was to determine the estimates of the expression $|a_4 - a_2a_3|$ for the classes $\mathcal{C}_0$, $\mathcal{C}(k)$, and $\mathcal{C}$. The functional $|a_4 - a_2a_3|$ has been estimated for many classes. Babalola, in [1], derived the exact bounds of $|a_4 - a_2a_3|$ for the class of starlike functions, for the class of convex functions and for the class of functions whose derivative has a positive real part; these values are equal to 2, $4/9\sqrt{3}$, and $5\sqrt{5}/18\sqrt{3}$, respectively. In [14], Mishra et al. proved that this bound in the class of starlike functions with respect to symmetric points is $1/2$ and in the class of convex functions with respect to symmetric points is $4/27$. Krishna et al. published the same results in [21]. In [17], Raza and Malik, found that $|a_4 - a_2a_3| \le 1/6$ for the class of lemniscate starlike functions (for the definition of the class see [19]). All these authors used this functional $|a_4 - a_2a_3|$ to estimate the third Hankel determinant $H_3(1)$.

Taking into account (1.2), we can write

$$\frac{e^{i\beta}zf'(z)}{g(z)} = h(z)\cos\beta + i\sin\beta, \tag{1.6}$$

where $h \in \mathcal{P}$. If $g \in \mathcal{S}^*$ and $h \in \mathcal{P}$ are given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.7}$$

and

$$h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.8}$$

then (1.6) leads to

$$z+\sum_{n=2}^{\infty} n a_n z^n = \left(z+\sum_{n=2}^{\infty} b_n z^n\right)\left(1+e^{-i\beta}\cos\beta\sum_{n=1}^{\infty} p_n z^n\right). \tag{1.9}$$

Therefore,

$$n a_n = b_n + e^{-i\beta}\cos\beta\left(p_{n-1}+\sum_{k=2}^{n-1} b_k p_{n-k}\right). \tag{1.10}$$

2. Preliminary results

We shall need the following results. The first one is known as Caratheodory's lemma (for example see [2]). The second one is by Libera and Złotkiewicz [10, 11].

Lemma 2.1 **([2])** *If $h\in\mathcal{P}$ is given by (1.8), then the sharp estimate $|p_n|\le 2$ holds for $n\ge 1$.*

Lemma 2.2 **([10, 11])** *Let h be given by (1.8) and $p_1\in[0,2]$. Then $h\in\mathcal{P}$ if and only if*

$$2p_2 = {p_1}^2 + (4-{p_1}^2)x$$

and

$$4p_3 = {p_1}^3 + 2(4-{p_1}^2)p_1 x - (4-{p_1}^2)p_1 x^2 + 2(4-{p_1}^2)(1-|x|^2)z$$

for some complex numbers x, z such that $|x|\le 1$, $|z|\le 1$.

Remark. Given $g\in\mathcal{S}^*$ defined by (1.7), the functional $|b_4-\mu b_2 b_3|$, $\mu\in\mathbb{R}$ is invariant under rotation. This means that for $g_\phi(z)=e^{-i\phi}g(ze^{i\phi})$, $\phi\in\mathbb{R}$ of the form $g_\phi(z)=z+c_2z^2+c_3z^3+\ldots$ we have $|b_4-\mu b_2 b_3| = |c_4-\mu c_2 c_3|$. Similarly, it can be proved that $|p_3-\mu p_1 p_2|$ for $h\in\mathcal{P}$ defined by (1.8) is invariant under rotation.

To obtain our results, we also need a few sharp estimates.

Lemma 2.3 *Let $h\in\mathcal{P}$ be given by (1.8) and $\mu\in[1/2,1]$, then*

$$|p_3-\mu p_1 p_2| \le \begin{cases} \frac{1}{4}\mu^2 p^3 - \frac{1}{2}\mu(2-\mu)p^2 + 2, & p\in\left[0,\frac{2}{2-\mu}\right],\\ (3-2\mu)p-(1-\mu)p^3, & p\in\left[\frac{2}{2-\mu},2\right],\end{cases}$$

where $p=|p_1|$.

Proof From Lemma 2.2, we have

$$|p_3-\mu p_1 p_2| = \tfrac{1}{4}\left|(1-2\mu){p_1}^3 + 2(1-\mu)(4-{p_1}^2)p_1 x - (4-{p_1}^2)p_1 x^2 \right. \tag{2.1}$$
$$\left. +2(4-{p_1}^2)(1-|x|^2)z\right|.$$

Applying the triangle inequality in (2.1) with $|x|=\varrho$, $\varrho\in[0,1]$ and $|p_1|=p$, $p\in[0,2]$, we obtain

$$|p_3-\mu p_1 p_2| \le \tfrac{1}{4}\left[(2\mu-1)p^3 + 2(1-\mu)(4-p^2)p\varrho + (4-p^2)p\varrho^2 + 2(4-p^2)(1-\varrho^2)\right]$$
$$= \tfrac{1}{4}\left[(2\mu-1)p^3 + 2(4-p^2) + 2(1-\mu)(4-p^2)p\varrho - (4-p^2)(2-p)\varrho^2\right] \equiv w(\varrho)$$

(with the equality when $x = -\varrho$ and $z = -1$). If $p < \frac{2}{2-\mu}$, then

$$w(\varrho) \le w\left(\tfrac{2}{2-\mu}\right) = \tfrac{1}{4}\mu^2 p^3 - \tfrac{1}{2}\mu(2-\mu)p^2 + 2.$$

If $p \ge \frac{2}{2-\mu}$, then

$$w(\varrho) \le w(1) = (3-2\mu)p - (1-\mu)p^3.$$

Therefore, for $\mu \in [1/2, 1]$, we get the desired result. □

In the second case, the equality holds when $\varrho = 1$, i.e., $x = -1$. Then $p_2 = {p_1}^2 - 2$. This means that the extremal function is

$$h_t(z) = \frac{1-z^2}{1-2tz+z^2}, \quad t \in [-1,1].$$

It is easy to check that

$$\begin{aligned} \max\{|p_3 - \mu p_1 p_2| : h \in \mathcal{P}\} &= \max\{|p_3 - \mu p_1 p_2| : |p_1| \in [0,2]\} \\ &= \max\{2, 4\mu - 2\} = 2, \end{aligned} \tag{2.2}$$

which is the result obtained by Hayami and Owa [4].

From Lemma 2.3, we can easily get the following corollary:

Corollary 2.4 *Let $h \in \mathcal{P}$ be given by (1.8), then $\left|p_3 - \frac{2}{3}p_1 p_2\right| \le G(p)$, where*

$$G(p) = \begin{cases} \frac{1}{9}p^3 - \frac{4}{9}p^2 + 2, & p \in [0, 3/2], \\ \frac{5}{3}p - \frac{1}{3}p^3, & p \in [3/2, 2] \end{cases} \tag{2.3}$$

and $p = |p_1|$.

Lemma 2.5 *Let $g \in \mathcal{S}^*$ be given by (1.7), then $\left|b_4 - \frac{2}{3}b_2 b_3\right| \le H(q)$, where*

$$H(q) = \begin{cases} \frac{1}{48}(2+q)(9q^2 - 8q + 16), & q \in [0, 4/5], \\ \frac{1}{3}q(4-q^2), & q \in [4/5, 2] \end{cases} \tag{2.4}$$

and $q = |b_2|$, $q \in [0,2]$.

Proof Every function $g \in \mathcal{S}^*$ satisfies in Δ the equality

$$zg'(z) = g(z)Q(z), \tag{2.5}$$

where $Q \in \mathcal{P}$. Let $Q(z) = 1 + \sum_{k=1}^{\infty} q_n z^n$. Equating the coefficients in (2.5) gives

$$b_2 = q_1, \quad b_3 = \tfrac{1}{2}\left(q_2 + {q_1}^2\right), \quad b_4 = \tfrac{1}{3}q_3 + \tfrac{1}{2}q_1 q_2 + \tfrac{1}{6}{q_1}^3. \tag{2.6}$$

Applying (2.6), we get

$$\left|b_4 - \tfrac{2}{3}b_2 b_3\right| = \tfrac{1}{3}\left|q_3 + \tfrac{1}{2}q_1 q_2 - \tfrac{1}{2}{q_1}^3\right|. \tag{2.7}$$

Now, we use Lemma 2.2 to get

$$\left|b_4 - \tfrac{2}{3}b_2b_3\right| = \tfrac{1}{12}\left|(4-{q_1}^2)\left[3q_1x - q_1x^2 + 2(1-|x|^2)z\right]\right|.$$

Since the functional $\left|b_4 - \frac{2}{3}b_2b_3\right|$ is invariant under rotation, we may write $q_1 = q$, $q \in [0,2]$. Hence, applying the triangle inequality with $|x| = \varrho$, $\varrho \in [0,1]$, we get

$$\begin{aligned}\left|b_4 - \tfrac{2}{3}b_2b_3\right| &= \tfrac{1}{12}(4-q^2)\left|3qx - qx^2 + 2(1-|x|^2)z\right| \\ &\le \tfrac{1}{12}(4-q^2)\left[3q\varrho + q\varrho^2 + 2(1-\varrho^2)\right] = \tfrac{1}{12}(4-q^2)\left[(q-2)\varrho^2 + 3q\varrho + 2\right]\end{aligned}$$

(with the equality when $x = -\varrho$ and $z = -1$).

Let $w(\varrho) = (q-2)\varrho^2 + 3q\varrho + 2$. If $q \in [0, 4/5)$, then

$$w(\varrho) \le w\left(\frac{3q}{4-2q}\right) = \frac{9q^2 - 8q + 16}{4(2-q)}.$$

If $q \in [4/5, 2]$, then

$$w(\varrho) \le w(1) = 4q.$$

Therefore,

$$\left|b_4 - \tfrac{2}{3}b_2b_3\right| \le \begin{cases} \frac{1}{48}(2+q)(9q^2 - 8q + 16), & q \in [0, 4/5], \\ \frac{1}{3}q(4-q^2), & q \in [4/5, 2]. \end{cases}$$

This completes the proof of Lemma 2.5. □

In the second case, the equality holds when $\varrho = 1$, i.e., $x = -1$. Then $b_3 = {b_2}^2 - 1$, which means that the extremal function is

$$g_t(z) = \frac{z}{1 - 2tz + z^2}, \quad t \in [-1, 1].$$

3. Main results

Theorem 3.1 *If $f \in \mathcal{C}_0$ is given by (1.1), then*

$$|a_4 - a_2a_3| \le 2.$$

This result is sharp.

Proof From (1.4), we can write

$$zf'(z) = g(z)h(z), \tag{3.1}$$

where $h \in \mathcal{P}$. Let g and h be given by (1.7) and (1.8), respectively. Equating the coefficients in (3.1) gives

$$2a_2 = b_2 + p_1, \quad 3a_3 = b_3 + b_2p_1 + p_2, \quad 4a_4 = b_4 + b_3p_1 + b_2p_2 + p_3. \tag{3.2}$$

Therefore, using (3.2), we have

$$\begin{aligned}|a_4 - a_2a_3| = \Big|&\tfrac{1}{4}\left(b_4 - \tfrac{2}{3}b_2b_3\right) + \tfrac{1}{4}\left(p_3 - \tfrac{2}{3}p_1p_2\right) + \tfrac{1}{12}p_1\left(b_3 - {b_2}^2\right) \\ &+ \tfrac{1}{12}b_2\left(p_2 - {p_1}^2\right) - \tfrac{1}{12}p_1b_2\left(p_1 + b_2\right)\Big|.\end{aligned}$$

Applying the triangle inequality, we obtain

$$|a_4 - a_2a_3| \le \tfrac{1}{4}\left|b_4 - \tfrac{2}{3}b_2b_3\right| + \tfrac{1}{4}\left|p_3 - \tfrac{2}{3}p_1p_2\right| + \tfrac{1}{12}|p_1||b_3 - {b_2}^2| \\ + \tfrac{1}{12}|b_2||p_2 - {p_1}^2| + \tfrac{1}{12}|p_1||b_2||p_1 + b_2|. \tag{3.3}$$

It is well known (see [12]) that

$$|p_2 - {p_1}^2| \le 2 \tag{3.4}$$

for $h \in \mathcal{P}$. Moreover, the Fekete-Szegö inequality for $g \in \mathcal{S}^*$ (see for example [7]) gives

$$|b_3 - {b_2}^2| \le 1. \tag{3.5}$$

From (3.3), using (3.4), (3.5), Lemma 2.5, and Corollary 2.4 and writing $|p_1| = p$ and $|b_2| = q$, $p, q \in [0, 2]$, we get

$$|a_4 - a_2a_3| \le F(p, q),$$

where

$$F(p, q) = \tfrac{1}{4}H(q) + \tfrac{1}{4}G(p) + \tfrac{1}{12}p + \tfrac{1}{6}q + \tfrac{1}{12}pq(p + q), \quad p, q \in [0, 2] \tag{3.6}$$

and $H(q), G(p)$ are given by (2.4) and (2.3).

We will show that $F(p, q) \le 2$ for $p, q \in [0, 2]$. It is easy to check that H is increasing for $q \in [0, 2/\sqrt{3}]$. Suppose that $q \in [2/\sqrt{3}, 2]$. Then

$$F(p, q) = \tfrac{1}{12}q(4 - q^2) + \tfrac{1}{4}G(p) + \tfrac{1}{12}p + \tfrac{1}{6}q + \tfrac{1}{12}p^2q + \tfrac{1}{12}pq^2,$$

so

$$\frac{\partial F}{\partial q} = \tfrac{1}{12}\left(6 - 3q^2 + p^2 + 2pq\right).$$

Thus, for $q \in [2/\sqrt{3}, \sqrt{2}]$, $p \in [0, 2]$, we have $\frac{\partial F}{\partial q} \ge 0$, so $F(p, q)$, with a fixed p, is increasing as a function of q.

For this reason,

$$\max\{F(p, q) : (p, q) \in [0, 2] \times [0, 2]\} = \max\left\{F(p, q) : (p, q) \in [0, 2] \times [\sqrt{2}, 2]\right\}. \tag{3.7}$$

Assume now that $q \in [\sqrt{2}, 2]$. If $p \in [0, 3/2]$, then

$$F(p, q) = \tfrac{1}{36}(p^3 - 4p^2 + 3p + 18 - 3q^3 + 18q + 3p^2q + 3pq^2).$$

Therefore,

$$\frac{\partial F}{\partial p} = \tfrac{1}{36}(3p^2 - 8p + 3 + 6pq + 3q^2) \ge \tfrac{1}{36}\left(3p^2 - 8p + 3 + 6\sqrt{2}p + 6\right) \\ = \tfrac{1}{36}\left(3p^2 + (6\sqrt{2} - 8)p + 9\right) > 0.$$

So (3.7) remains true even for $(p, q) \in [3/2, 2] \times [\sqrt{2}, 2]$.

Now, let $p \in [3/2, 2]$. In this case,

$$F(p,q) = \tfrac{1}{12}(p+q)\left[6-(p-q)^2\right] \le 2$$

with the equality if $p = q = 2$.

The equalities $|p_1| = 2$ and $|b_2| = 2$ hold only for the functions

$$h(z) = \frac{1+ze^{i\theta}}{1-ze^{i\theta}} \quad \text{and} \quad g(z) = \frac{z}{(1-ze^{i\phi})^2}, \quad \theta, \phi \in \mathbb{R}$$

respectively. This means that the equality in Theorem 3.1 holds for f given by

$$f'(z) = \frac{1+ze^{i\theta}}{1-ze^{i\theta}} \, \frac{1}{(1-ze^{i\phi})^2}.$$

For this f, we have

$$a_4 - a_2a_3 = -e^{i(2\theta+\phi)} - \tfrac{5}{6}e^{i(\theta+2\phi)} - \tfrac{1}{6}e^{3i\theta}.$$

Thus,

$$|a_4 - a_2a_3| = \left|1 + \cos(\phi-\theta) + \tfrac{2}{3}i\sin(\phi-\theta)\right|.$$

This expression is less than or equal to 2; the equality holds only when $\theta = \phi$. So we obtain that equality in Theorem 3.1 holds only for the function $f(z) = \frac{z}{(1-ze^{i\phi})^2}$ with arbitrary $\phi \in \mathbb{R}$. □

Theorem 3.2 *If $f \in \mathcal{C}$ is given by (1.1), then*

$$|a_4 - a_2a_3| < 2.5.$$

Proof From (1.10) we have

$$2a_2 = b_2 + p_1e^{-i\beta}\cos\beta, \quad 3a_3 = b_3 + (b_2p_1 + p_2)e^{-i\beta}\cos\beta, \tag{3.8}$$
$$4a_4 = b_4 + (b_3p_1 + b_2p_2 + p_3)e^{-i\beta}\cos\beta.$$

Hence, using (3.8), we get

$$\begin{aligned}|a_4 - a_2a_3| = \Big|&\tfrac{1}{4}\left(b_4 - \tfrac{2}{3}b_2b_3\right) + \tfrac{1}{4}\left(p_3 - \tfrac{2}{3}p_1p_2e^{-i\beta}\cos\beta\right)e^{-i\beta}\cos\beta \\ &+ \tfrac{1}{12}p_1\left(b_3 - {b_2}^2\right)e^{-i\beta}\cos\beta + \tfrac{1}{12}b_2\left(p_2 - {p_1}^2e^{-i\beta}\cos\beta\right)e^{-i\beta}\cos\beta \\ &- \tfrac{1}{12}p_1b_2\left(b_2 + p_1e^{-i\beta}\cos\beta\right)e^{-i\beta}\cos\beta\Big|.\end{aligned} \tag{3.9}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}|a_4 - a_2a_3| \le\ &\tfrac{1}{4}\left|b_4 - \tfrac{2}{3}b_2b_3\right| + \tfrac{1}{4}\left|p_3 - \tfrac{2}{3}p_1p_2e^{-i\beta}\cos\beta\right| + \tfrac{1}{12}|p_1|\left|b_3 - {b_2}^2\right| \\ &+ \tfrac{1}{12}|b_2|\left|p_2 - {p_1}^2e^{-i\beta}\cos\beta\right| + \tfrac{1}{12}|p_1||b_2|\left|b_2 + p_1e^{-i\beta}\cos\beta\right|.\end{aligned} \tag{3.10}$$

Using Lemma 2.2, we have

$$\begin{aligned}\left|p_2 - {p_1}^2 e^{-i\beta}\cos\beta\right| &= \left|(p_2 - {p_1}^2)\cos\beta + ip_2\sin\beta\right| \\ &= \left|\left(\tfrac{1}{2}(4-{p_1}^2)x - \tfrac{1}{2}{p_1}^2\right)\cos\beta + i\left(\tfrac{1}{2}(4-{p_1}^2)x + \tfrac{1}{2}{p_1}^2\right)\sin\beta\right| \\ &= \left|\tfrac{1}{2}(4-{p_1}^2)xe^{i\beta} - \tfrac{1}{2}{p_1}^2 e^{-i\beta}\right|.\end{aligned}$$

Since the functional $\left|p_2 - {p_1}^2 e^{-i\beta}\cos\beta\right|$ is invariant under rotation, we can assume (for a moment) that p_1 is a positive real number. In this case,

$$\left|\tfrac{1}{2}(4-{p_1}^2)xe^{i\beta} - \tfrac{1}{2}{p_1}^2 e^{-i\beta}\right| \le \tfrac{1}{2}(4-{p_1}^2) + \tfrac{1}{2}{p_1}^2 = 2.$$

Hence, in general (for an arbitrary p_1)

$$\left|p_2 - {p_1}^2 e^{-i\beta}\cos\beta\right| \le 2. \tag{3.11}$$

From Corollary 2.4 and Lemma 2.1, we get

$$\begin{aligned}\left|p_3 - \tfrac{2}{3}p_1p_2e^{-i\beta}\cos\beta\right| &= \left|\left(p_3 - \tfrac{2}{3}p_1p_2\right)\cos\beta + ip_3\sin\beta\right| \\ &\le G(p)\cos\beta + 2|\sin\beta| \\ &\le G(p) + 2, \qquad (3.12)\end{aligned}$$

where $G(p)$ is given by (2.3) and $p = |p_1|$. Taking into account (3.5), (3.11), (3.12), and Lemma 2.5 and writing $|p_1| = p$, $|b_2| = q$, from (3.10), we obtain

$$|a_4 - a_2a_3| \le \tfrac{1}{4}H(q) + \tfrac{1}{4}G(p) + \tfrac{1}{2} + \tfrac{1}{12}p + \tfrac{1}{6}q + \tfrac{1}{12}pq(p+q) = F(p,q) + \tfrac{1}{2},$$

where $F(p,q)$ is given by (3.6). Since $F(p,q) \le 2$ (see proof of Theorem 3.1), we obtain the declared bound. □

However, the result in Theorem 3.2 is not sharp, it is the best known estimate for the whole class $\mathcal{C}$. Moreover, we conjecture that the exact bound is 2. This presumption is supported by the following theorem.

Theorem 3.3 *If $f \in \mathcal{C}(k)$ is given by (1.1), then*

$$|a_4 - a_2a_3| \le 2.$$

This result is sharp.

Proof From (3.9) for the Koebe function given by (1.3), we have

$$\begin{aligned}|a_4 - a_2a_3| &= \cos\beta\left|\tfrac{1}{4}\left(p_3 - \tfrac{2}{3}p_1p_2e^{-i\beta}\cos\beta\right) - \tfrac{1}{12}p_1 + \tfrac{1}{6}\left(p_2 - {p_1}^2e^{-i\beta}\cos\beta\right)\right. \\ &\quad \left. - \tfrac{1}{6}p_1\left(2 + p_1e^{-i\beta}\cos\beta\right)\right| \\ &= \cos\beta\left|\left[\tfrac{1}{4}\left(p_3 - \tfrac{2}{3}p_1p_2\right) + \tfrac{1}{6}\left(p_2 - {p_1}^2\right) - \tfrac{1}{6}{p_1}^2 - \tfrac{5}{12}p_1\right]\cos\beta\right. \qquad (3.13) \\ &\quad \left. + i\left(\tfrac{1}{4}p_3 + \tfrac{1}{6}p_2 - \tfrac{5}{12}p_1\right)\sin\beta\right|.\end{aligned}$$

Using Lemma 2.1 it is easy to check that

$$\left|\tfrac{1}{4}p_3 + \tfrac{1}{6}p_2 - \tfrac{5}{12}p_1\right| \leq \tfrac{5}{3}. \tag{3.14}$$

Applying Corollary 2.4 and (3.4), we get

$$\left|\tfrac{1}{4}\left(p_3 - \tfrac{2}{3}p_1p_2\right) + \tfrac{1}{6}\left(p_2 - {p_1}^2\right) - \tfrac{1}{6}{p_1}^2 - \tfrac{5}{12}p_1\right| \leq \tfrac{1}{4}G(p) + \tfrac{1}{3} + \tfrac{1}{6}p^2 + \tfrac{5}{12}p \equiv w(p),$$

where $G(p)$ is given by (2.3) and $|p_1| = p$, $p \in [0, 2]$. We have

$$w(p) = \begin{cases} \tfrac{1}{36}p^3 + \tfrac{1}{18}p^2 + \tfrac{5}{12}p + \tfrac{5}{6}, & p \in [0, 3/2], \\ -\tfrac{1}{12}p^3 + \tfrac{1}{6}p^2 + \tfrac{5}{6}p + \tfrac{1}{3}, & p \in [3/2, 2]. \end{cases}$$

The function w is increasing, so $w(p) \leq w(2) = 2$. Therefore,

$$\max\{w(p) : p \in [0, 2]\} = 2. \tag{3.15}$$

Applying (3.14) and (3.15) in (3.13), we obtain

$$|a_4 - a_2a_3|^2 \leq \cos^2\beta\left(4\cos^2\beta + \tfrac{25}{9}\sin^2\beta\right) \leq \cos^2\beta\left(4\cos^2\beta + 4\sin^2\beta\right) = 4\cos^2\beta$$

Hence,

$$|a_4 - a_2a_3| \leq 2\cos\beta \leq 2$$

and we get the desired result. □

Remark. In [15], Prajapat et al. proved that $|a_2a_3 - a_4| \leq 3$ in the class $\mathcal{C}_0$. Our results $|a_2a_3 - a_4| \leq 2$ for the classes $\mathcal{C}_0$ and $\mathcal{C}(k)$ are sharp. Obtaining a sharp estimate for the class $\mathcal{C}$ is still an open problem.

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