

Coefficient estimates for a class containing quasi-convex functions

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Abstract: In the present study, we introduce the classes $\mathcal{Q}_{CV}(\mu, A, B)$ and $\mathcal{Q}_{ST}(\eta, A, B)$. Furthermore, we obtain coefficient bounds of these classes.

Key words: Analytic function, subordination, convex function, starlike function, quasi-convex function, coefficient estimates

1. Introduction and definitions

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane \mathbb{C} . Let \mathcal{A} be the class of all functions $f \in \mathcal{H}(\mathbb{D})$ which are normalized by $f(0) = f'(0) - 1 = 0$ and have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

It is well known that a function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1 = z_2$ if $f(z_1) = f(z_2)$. Let \mathcal{S} denote the subclass of \mathcal{A} which are univalent in \mathbb{D} .

Let f and F be analytic functions in the unit disk \mathbb{D} . A function f is said to be subordinate to F , written as $f \prec F$ or $f(z) \prec F(z)$, if there exists a Schwarz function $\omega : \mathbb{D} \rightarrow \mathbb{D}$ with $\omega(0) = 0$ such that $f(z) = F(\omega(z))$. In particular, if F is univalent in \mathbb{D} , we have the following equivalence:

$$f(z) \prec F(z) \iff [f(0) = F(0) \wedge f(\mathbb{D}) \subseteq F(\mathbb{D})].$$

Let \mathcal{ST} and \mathcal{CV} be the usual subclasses of functions whose members are univalent starlike and univalent convex in \mathbb{D} , respectively. We also denote the class of starlike functions of order α and the class of convex functions of order α by $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$, respectively, where $0 \leq \alpha < 1$.

Also, we note that $\mathcal{ST} := \mathcal{ST}(0)$ and $\mathcal{CV} := \mathcal{CV}(0)$.

Janowski [5] introduced the classes by $\mathcal{ST}(A, B)$ and $\mathcal{CV}(A, B)$

$$\mathcal{ST}(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\} \quad (1.2)$$

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and

$$\mathcal{CV}(A, B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \right\}, \tag{1.3}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$.

Also, we note that $\mathcal{ST}(\alpha) := \mathcal{ST}(1 - 2\alpha, -1)$, $\mathcal{ST} := \mathcal{ST}(1, -1)$ and $\mathcal{CV}(\alpha) := \mathcal{CV}(1 - 2\alpha, -1)$, $\mathcal{CV} := \mathcal{CV}(1, -1)$.

A function $f \in \mathcal{A}$ is said to be close-to-star if and only if there exists $g \in \mathcal{ST}$ such that $\Re\{f(z)/g(z)\} > 0$ for all $z \in \mathbb{D}$. Also, a function $f \in \mathcal{A}$ is said to be close-to-convex if and only if there exists $g \in \mathcal{CV}$ such that $\Re\{f'(z)/g'(z)\} > 0$ for all $z \in \mathbb{D}$. The classes of close-to-star and close-to-convex functions are denoted by \mathcal{CST} and \mathcal{CCV} , respectively. The class of close-to-star was introduced by Reade in [8] and the class of close-to-convex was introduced by Kaplan in [6]. Similarly, we denote the class of close-to-star functions of order β and close-to-convex functions of order β by $\mathcal{CST}(\beta)$ and $\mathcal{CCV}(\beta)$, respectively, where $0 \leq \beta < 1$ (see Goodman [3]). The class of close-to-convex functions of order β was introduced by Goodman in [4].

Clearly, we note that $\mathcal{CST} := \mathcal{CST}(0)$ and $\mathcal{CCV} := \mathcal{CCV}(0)$.

Definition 1.1 A function $f(z)$ in the form (1.1) is Janowski type close-to-starlike in \mathbb{D} , there is a starlike function $g(z)$ such that

$$\frac{f(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{1.4}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$. We denote it by $\mathcal{CST}(A, B)$.

Similarly, a function $f(z)$ in the form (1.1) is Janowski type close-to-convex in \mathbb{D} , there is a convex function $g(z)$ such that

$$\frac{f'(z)}{g'(z)} \prec \frac{1 + Az}{1 + Bz} \tag{1.5}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$. We denote it by $\mathcal{CCV}(A, B)$.

These classes are introduced by Reade [8] in 1955. We can easily write that

$$f \in \mathcal{CCV}(A, B) \iff zf' \in \mathcal{CST}(A, B).$$

A function $f \in \mathcal{A}$ is said to be quasi-convex if there exists a convex function $g(z)$ such that $\Re\left\{\left(zf'(z)\right)' / g'(z)\right\} > 0$ for all $z \in \mathbb{D}$. The class of quasi-convex functions is denoted by \mathcal{QCV} . The class of quasi-convex was introduced by Noor and Thomas in [7]. Similarly, we denote the class of quasi-convex functions of order γ by $\mathcal{QCV}(\gamma)$, where $0 \leq \gamma < 1$.

Definition 1.2 A function $f(z)$ in the form (1.1) is Janowski type quasi-convex in \mathbb{D} , there is a convex function $g(z)$ such that

$$\frac{\left(zf'(z)\right)'}{g'(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{1.6}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$. We denote it by $\mathcal{QCV}(A, B)$.

Definition 1.3 A function $f(z)$ in the form (1.1) is in the class $\mathcal{Q}_{\mathcal{CV}}(\mu, A, B)$ if there exists a function $g(z) \in \mathcal{CV}$ such that

$$\frac{f'(z) + \mu z f''(z)}{g'(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{1.7}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$, $0 \leq \mu \leq 1$.

Definition 1.4 A function $f(z)$ in the form (1.1) is in the class $\mathcal{Q}_{\mathcal{ST}}(\eta, A, B)$ if there exists a function $g(z) \in \mathcal{ST}$ such that

$$\frac{(1 - \eta) f(z) + \eta z f'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz} \tag{1.8}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$, $0 \leq \eta \leq 1$.

By specializing μ, η , and A, B , we obtain the following subclasses studied by earlier authors:

- (1) If $\mu = 1$, then $\mathcal{Q}_{\mathcal{CV}}(1, A, B) = \mathcal{QCV}(A, B)$ is the class of Janowski type quasi-convex functions,
- (2) If $\mu = 1$, $A = 1 - 2\gamma$, and $B = -1$, then $\mathcal{Q}_{\mathcal{CV}}(1, 1 - 2\gamma, -1) = \mathcal{QCV}(\gamma)$ is the class of quasi-convex functions of order γ ,
- (3) If $\mu = 1$, $A = 1$, and $B = -1$, then $\mathcal{Q}_{\mathcal{CV}}(1, 1, -1) = \mathcal{QCV}$ is the class of quasi-convex functions,
- (4) If $\mu = 0$, then $\mathcal{Q}_{\mathcal{CV}}(0, A, B) = \mathcal{CCV}(A, B)$ is the class of Janowski type close-to-convex functions,
- (5) If $\mu = 0$, $A = 1 - 2\beta$, and $B = -1$, then $\mathcal{Q}_{\mathcal{CV}}(0, 1 - 2\beta, -1) = \mathcal{CCV}(\beta)$ is the class of close-to-convex functions of order β ,
- (6) If $\mu = 0$, $A = 1$, and $B = -1$, then $\mathcal{Q}_{\mathcal{CV}}(0, 1, -1) = \mathcal{CCV}$ is the class of close-to-convex functions,
- (7) If $\eta = 0$, then $\mathcal{Q}_{\mathcal{ST}}(0, A, B) = \mathcal{CST}(A, B)$ is the class of Janowski type close-to-starlike functions,
- (8) If $\eta = 0$, $A = 1 - 2\beta$, and $B = -1$, then $\mathcal{Q}_{\mathcal{ST}}(0, 1 - 2\beta, -1) = \mathcal{CST}(\beta)$ is the class of close-to-starlike functions of order β ,
- (9) If $\eta = 0$, $A = 1$, and $B = -1$, then $\mathcal{Q}_{\mathcal{ST}}(0, 1, -1) = \mathcal{CST}$ is the class of close-to-starlike functions,

If we let $g(z) = f(z)$ in the Definitions 1.1, 1.2, 1.3, and 1.4, we have

$$\mathcal{CV}(A, B) \subset \mathcal{QCV}(A, B) \subset \mathcal{CCV}(A, B) \quad \text{and} \quad \mathcal{ST}(A, B) \subset \mathcal{CST}(A, B).$$

Coefficient bounds on the classes convex and starlike functions of complex order is studied in [1, 2, 9].

Lemma 1.5 [9] If the function $h(z)$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$$

is analytic in \mathbb{D} and

$$h(z) \prec \frac{1 + Az}{1 + Bz},$$

then

$$\Re h(z) > \frac{1 - A}{1 - B} \quad \text{and} \quad |h_n| \leq \frac{2(A - B)}{1 - B}, \tag{1.9}$$

where $z \in \mathbb{D}$, $-1 \leq B < A \leq 1$.

In this paper, we define $\mathcal{Q}_{CV}(\mu, A, B)$ and $\mathcal{Q}_{ST}(\eta, A, B)$ and we obtain coefficient bounds for these classes. Relevant connections of the various function classes investigated in this paper with those considered by earlier authors on the subject are also mentioned.

2. Main results and their consequences

Theorem 2.1 *If $f(z) \in \mathcal{Q}_{CV}(\mu, A, B)$, then we have*

$$|a_n| \leq \frac{1}{1 + (n - 1)\mu} \left[1 + \frac{(n - 1)(A - B)}{1 - B} \right]. \tag{2.1}$$

The extremal function $f(z)$ satisfying the inequality (2.1) is given as

$$f'(z) + \mu z f''(z) = 1 + \frac{2(A - B)}{1 - B} \left(\frac{z}{1 - z} \right) \frac{1}{(1 - z)^2}$$

that is,

$$f'(z) + \mu z f''(z) = \sum_{n=1}^{\infty} \left[n + \frac{n(n - 1)(A - B)}{1 - B} \right] z^{n-1}.$$

Proof Suppose that $f(z) \in \mathcal{Q}_{CV}(\mu, A, B)$. Then, if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{CV}$ and $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ as in Lemma 1.5 such that

$$\frac{f'(z) + \mu z f''(z)}{g'(z)} = h(z) \tag{2.2}$$

for all $z \in \mathbb{D}$. From (2.2), we obtain

$$f'(z) + \mu z f''(z) = h(z) g'(z)$$

or

$$1 + \sum_{n=2}^{\infty} n [1 + (n - 1)\mu] a_n z^{n-1} = 1 + \sum_{n=2}^{\infty} \left(\sum_{k=1}^n b_{n-k+1} h_{k-1} (n - k + 1) \right) z^{n-1}. \tag{2.3}$$

From coefficient equality of the term z^{n-1} on both sides of (2.3), we obtain

$$n [1 + (n - 1)\mu] a_n = n b_n + (n - 1) b_{n-1} h_1 + (n - 2) b_{n-2} h_2 + \dots + h_{n-1}.$$

Since $g(z)$ is convex in \mathbb{D} and $|b_n| \leq 1$ and $|h_n| \leq \frac{2(A-B)}{1-B} = \epsilon$ in Lemma 1.5, we have

$$n[1 + (n - 1)\mu] |a_n| \leq n + \epsilon \sum_{k=1}^{n-1} k$$

and this inequality is equivalent to (2.1). □

In Theorem 2.1, if we choose special values for μ, A, B , we can find the following corollaries.

Corollary 2.2 *If $f(z) \in \mathcal{QCV}(A, B)$, then we have*

$$|a_n| \leq \frac{1}{n} \left[1 + \frac{(n - 1)(A - B)}{1 - B} \right].$$

Proof We let $\mu = 1$ in Theorem 2.1. □

Corollary 2.3 *If $f(z) \in \mathcal{QCV}(\gamma)$, then we have*

$$|a_n| \leq \frac{1}{n} [n(1 - \gamma) + \gamma].$$

Proof We let $\mu = 1, A = 1 - 2\gamma, B = -1$ in Theorem 2.1. □

Corollary 2.4 *If $f(z) \in \mathcal{QCV}$, then we have*

$$|a_n| \leq 1.$$

Proof We let $\mu = 1, A = 1, B = -1$ in Theorem 2.1. □

Corollary 2.5 *If $f(z) \in \mathcal{CCV}(A, B)$, then we have*

$$|a_n| \leq \left[1 + \frac{(n - 1)(A - B)}{1 - B} \right].$$

Proof We let $\mu = 0$ in Theorem 2.1. □

Corollary 2.6 *If $f(z) \in \mathcal{CCV}(\beta)$, then we have*

$$|a_n| \leq n(1 - \beta) + \beta.$$

Proof We let $\mu = 0, A = 1 - 2\beta, B = -1$ in Theorem 2.1. □

Corollary 2.7 *If $f(z) \in \mathcal{CCV}$, then we have*

$$|a_n| \leq n.$$

Proof We let $\mu = 0, A = 1, B = -1$ in Theorem 2.1. □

Theorem 2.8 *If $f(z) \in \mathcal{Q}_{\mathcal{ST}}(\eta, A, B)$, then we have*

$$|a_n| \leq \frac{n}{1 + (n-1)\eta} \left[1 + \frac{(n-1)(A-B)}{1-B} \right]. \tag{2.4}$$

The extremal function $f(z)$ satisfying the inequality (2.4) is given as

$$(1 - \eta) f(z) + \eta z f'(z) = 1 + \frac{2(A-B)}{1-B} \left(\frac{z}{1-z} \right) \frac{z}{(1-z)^2}$$

that is,

$$(1 - \eta) f(z) + \eta z f'(z) = \sum_{n=1}^{\infty} \left[n + \frac{n(n-1)(A-B)}{1-B} \right] z^n.$$

Proof Suppose that $f(z) \in \mathcal{Q}_{\mathcal{ST}}(\eta, A, B)$. Then, if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{ST}$ and $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ as in Lemma 1.5 such that

$$\frac{(1 - \eta) f(z) + \eta z f'(z)}{g(z)} = h(z) \tag{2.5}$$

for all $z \in \mathbb{D}$. From (2.5), we obtain

$$(1 - \eta) f(z) + \eta z f'(z) = h(z) g(z)$$

or

$$z + \sum_{n=2}^{\infty} [1 + (n-1)\eta] a_n z^n = z + \sum_{n=2}^{\infty} \left(\sum_{k=1}^n b_{n-k+1} h_{k-1} \right) z^n. \tag{2.6}$$

From coefficient equality of the term z^n on both sides of (2.6), we obtain

$$[1 + (n-1)\eta] a_n = b_n + b_{n-1} h_1 + b_{n-2} h_2 + \dots + h_{n-1}.$$

Using $g(z)$ is starlike in \mathbb{D} and Lemma 1.5, we write $|b_n| \leq n$ and $|h_n| \leq \frac{2(A-B)}{1-B} = \epsilon$. Hence, we have

$$[1 + (n-1)\eta] |a_n| \leq n + \epsilon \sum_{k=1}^{n-1} k$$

that is,

$$|a_n| \leq \frac{n}{1 + (n-1)\eta} \left[1 + \frac{(n-1)(A-B)}{1-B} \right].$$

Thus, we have completed the proof of Theorem 2.8. □

Similarly, In Theorem 2.8, if we choose special values for η, A, B , we can find the following corollaries.

Corollary 2.9 *If $f(z) \in \mathcal{CST}(A, B)$, then we have*

$$|a_n| \leq n \left[1 + \frac{(n-1)(A-B)}{1-B} \right].$$

Proof We let $\eta = 0$ in Theorem 2.8. □

Corollary 2.10 *If $f(z) \in \mathcal{CST}(\beta)$, then we have*

$$|a_n| \leq n^2(1 - \beta) + n\beta.$$

Proof We let $\eta = 0$, $A = 1 - 2\beta$, $B = -1$ in Theorem 2.8. □

Corollary 2.11 *If $f(z) \in \mathcal{CST}$, then we have*

$$|a_n| \leq n^2.$$

Proof We let $\eta = 0$, $A = 1$, $B = -1$ in Theorem 2.8. □

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