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Natural mates of Frenet curves in Euclidean 3-space

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Abstract: For each Frenet curve α in the Euclidean 3-space $\mathbb{E}^3$, there exists a unique unit speed curve β tangent to the principal normal vector field of α. We simply call this curve β the *natural mate* of α. The main purpose of this paper is to prove some relationships between a Frenet curve and its natural mate. In particular, we obtain some necessary and sufficient conditions for the natural mate of a Frenet curve to be a helix, a spherical curve, or a curve of constant curvature. Several applications of our main results are also presented.

Key words: Rectifying curves, natural mate, spherical curves, conjugate mate, helix, slant helix

1. Introduction

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve in the Euclidean 3-space $\mathbb{E}^3$ with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$. Important classes of Frenet curves are helices, spherical curves, and rectifying curves. A helix is defined by the property that its tangent vector field makes a constant angle with a fixed direction. An important theorem that characterizes a helix is that the ratio of its torsion and curvature is constant (Lancret's theorem, cf. [14]). In contrast, the ratio of torsion and curvature of a rectifying curve is a nontrivial linear function of its arc-length [1]. Helices are an important class of curves since they have numerous applications ranging from physical and medical sciences to engineering, biology, and even computer designs [27]. Rectifying curves are also important in mathematics, mechanics, and medical imaging (cf. [1–7] and [13]).

Watson and Crick observed important roles of helices in nucleic acids for the first time in [26]. They observed that in a molecular model of DNA there are two side by side in opposing direction helices linked by hydrogen bonds (this phenomenon was also reported in [18, 19, 24, 25]). Helices exist quite extensively in the structure of proteins, in particular as α-helices, and there are many studies on them (see [16, 21, 22]). Helices have also been used in physics for studying different shapes of springs and helical gears as well as for elastic rods (see [8, 11]).

One interesting topic in the theory of curves in $\mathbb{E}^3$ is to find new characterizations of important curves, especially for helices, slant helices, spherical curves, and rectifying curves. In this paper, we are interested in deriving some new characterizations of these curves. In order to do so we recall from [4] that for any given unit speed Frenet curve α in $\mathbb{E}^3$ there is a unique unit speed curve β tangent to the principal normal vector field of α. We simply call this curve β the *natural mate* of α.

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In this paper we establish some relationships between a Frenet curve and its natural mate. In particular, we obtain necessary and sufficient conditions for the natural mate of a Frenet curve to be a helix, a spherical curve, or a curve of constant curvature. Several applications of our results are also presented.

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with the Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$ such that $\kappa > 0$. Then α is called a *Frenet curve* if $\tau \neq 0$. The Frenet–Serret equations for α are given by

$$T' = \kappa N, \quad N' = -\kappa T + \tau B\ , \quad B' = -\tau N. \tag{2.1}$$

A regular curve $\alpha : I \to \mathbb{E}^3$ is called a *helix* if its unit tangent vector field T makes a constant angle with a constant vector in $\mathbb{E}^3$, and α is called a *slant helix* if its principal normal field N makes a constant angle with a constant vector in $\mathbb{E}^3$ (cf. [10]).

It is known that a Frenet curve α is a helix if and only if the ratio $\tau : \kappa$ is a constant (cf. [14]). Moreover, it is also known that α is a slant helix if and only if (cf. [10])

$$\sigma = \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{(\tau^2 + \kappa^2)^{\frac{3}{2}}} = \text{constant}. \tag{2.2}$$

The *centrode* of the unit speed curve $\alpha : I \to \mathbb{E}^3$ is given by the *Darboux vector*:

$$\delta(s) = \tau T + \kappa B, \tag{2.3}$$

which is the angular velocity vector of the motion of a particle along the curve α, and it obeys the *laws of motion*:

$$T' = \delta \times T, \quad N' = \delta \times N, \quad B' = \delta \times B. \tag{2.4}$$

The direction of the Darboux vector is that of the instantaneous axis of rotation and it length, denoted traditionally in classical mechanics as

$$\omega = \sqrt{\tau^2 + \kappa^2}, \tag{2.5}$$

is called the (scalar) *angular velocity* (see, e.g., [12, page 12]).

The *co-centrode* of α in the sense of [7] is defined to be the *co-Darboux vector* δ^* as

$$\delta^*(s) = -\kappa T + \tau B. \tag{2.6}$$

Notice that the co-Darboux vector of α defined by (2.6) is exactly the derivative of the principal normal vector of the curve. Also, the co-centrode of α in the sense of [7] is different from the one defined in [3].

It is known that a Frenet curve α in $\mathbb{E}^3$ is a spherical curve if and only if

$$(p'q)' + \frac{p}{q} = 0 \tag{2.7}$$

holds, where $p = \kappa^{-1}$, $q = \tau^{-1}$. Moreover, if the Frenet curve $\alpha : I \to \mathbb{E}^3$ is a spherical curve lying on a sphere of radius a, then we have (cf. [14])

$$p^2 + (p'q)^2 = a^2. \tag{2.8}$$

A Frenet curve $\alpha : I \to \mathbb{E}^3$ is called a *rectifying curve* if $\langle \alpha(s), N(s)\rangle = 0$, $\forall s \in I$ (cf. [1, 2, 9]). The centrodes of unit speed curves were used to characterize rectifying curves in [3, Theorems 1 and 2] and in [6, Theorem 1]. Moreover, it is known that a Frenet curve is a rectifying curve if and only if the ratio $\tau : \kappa$ is a linear function in term of the arc length function (cf. [1]).

3. Natural mates of Frenet curves

First we recall the following result from [4, 23].

Theorem 1. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve with Frenet-Serret apparatus $\{\kappa, \tau, T, N, B\}$. Then there is a unit speed curve $\beta : I \to \mathbb{E}^3$ with Frenet-Serret apparatus $\{\bar\kappa, \bar\tau, \bar T, \bar N, \bar B\}$, where*

$$\bar\kappa = \omega, \;\; \bar\tau = \sigma\omega = \frac{\kappa^2}{\omega^2}\left(\frac{\tau}{\kappa}\right)', \; \bar T = N, \; \bar N = \frac{\delta^*}{\omega}, \; \bar B = \frac{\delta}{\omega}, \tag{3.2}$$

and δ, ω, δ^* are defined by (2.3)*,* (2.5)*, and* (2.6)*, respectively.*

Conversely, every unit speed curve $\beta : I \to \mathbb{E}^3$ tangent to the principal normal vector field N of α is obtained in this way.

Proof Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve. It is easy to verify that the set $\{N, \frac{\delta^*}{\omega}, \frac{\delta}{\omega}\}$ is an orthonormal frame along the curve α in $\mathbb{E}^3$. Now, using

$$\left(\frac{\kappa}{\omega}\right)' = -\tau\sigma \quad \text{and} \quad \left(\frac{\tau}{\omega}\right)' = \kappa\sigma$$

and the Frenet formulas, we get

$$N' = \delta^*, \quad \left(\frac{\delta^*}{\omega}\right)' = -\omega N + \sigma\omega\left(\frac{\delta}{\omega}\right), \quad \left(\frac{\delta}{\omega}\right)' = -\sigma\omega\left(\frac{\delta^*}{\omega}\right). \tag{3.3}$$

In view of existence theorem, the equations in (3.3) guarantee that there exists a unit speed curve $\beta : I \to \mathbb{E}^3$ with Frenet–Serret apparatus $\{\bar\kappa, \bar\tau, \bar T, \bar N, \bar B\}$ described as in the theorem.

Conversely, assume that $\beta : I \to \mathbb{E}^3$ is a unit speed curve such that β is tangent to N at each $s \in I$. Then it follows from the Frenet frame of α that the Frenet–Serret apparatus of β must be given in the way given above.

Remark 1. *For a unit speed Frenet curve α in $\mathbb{E}^3$, the curve β in Theorem 1 is given by $\beta(s) = \int N(s)ds$. We call such a curve β the natural mate of α for short. Clearly, the curve β is orthogonal to α since it is tangent to the principal normal vector field N of α. On the other hand, the authors in [4] defined a principal-direction curve of a Frenet curve α as an integral curve of the principal normal vector field of α, which is the same as a natural mate of α. However, the definition of principal-direction curves in [4] is confusing since integral curves are defined only for vector fields on a region containing a curve, not along a curve. For this reason we use the terminology "natural mate" instead to avoid this confusion.*

The following two corollaries are easy consequences of Theorem 1.

Corollary 1. [4, Theorem 4.1] *A Frenet curve $\alpha : I \to \mathbb{E}^3$ is a helix if and only if its natural mate β is a planar curve.*

Corollary 2. [4, Theorem 4.5] *A Frenet curve $\alpha : I \to \mathbb{E}^3$ is a slant helix if and only if its natural mate β is a helix.*

By applying Theorem 1, we also have the following.

Corollary 3. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ with curvature κ and torsion τ is a spherical curve (that lies on a sphere of radius a) if and only if the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of its natural mate satisfy*

$$\frac{\bar{\kappa}'}{\bar{\kappa}} = \left(\frac{\tau}{\kappa}\right)\bar{\tau} \pm \tau\sqrt{a^2\kappa^2 - 1}. \tag{3.4}$$

Proof Suppose $\alpha : I \to \mathbb{E}^3$ is a spherical curve with curvature κ and torsion τ that lies on a sphere of radius a. Then we have $p^2 + (p'q)^2 = a^2$, with $p = \kappa^{-1}$ and $q = \tau^{-1}$. That is,

$$\frac{1}{\kappa^2} + \frac{\kappa'^2}{\kappa^4\tau^2} = a^2,$$

which gives

$$\kappa' = \pm\tau\kappa\sqrt{a^2\kappa^2 - 1}. \tag{3.5}$$

Now differentiating

$$\bar{\kappa} = \kappa\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}$$

and using the expression for $\bar{\tau}$, we get condition (3.4).

Conversely, suppose the curvatures and torsion of the Frenet curve α and its mate satisfy Equation (3.4). By differentiating $\bar{\kappa} = \omega = \sqrt{\tau^2 + \kappa^2}$, we get

$$\bar{\kappa}' = \frac{\tau\tau' + \kappa\kappa'}{\omega},$$

which together with the value of $\bar{\tau}$ and Equation (3.4) gives

$$\frac{\bar{\kappa}'}{\bar{\kappa}} = \frac{\tau\tau' + \kappa\kappa'}{\omega^2} = \frac{\tau\kappa}{\omega^2}\left(\frac{\tau}{\kappa}\right)' \pm \tau\sqrt{a^2\kappa^2 - 1}.$$

The above equation simplifies to

$$\frac{\kappa'}{\kappa} = \pm\tau\sqrt{a^2\kappa^2 - 1}, \tag{3.6}$$

that is, $-\kappa p' q = \pm\sqrt{a^2\kappa^2 - 1}$, which gives

$$-p'q = \pm\frac{\sqrt{a^2\kappa^2 - 1}}{\kappa}.$$

Differentiating the above equation and using equation (3.6) gives

$$-\left(p'q\right)' = \pm\frac{\kappa'}{\kappa^2\sqrt{a^2\kappa^2 - 1}} = \frac{\tau}{\kappa},$$

and consequently, we have

$$(p'q)' + \frac{p}{q} = -\frac{\tau}{\kappa} + \frac{\tau}{\kappa} = 0,$$

which proves that α is a spherical curve.

Also, by applying Theorem 1, we have the following new characterization of rectifying curves in terms of natural mates.

Corollary 4. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ with curvature κ and torsion τ is a rectifying curve if and only if the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of its natural mate satisfy*

$$\kappa^2 = c\bar{\tau}\bar{\kappa}^2 \tag{3.7}$$

for a nonzero constant c.

Proof Suppose $\alpha : I \to \mathbb{E}^3$ is a rectifying curve with curvature κ and torsion τ. Then we have (cf. [1])

$$\alpha(s) = (s+b)T(s) + cB(s),$$

where b and c are constants and $c \neq 0$. Differentiating the above equation yields $c\tau = (s+b)\kappa$. Thus, we find

$$\left(\frac{\tau}{\kappa}\right)' = \frac{1}{c}.$$

Consequently, using expressions for $\bar{\kappa}$ and $\bar{\tau}$, we get $\kappa^2 = c\bar{\tau}\bar{\kappa}^2$ and hence condition (3.7) holds.

Conversely, suppose the curvature κ and the torsion τ of the Frenet curve α and the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of its natural mate satisfy condition (3.7), which together with the expressions for $\bar{\kappa}$ and $\bar{\tau}$ give

$$\left(\frac{\tau}{\kappa}\right)' = \frac{1}{c},$$

that is

$$\frac{\tau}{\kappa} = \frac{1}{c}s + \bar{c},$$

where $\bar{c}$ is a constant. This proves that τ/κ is a linear function of s and hence α is a rectifying curve according to [1, Theorem 2].

4. Frenet curves with spherical natural mates

In this section, we study the following.

Question. *"Which Frenet curve has a spherical natural mate?"*

This question is answered by the following theorem.

Theorem 2. *If a Frenet curve $\alpha : I \to \mathbb{E}^3$ has constant curvature $\kappa = c > 0$, then its natural mate β lies on a sphere of radius c^{-1}. The converse holds if the natural mate β has nonzero torsion, that is $\bar{\tau} \neq 0$.*

Proof Assume that the Frenet curve $\alpha : I \to \mathbb{E}^3$ has constant curvature $\kappa = c$. Then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of the natural mate $\beta : I \to \mathbb{E}^3$ are given by

$$\bar{\kappa} = \sqrt{c^2 + \tau^2}, \quad \bar{\tau} = \frac{c\tau'}{c^2 + \tau^2}.$$

Case (1): If the torsion of the curve α is a constant, i.e. $\tau = \bar{c}\,(const.)$, then we have

$$\bar{\kappa} = \sqrt{c^2 + \bar{c}^2}, \quad \bar{\tau} = 0,$$

that is, the natural mate β of α is an arc of a small circle on the sphere of radius c^{-1} at a height $\bar{c}/(c\sqrt{c^2 + \bar{c}^2}\,)$ from the center of the sphere, and then it is a spherical curve that lies on a sphere of radius c^{-1}.

Case (2): If τ is nonconstant, then by putting $\bar{p} = \bar{\kappa}^{-1}$ and $\bar{q} = \bar{\tau}^{-1}$, we have

$$\bar{p}' = -\frac{\tau\tau'}{(c^2 + \tau^2)^{\frac{3}{2}}}.$$

Consequently, we get

$$\bar{p}'\,\bar{q} = -\frac{1}{c}\frac{\tau}{\sqrt{c^2 + \tau^2}},$$

that is,

$$(\bar{p}'\,\bar{q})' = -\frac{c\tau'}{(c^2 + \tau^2)^{\frac{3}{2}}}.$$

Also, we have

$$\frac{\bar{p}}{\bar{q}} = \frac{c\tau'}{(c^2 + \tau^2)^{\frac{3}{2}}}.$$

After combining the above two equations, we conclude that

$$(\bar{p}'\,\bar{q})' + \frac{\bar{p}}{\bar{q}} = 0.$$

Thus, by Equation (2.7), we know that the natural mate of α is a spherical curve. Now, to find the radius of the sphere, we have

$$\bar{p}^2 + (\bar{p}'\,\bar{q})^2 = \frac{1}{c^2 + \tau^2} + \frac{\tau^2}{c^2\,(c^2 + \tau^2)} = \frac{1}{c^2},$$

which in view of Equation (2.8) implies that it lies on the sphere of radius c^{-1}.

Conversely, suppose that the natural mate β of the Frenet curve α is a spherical curve with nonzero torsion lying on the sphere of radius c^{-1}. Then Equation (2.8) written in terms of $\bar{\kappa}$ and $\bar{\tau}$ gives

$$\frac{1}{\bar{\kappa}^2} + \frac{\bar{\kappa}'^2}{\bar{\kappa}^4\,\bar{\tau}^2} = \frac{1}{c^2}.$$

Hence, we obtain

$$\frac{\bar{\kappa}'}{\bar{\kappa}\sqrt{\bar{\kappa}^2 - c^2}} = \pm\frac{\bar{\tau}}{c}.$$

Integrating the above equation gives

$$\sec^{-1}\left(\frac{\bar{\kappa}}{c}\right) = \pm\int \bar{\tau} ds \quad \text{or} \quad \bar{\kappa} = c\sec\left(\int \bar{\tau} ds\right). \tag{4.1}$$

In general, from Theorem 1, the torsion of the natural mate is

$$\bar{\tau} = \frac{\left(\frac{\tau}{\kappa}\right)'}{1+\left(\frac{\tau}{\kappa}\right)^2},$$

which, by the substitution $\tan(\theta + b) = \tau/\kappa$ for some constant b, gives $\bar{\tau} = \theta'$. Consequently, Equation (4.1) gives $\bar{\kappa} = c\sec(\theta + \theta_0)$, where θ_0 is a constant. Now, by choosing $b = \theta_0$, we find

$$\bar{\kappa} = c\sec(\theta + b). \tag{4.2}$$

On the other hand, since $\bar{\kappa} = \sqrt{\tau^2 + \kappa^2}$ and $\tau = \kappa\tan(\theta + b)$, we also have

$$\bar{\kappa} = \kappa\sec(\theta + b). \tag{4.3}$$

Consequently, after comparing expressions of $\bar{\kappa}$ in (4.2) and (4.3), we obtain $\kappa = c$, which completes the proof of the theorem.

Remark 2. *It follows from the proof of Theorem 2 that the natural mate β of the Frenet curve α lies in an arc of a circle if and only if it has constant torsion and constant curvature > 0.*

Remark 3. *Salkowski curves in $\mathbb{E}^3$ are curves with constant curvature but nonconstant torsion with explicit parametrization. Such curves were initially studied and constructed in 1909 by Salkowski [20]. Similar curves with constant torsion but nonconstant curvature are known as anti-Salkowski curves (see, e.g., [15]).*

5. Natural mates with constant curvature

It follows from (2.5) and $\bar{\kappa} = \omega$ in (3.2) that if there exists a function $\varphi(s)$ such that the curvature κ and the torsion τ of a Frenet curve α satisfies

$$\kappa = c\cos(\varphi(s)) \quad \text{and} \quad \tau = c\sin(\varphi(s)), \tag{5.1}$$

then the natural mate of α has constant curvature $\bar{\kappa}$.

Conversely, we have the following.

Theorem 3. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve and let $\beta : I \to \mathbb{E}^3$ be its natural mate. If the natural mate β has constant curvature $\bar{\kappa} = c > 0$, then the curvature κ and the torsion τ of α are related to the torsion $\bar{\tau}$ of the natural mate by*

$$\kappa = c\cos\left(\int \bar{\tau} ds\right) \text{ and } \tau = c\sin\left(\int \bar{\tau} ds\right). \tag{5.2}$$

Proof Suppose that the curvature of the natural mate β of α is a constant $c > 0$. Then by the expression for $\bar{\kappa}$ in (3.2), we get

$$\kappa^2 + \tau^2 = c^2, \quad \kappa\kappa' + \tau\tau' = 0. \tag{5.3}$$

Thus, the expression for $\bar{\tau}$ in (3.2) yields $\bar{\tau} = (\kappa\tau' - \tau\kappa')/c^2$, which in view of equation (5.3) gives

$$\bar{\tau} = \frac{1}{\kappa c^2}\left(\kappa^2\tau' - \kappa\tau\kappa'\right) = \frac{\tau'}{\kappa} = \frac{\tau'}{\sqrt{c^2 - \tau^2}}. \tag{5.4}$$

Hence, we obtain $\int \bar{\tau} ds = \sin^{-1}(\tau/c)$, or equivalently, $\tau = c\sin\left(\int \bar{\tau} ds\right)$. Therefore, after using $\kappa^2 + \tau^2 = c^2$, we also obtain $\kappa = c\cos\left(\int \bar{\tau} ds\right)$.

Remark 4. *Theorem 3 implies that in order for the natural mate β of a Frenet curve α to have constant curvature, the function φ in (5.1) cannot be an arbitrary function.*

Corollary 5. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve with curvature κ and torsion τ and let $\beta : I \to \mathbb{E}^3$ be its natural mate with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$. If the natural mate has constant curvature c, then we have*

$$\sin^{-1}\left(\frac{\tau}{c}\right) = \cos^{-1}\left(\frac{\kappa}{c}\right), \quad \kappa = -\int \tau\bar{\tau} ds, \quad \tau = \int \kappa\bar{\tau} ds. \tag{5.5}$$

Proof The first equality of (5.5) follows immediately from (5.2). The second and the third equalities of (5.5) follow from (5.3) and (5.4).

By applying Theorem 3, we also have the following characterization of a spherical curve whenever the natural mate has constant curvature c.

Theorem 4. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve whose natural mate β has constant curvature $c > 0$. Then α is a spherical curve if and only if there is a positive constant $a \geq c$ such that with respect to a suitable arc-length parametrization s the torsion $\bar{\tau}$ of β satisfies*

$$\bar{\tau} = \pm\frac{c^2\sqrt{a^2 - c^2}\cos(cs)}{c^2 + (a^2 - c^2)\sin^2(cs)}. \tag{5.6}$$

Proof Assume that the natural mate β of a Frenet curve α has the constant curvature $\bar{\kappa} = c > 0$ and torsion $\bar{\tau}$ given by (5.6). Then, by Theorem 3, as $\bar{\kappa} = c$, a constant, we have the following expressions for curvature and torsion of α:

$$\kappa = c\cos\left(\int \bar{\tau} ds\right) \text{ and } \tau = c\sin\left(\int \bar{\tau} ds\right), \tag{5.7}$$

which gives $p = c^{-1}\sec\left(\int \bar{\tau} ds\right)$ and $q = c^{-1}\csc\left(\int \bar{\tau} ds\right)$. Thus, by differentiating p we get

$$p' = \frac{\bar{\tau}}{c}\sec\left(\int \bar{\tau} ds\right)\tan\left(\int \bar{\tau} ds\right). \tag{5.8}$$

Since

$$\tan\left(\pm\int \frac{c^2\sqrt{a^2 - c^2}\cos(cs)}{c^2 + (a^2 - c^2)\sin^2(cs)} ds\right) = \pm\frac{\sqrt{a^2 - c^2}}{c}\sin(cs),$$

we obtain from the expression for $\bar{\tau}$ in (5.6) that

$$\tan\left(\int \bar{\tau} ds\right) = \pm\frac{\sqrt{a^2 - c^2}}{c}\sin(cs). \tag{5.9}$$

Now, by differentiating the last equation and using (5.6), we find

$$\sec^2\left(\int \bar{\tau} ds\right) = \frac{1}{c^2}\left(c^2 + (a^2 - c^2)\sin^2(cs)\right). \tag{5.10}$$

Thus, using (5.6), (5.7), (5.8), and (5.10), we find $p'q = \pm c^{-2}\sqrt{a^2 - c^2}\cos(cs)$, which gives

$$(p'q)' = \mp\frac{\sqrt{a^2 - c^2}}{c}\sin(cs). \tag{5.11}$$

Also, we find from (5.7) and (5.9) that

$$\frac{p}{q} = \tan\left(\int \bar{\tau} ds\right) = \pm\frac{\sqrt{a^2 - c^2}}{c}\sin(cs). \tag{5.12}$$

Now, combining (5.11) and (5.12) gives $(p'q)' + p/q = 0$. Hence, α is a spherical curve.

Conversely, let us assume that α is a spherical curve and that the natural mate β of α has constant curvature $\bar{\kappa} = c > 0$. Then it follows from (2.8) that $p^2 + (p'q)^2 = b^2$, where b is a positive constant. Hence, using the expressions of p and p' of (5.7) and (5.8), we have

$$\frac{1}{c^4}\sec^2\left(\int \bar{\tau} ds\right)\left\{\bar{\tau}^2\sec^2\left(\int \bar{\tau} ds\right) + c^2\right\} = b^2.$$

Thus, we have

$$\bar{\tau}^2\sec^2\left(\int \bar{\tau} ds\right) + c^2 = b^2c^4\cos^2\left(\int \bar{\tau} ds\right) = a^2\cos^2\left(\int \bar{\tau} ds\right)$$

with $a = bc^2$, that is,

$$\bar{\tau}^2\sec^2\left(\int \bar{\tau} ds\right) = a^2\cos^2\left(\int \bar{\tau} ds\right) - c^2.$$

The above expression implies $a \geq c$ and thus we have

$$\bar{\tau}\sec\left(\int \bar{\tau} ds\right) = \pm\cos\left(\int \bar{\tau} ds\right)\sqrt{a^2 - c^2\sec^2\left(\int \bar{\tau} ds\right)},$$

that is,

$$\frac{c\bar{\tau}\sec^2\left(\int \bar{\tau} ds\right)}{\sqrt{a^2 - c^2\sec^2\left(\int \bar{\tau} ds\right)}} = \pm c.$$

Now, by integrating the last equation, we obtain

$$\sin^{-1}\left(\frac{c}{\sqrt{a^2 - c^2}}\tan\left(\int \bar{\tau} ds\right)\right) = \pm cs + s_0,$$

for some constant s_0. Hence, after applying a suitable translation in s, we get

$$\tan\left(\int \bar{\tau} ds\right) = \pm\frac{\sqrt{a^2 - c^2}}{c}\sin(cs),$$

which is equivalent to

$$\int \bar{\tau} ds = \pm \tan^{-1}\left(\frac{\sqrt{a^2-c^2}}{c}\sin(cs)\right).$$

Consequently, after differentiating the last equation, we obtain (5.6).

Another result for a spherical curve whose natural mate has constant curvature is the following consequence of Corollary 3.

Corollary 6. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve with curvature κ and torsion τ. Assume that the natural mate β of α has constant curvature $c > 0$. Then α lies on a sphere of radius a if and only if either $\tau = 0$ or $\bar{\tau} = \mp\kappa\sqrt{a^2\kappa^2 - 1}$.*

Proof Corollary 3 states that α lies on a sphere of radius a if and only if (3.4) holds. When the natural mate has constant curvature $\bar{\kappa} = c$, Equation (3.4) reduces to $\bar{\tau}\tau = \mp\kappa\tau\sqrt{a^2\kappa^2 - 1}$. Hence, α lies on a sphere of radius a if and only if either $\tau = 0$ or $\bar{\tau} = \mp\kappa\sqrt{a^2\kappa^2 - 1}$.

6. Conjugate mates for Frenet curves

We provide the following easy result for later use.

Proposition 1. *Assume that $\alpha : I \to \mathbb{E}^3$ is a Frenet curve with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$ with torsion $\tau < 0$ (resp., torsion $\tau > 0$). Then there exists a Frenet curve $\bar{\alpha} : I \to \mathbb{E}^3$ with Frenet–Serret apparatus $\{-\tau, \kappa, B, N, -T\}$ (resp., $\{\tau, -\kappa, B, -N, T\}$).*

Proof If the torsion of α is negative, then by Equation (2.4) we have

$$B' = -\tau N, \quad N' = -(-\tau)B + \kappa(-T), \quad (-T)' = -\kappa N.$$

Hence, there is a unique unit speed curve $\bar{\alpha} : I \to \mathbb{E}^3$ with curvature $-\tau$, torsion κ, and Frenet frame $\{B, N, -T\}$ according to the existence theorem. Moreover, as torsion $\kappa \neq 0$, it is a Frenet curve. If the torsion τ of α is positive, then a similar argument shows that there exists a unique unit speed curve with curvature τ and torsion $-\kappa$ and with Frenet frame $\{B, -N, T\}$.

For the Frenet curve $\alpha : I \to \mathbb{E}^3$, the curve $\bar{\alpha}$ in Proposition 1 is given by $\bar{\alpha} = \int B ds$, which was called the *binormal-direction curve of the curve* α in [4]. In this paper, we call $\bar{\alpha}$ the *conjugate mate* of α for short.

Notice that the curves α and $\bar{\alpha}$ are orthogonal curves and that α is a helix if and only if its conjugate mate $\bar{\alpha}$ is a helix. Similarly, we also have the following.

Corollary 7. *A Frenet curve $\alpha : I \to \mathbb{E}^3$ with negative torsion (or with positive torsion) is a slant helix if and only if its conjugate mate $\bar{\alpha}$ is a slant helix.*

Proof Let α be a Frenet curve with curvature κ and torsion τ and let $\bar{\kappa}$ and $\bar{\tau}$ be the curvature and torsion of its conjugate mate $\bar{\alpha}$. If $\tau < 0$, then we have $\bar{\kappa} = -\tau$ and $\bar{\tau} = \kappa$. Thus, we have

$$\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)' = -\left(\frac{\kappa}{\tau}\right)' = \frac{\kappa\tau' - \tau\kappa'}{\tau^2} = \frac{\kappa^2}{\tau^2}\left(\frac{\tau}{\kappa}\right)',$$

which gives

$$\frac{\bar{\kappa}^2\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)'}{\left(\bar{\tau}^2+\bar{\kappa}^2\right)^{\frac{3}{2}}}=\frac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\left(\tau^2+\kappa^2\right)^{\frac{3}{2}}}.$$

Hence, in view of Equation (2.2), the above equation implies that α is a slant helix if and only if its conjugate mate $\bar{\alpha}$ is a slant helix.

A similar argument applies to the case $\tau > 0$.

Combining Theorem 1 and Proposition 1, we obtain immediately the following.

Corollary 8. *Let $\alpha : I \to \mathbb{E}^3$ be a Frenet curve with negative torsion (or with positive torsion). Then there exists a unique pair of a unit speed curve $\beta : I \to \mathbb{E}^3$ and a Frenet curve $\bar{\alpha} : I \to \mathbb{E}^3$, such that the curves α, β, and $\bar{\alpha}$ are mutually orthogonal.*

Finally, we prove the next result for Frenet curves with nonzero constant torsion.

Theorem 5. *If a Frenet curve α in $\mathbb{E}^3$ has constant torsion $\tau \neq 0$, then its natural mate β lies on a sphere of radius c^{-1}.*

Proof Let α be a Frenet curve with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$. Assume that the torsion of α is a negative constant, say $\tau = -c$. Then Proposition 1 implies that α has a conjugate mate $\bar{\alpha}$ with Frenet–Serret apparatus$\{-\tau, \kappa, B, N, -T\}$ so that $\bar{\alpha}$ is a Frenet curve with constant curvature $\kappa^* = c > 0$.

Now we claim that α and $\bar{\alpha}$ have the same natural mate. According to Theorem 1, the natural mate β of α has Frenet–Serret apparatus $\left\{\bar{\kappa}, \bar{\tau}, \bar{T}, \bar{N}, \bar{B}\right\}$ as in (3.2).

Suppose that $\bar{\beta}$ is the natural mate of the conjugate mate $\bar{\alpha}$. Let $\{\kappa^*, \tau^*, T^*, N^*, B^*\}$ be the Frenet–Serret apparatus of $\bar{\beta}$. Then, by Theorem 1, we have

$$\kappa^* = \sqrt{(-\tau)^2+\kappa^2} = \bar{\kappa}, \quad \tau^* = \frac{\tau^2\left(\frac{\kappa}{-\tau}\right)'}{\tau^2+\kappa^2}. \tag{6.1}$$

Clearly, we have

$$\tau^2\left(\frac{\kappa}{-\tau}\right)' = -\tau^2\left(\frac{\kappa}{\tau}\right)' = \kappa\tau' - \tau\kappa' = \kappa^2\left(\frac{\tau}{\kappa}\right)'.$$

Combining this with the second equation in (6.1) gives

$$\tau^* = \frac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\tau^2+\kappa^2} = \bar{\tau}. \tag{6.2}$$

By Proposition 1, we also have

$$N^* = \frac{1}{\sqrt{\tau^2+\kappa^2}}\left(-(-\tau)B+\kappa(-T)\right) = \bar{N} \tag{6.3}$$

and similarly

$$B^* = \frac{1}{\sqrt{\tau^2+\kappa^2}}\left(\kappa B+(-\tau)(-T)\right) = \bar{B}. \tag{6.4}$$

Therefore, we get $\beta = \bar{\beta}$ from (6.1), (6.2), (6.3), and (6.4) and the uniqueness part of the fundamental theorem of curves.

Since the torsion τ of α is a constant $-c < 0$, the conjugate mate $\bar{\alpha}$ of α has constant curvature $c > 0$. Consequently, the natural mate $\bar{\beta} = \beta$ is a spherical curve lying on the sphere of radius c^{-1} according to Theorem 2.

A similar argument applies to the case $\tau > 0$.

Example 1. Let $\alpha : (0, 2\pi) \to \mathbb{E}^3$ be the unit speed Frenet curve defined by

$$\alpha(s) = \left(\sqrt{1 - \sin s}, \sqrt{1 + \sin s}, \frac{1}{\sqrt{2}} \cos^{-1}(\sin s)\right).$$

The Frenet–Serret apparatus of α is given by

$$\left\{\frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \left(\frac{-\cos s}{2\sqrt{1 - \sin s}}, \frac{\cos s}{2\sqrt{1 + \sin s}}, \frac{-\cos s}{\sqrt{2 - 2\sin^2 s}}\right),\right.$$
$$\left.\left(-\frac{\sqrt{1 - \sin s}}{\sqrt{2}}, -\frac{\sqrt{1 + \sin s}}{\sqrt{2}}, 0\right), \left(\frac{-\cos s}{2\sqrt{1 - \sin s}}, \frac{\cos s}{2\sqrt{1 + \sin s}}, \frac{\cos s}{\sqrt{2 - 2\sin^2 s}}\right)\right\}.$$

By Proposition 1, the conjugate mate $\bar{\alpha} : (0, 2\pi) \to \mathbb{E}^3$ of α has Frenet–Serret apparatus

$$\left\{\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \left(\frac{-\cos s}{2\sqrt{1 - \sin s}}, \frac{\cos s}{2\sqrt{1 + \sin s}}, \frac{\cos s}{\sqrt{2 - 2\sin^2 s}}\right),\right.$$
$$\left.\left(-\frac{\sqrt{1 - \sin s}}{\sqrt{2}}, -\frac{\sqrt{1 + \sin s}}{\sqrt{2}}, 0\right), \left(\frac{\cos s}{2\sqrt{1 - \sin s}}, \frac{-\cos s}{2\sqrt{1 + \sin s}}, \frac{\cos s}{\sqrt{2 - 2\sin^2 s}}\right)\right\}.$$

By Theorem 1, the natural mate β of α has Frenet–Serret apparatus

$$\left\{\frac{1}{2}, 0, \left(-\frac{\sqrt{1 - \sin s}}{\sqrt{2}}, -\frac{\sqrt{1 + \sin s}}{\sqrt{2}}, 0\right), \left(\frac{\cos s}{\sqrt{2 - 2\sin s}}, \frac{-\cos s}{\sqrt{2 + 2\sin s}}, 0\right), (0, 0, 1)\right\}.$$

By integrating $N(s)$ and $B(s)$, respectively, we obtain

$$\beta(s) = \left\{-\frac{\cos(\frac{\pi}{4} - \frac{s}{2})\sqrt{2 - 2\sin s}}{\cos(\frac{\pi}{4} + \frac{s}{2})}, \frac{\cos(\frac{\pi}{4} + \frac{s}{2})\sqrt{2 + 2\sin s}}{\cos(\frac{\pi}{4} - \frac{s}{2})}, 0\right\},$$

$$\bar{\alpha}(s) = \left\{\sqrt{1 - \sin s}, \sqrt{1 + \sin s}, \frac{s \cos s}{\sqrt{1 + \cos 2s}}\right\}.$$

Notice that α and $\bar{\alpha}$ of Example 1 are arcs of helices in opposing directions as seen in the following figure and the natural mate is β, an arc of a circle of radius 2 (cf. [25]).

From Corollary 7, we know that the conjugate mate $\bar{\alpha}$ of a slant helix $\alpha(t)$ of negative torsion is a slant helix, and by Corollary 2 its natural mate β is a helix. Next, we provide an example of a slant helix α with negative torsion.

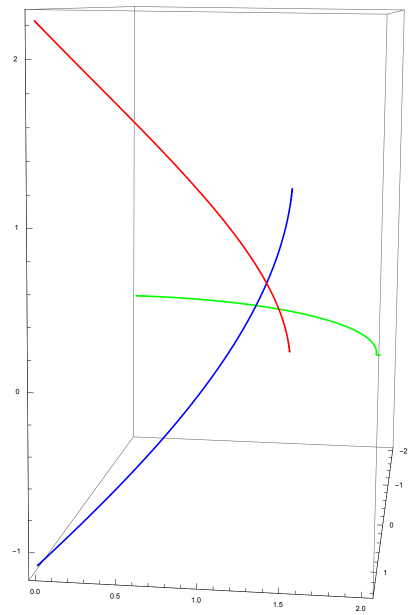

Figure 1. Graph of α (in red), $\bar{\alpha}$ (in blue), β (in black).

Example 2. Consider the unit speed curve $\alpha : (-\pi, \pi) \to \mathbb{E}^3$ given by

$$\alpha(s) = \frac{3}{4}\left(\cos s + \frac{\cos 3s}{9}, \sin s + \frac{\sin 3s}{9}, \frac{-2\cos s}{\sqrt{3}}\right).$$

The curvature and torsion of α are $\kappa = \sqrt{3}\cos s$, $\tau = -\sqrt{3}\sin s$. The Frenet frame of α is

$$T = \frac{3}{4}\left(-\sin s - \frac{\sin 3s}{3}, \cos s + \frac{\cos 3s}{3}, \frac{2\sin s}{\sqrt{3}}\right),$$

$$N = \left(\frac{-\sqrt{3}\cos 2s}{2}, \frac{-\sqrt{3}\sin 2s}{2}, \frac{1}{2}\right), \quad B = \left(\frac{3\cos s - \cos 3s}{4}, \sin^3 s, \frac{\sqrt{3}\cos s}{2}\right).$$

Since the natural mate β of α has curvature $\bar{\kappa} = \sqrt{3}$ and torsion $\bar{\tau} = -1$, the Frenet frame of β is given by

$$\bar{T} = N, \quad \bar{N} = \left(\frac{\sin 4s}{4}, \frac{-3 - \cos 4s}{4}, -\frac{\sqrt{3}\sin 2s}{2}\right),$$

$$\bar{B} = \left(\frac{3 - \cos 4s}{4}, \frac{-\cos s - \cos 3s}{2}, \frac{\sqrt{3}\cos 2s}{2}\right).$$

The conjugate mate $\bar{\alpha}$ of α has curvature $\kappa^* = \sqrt{3}\sin s$ and torsion $\tau^* = \sqrt{3}\cos s$ and the Frenet frame $\{T^*, N^*, B^*\}$ of $\bar{\alpha}$ is given by $\{B, N, -T\}$.

After integrating $N(s)$ and $B(s)$, respectively, we obtain

$$\beta(s) = \frac{1}{4}\left\{-\sqrt{3}\sin 2s, \sqrt{3}\cos^2 s, 2s\right\},$$

$$\bar{\alpha}(s) = \frac{1}{12}\left\{9\sin s - \sin 3s, -9\cos s + \cos 3s, 6\sqrt{3}\sin s\right\}.$$

The curves α, β, and $\bar{\alpha}$ of Example 2 are shown in the following computer generated graph, of which α and $\bar{\alpha}$ are slant helices moving in opposing directions and β is the helix.

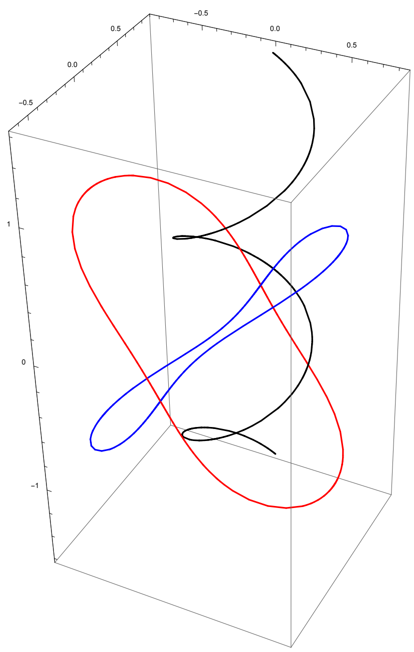

Figure 2. Graph of α (in red), $\bar{\alpha}$ (in blue), β (in black).

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