

## On dominated coloring of graphs and some Nordhaus–Gaddum-type relations

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**Abstract:** The dominated coloring of a graph  $G$  is a proper coloring of  $G$  such that each color class is dominated by at least one vertex. The minimum number of colors needed for a dominated coloring of  $G$  is called the dominated chromatic number of  $G$ , denoted by  $\chi_{dom}(G)$ . In this paper, dominated coloring of graphs is compared with (open) packing number of  $G$  and it is shown that if  $G$  is a graph of order  $n$  with  $diam(G) \geq 3$ , then  $\chi_{dom}(G) \leq n - \rho(G)$  and if  $\rho_0(G) = 2n/3$ , then  $\chi_{dom}(G) = \rho_0(G)$ , and if  $\rho(G) = n/2$ , then  $\chi_{dom}(G) = \rho(G)$ . The dominated chromatic numbers of the corona of two graphs are investigated and it is shown that if  $\mu(G)$  is the Mycielsky graph of  $G$ , then we have  $\chi_{dom}(\mu(G)) = \chi_{dom}(G) + 1$ . It is also proved that the Vizing-type conjecture holds for dominated colorings of the direct product of two graphs. Finally we obtain some Nordhaus–Gaddum-type results for the dominated chromatic number  $\chi_{dom}(G)$ .

**Key words:** Dominated coloring, dominated chromatic number, total domination number

### 1. Introduction

Let  $G = (V, E)$  be a simple, undirected, and finite graph. A set  $D \subseteq V$  is called a *dominating set* if the vertices in  $D$  dominate all the vertices in  $V \setminus D$ . The *domination number* of  $G$  is the smallest cardinality of a dominating set in  $G$ , denoted by  $\gamma(G)$ . For a graph  $G$  with no isolated vertices, a dominating set  $D$  is called a *total dominating set* provided that each vertex in  $V$  is adjacent to a vertex in  $D$ . The *total domination number*  $\gamma_t(G)$  is the minimum cardinality among all total dominating sets. For more studies refer to [7] and [8].

A *proper coloring* of  $G$  is an assignment of colors to the vertices of  $G$  such that no two adjacent vertices are assigned the same color. The *dominator coloring* of the graph  $G$  is a proper coloring such that each vertex of  $V$  dominates all the vertices of at least one color class. The *dominator chromatic number*  $\chi_d(G)$  is the minimum number of colors needed for a dominator coloring of  $G$  (see [2, 4, 9]). *Dominated coloring* of a graph is a proper coloring in which each color class is dominated by a vertex. The least number of colors needed for a dominated coloring of  $G$  is called the *dominated chromatic number* of  $G$  and is denoted by  $\chi_{dom}(G)$  [10]. We call this coloring a dom-coloring, for simplicity. In this manuscript, we study some properties of dom-coloring of graphs. First, we need to define some terminology and notations.

If a vertex  $v$  is adjacent to a vertex  $u$  in  $G$ , then we write  $v \sim u$ . An *open neighborhood* of a vertex  $v$

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is a set  $N(v) = \{u \in V : u \sim v\}$  and a *closed neighborhood* of  $v$  is  $N[v] = N(v) \cup \{v\}$ . A vertex  $u \in N(v)$  is called a *neighbor* of  $v$ .

The *diameter* of a connected graph  $G$ ,  $diam(G)$ , is the maximum distance between any two vertices in  $G$ . The  $k$ -th power of  $G$ ,  $G^k$ , is a graph whose vertex set is that of  $G$  and two vertices in it are adjacent if their distance in  $G$  is at most  $k$ . The *complement* of  $G$ ,  $G^c$ , is a graph with the same vertex set of  $G$ , and two vertices are adjacent if they are not adjacent in  $G$ .

A *matching* in a graph  $G$  is a set of edges in  $G$  that have no common vertices and the number of edges in a maximum matching is called the *matching number*, denoted by  $\alpha'(G)$ . An *independent set*  $S$  in  $G$  is a set of vertices such that the induced subgraph by  $S$ ,  $G[S]$ , has no edge. The *independence number*  $\alpha(G)$  is the size of the maximum independent set in  $G$ . A subset  $S \subseteq V(G)$  is called a *packing* (an *open packing*) in  $G$  if for distinct vertices  $u, v \in S$ ,  $N(u) \cap N(v) = \emptyset$  ( $N[u] \cap N[v] = \emptyset$ ). The *packing number*  $\rho(G)$  (*open packing number*  $\rho_0(G)$ ) is the maximum cardinality of a packing (an open packing) in  $G$ .

Let  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  be two arbitrary graphs. The *direct product* of  $G$  and  $H$ , denoted by  $G \times H$ , is a graph whose vertex set is the Cartesian product  $V(G) \times V(H)$  and whose edge set is  $E(G \times H) = \{(x_1, y_1)(x_2, y_2) : x_1x_2 \in E(G), y_1y_2 \in E(H)\}$ . The *corona* of  $G$  and  $H$ , denoted by  $G \circ H$ , is a graph made by a copy of  $G$  (which has  $n$  vertices) and  $n$  copies of  $H$  and joining the  $i$ -th vertex of  $G$  to every vertex in the  $i$ -th copy of  $H$ . We call  $G \circ K_1$  the *corona* of  $G$ . The *joint* of  $G$  and  $H$ , denoted by  $G + H$ , is a graph with vertex set  $V(H) \cup V(G)$  and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}.$$

We offer some necessary lemmas and observations. Assume that  $G = (V, E)$  is a simple graph with  $n$  vertices. It is well known  $\chi_{dom}(G) \geq \chi(G)$ . In the following we state, when they are equal?

**Observation 1.1** For any graph  $G$ ,  $\chi_{dom}(G) \geq \chi(G)$  and  $\chi_{dom}(G) = \chi(G)$  if  $diam(G) \leq 2$ .

**Proof** Let  $diam(G) \leq 2$ . Since two vertices have at most distance 2 from each other, the vertices of any color class have a common neighbor if the class has more than 1 vertex, and so every color class is dominated by a vertex. Therefore  $\chi_{dom}(G) \leq \chi(G)$  and  $\chi_{dom}(G) = \chi(G)$ . □

The converse of Observation 1.2 does not necessarily hold. For example consider for Path  $P_4$ ,  $\chi_{dom}(G) = \chi(G) = 2$  but  $diam(G) \geq 3$ .

Let  $G$  be a disconnected graph with  $k$  components  $G_i$  ( $1 \leq i \leq k$ ). It is clear that  $\chi_{dom}(G) = \sum_{i=1}^k \chi_{dom}(G_i)$ . Also it is easy to see that, for complete graph  $K_n$ ,  $\chi_{dom}(K_n) = n = \chi_{dom}(K_n^c)$ . Moreover, if  $G$  is not a complete graph and its complement, then there are two vertices in  $G$  with distance 2 and they are assigned the same color. Therefore we have:

**Observation 1.2** Let  $G$  be a graph. Then  $1 \leq \chi_{dom}(G) \leq n$ . Furthermore,  $\chi_{dom}(G) = n$  if and only if each component of  $G$  is a complete graph.

**Observation 1.3** [10] If  $G$  is a triangle-free graph, then  $\chi_{dom}(G) \leq 2\gamma(G)$ .

**Proposition 1.4** [10] *Let  $G$  be a graph with order at least 2. Then  $\chi_{dom}(G) \geq \gamma_t(G)$ . Moreover, if  $G$  is a triangle-free graph, then  $\chi_{dom}(G) = \gamma_t(G)$ .*

**Proposition 1.5** [10] *Let  $G$  be a graph without isolated vertices. Then  $\chi_{dom}(G) \leq \chi(G) \cdot \gamma(G)$ .*

## 2. Dominated colorings and dominator colorings in graphs

In this section we study the relationship between dominated coloring and some parameters of graphs.

Although it seems that there is no meaningful relation between the two types of colorings: dominated coloring and dominator coloring, one can find a relation between the dominated chromatic number and dominator chromatic number in some special cases.

**Theorem 2.1** *If for a graph  $G$ ,  $\gamma(G) = 1$ , then  $\chi_d(G) = \chi_{dom}(G) = \chi(G)$ .*

**Proof** Since  $\gamma(G) = 1$ ,  $G$  consists of one connected component. If  $G = K_1$ , the theorem is obvious. Otherwise, according to Proposition 1.5 we have  $\chi(G) \leq \chi_{dom}(G) \leq \chi(G) \cdot \gamma(G)$  and since  $\gamma(G) = 1$ , so  $\chi(G) = \chi_{dom}(G)$ . On the other hand,  $G$  has a vertex that is adjacent to all other vertices of the graph. If we color  $G$  with  $\chi$  colors (properly), this vertex will take a unique color. Therefore, this coloring is also a dominator coloring. In other words,  $\chi_d(G) \leq \chi(G)$  and so  $\chi_d(G) = \chi(G)$  and hence the proof is complete.  $\square$

**Theorem 2.2** [9] *Every tree  $T$  of order at least three admits a dominator coloring with  $\chi_d(T)$  colors such that all leaves of  $T$  have the same color.*

**Theorem 2.3** [4] *Let  $T$  be a nontrivial tree of order  $n$  and  $l$  leaves. Then  $\gamma_t \geq \frac{n+2-l}{2}$ .*

**Theorem 2.4** [5] *Let  $G$  be a connected graph of order  $n \geq 2$  such that either (a)  $G$  is  $C_4$ -free, or (b)  $G$  is claw-free and not a  $C_4$ . Then  $\chi_d(G) \geq \gamma(G) + 1$ .*

**Theorem 2.5** *For every tree  $T$  with at least two vertices,  $\frac{\chi_d(T)+1}{2} \leq \chi_{dom}(T) \leq 2\chi_d(T) - 2$ .*

**Proof** If tree  $T$  has two vertices, the result is obvious. Let  $T$  have  $n \geq 3$  vertices and  $l$  leaves. According to Theorem 2.2,  $n - l + 1 \geq \chi_d(T)$  since assigning the same colors to the leaves and a unique color to other vertices of the tree we will have a dominator coloring with  $n - l + 1$  colors. Moreover, using Proposition 1.4 and Theorem 2.3, we have  $\chi_{dom}(T) = \gamma_t(T) = \frac{n-l+2}{2} \geq \frac{\chi_d(T)+1}{2}$ . On the other hand, from Observation 1.3 and Theorem 2.4 for tree  $T$ , one can verify that  $\chi_{dom}(T) \leq 2\gamma(T) \leq 2(\chi_d(T) - 1)$  and thus the proof is complete.  $\square$

**Lemma 2.6** *For any graph  $G$ ,  $\chi_{dom}(G) \geq \chi(G \cup (G^2)^c)$ .*

**Proof** Assume that  $G$  is dom-colored with  $\chi_{dom}$  colors. Since every two vertices with a distance at least 3 from each other cannot be in the same color class, by connecting the vertices that have different colors we obtain a graph that includes  $G \cup (G^2)^c$  as its subgraph and that is properly colored. Thus  $\chi_{dom}(G) \geq \chi(G \cup (G^2)^c)$ .  $\square$

**Theorem 2.7** *For every graph  $G$ , if girth of  $G$  is at least 7, then  $\chi_{dom}(G) = \chi(G \cup (G^2)^c)$ .*

**Proof** According to Lemma 2.6, it is sufficient to prove that  $\chi_{dom}(G) \leq \chi(G \cup (G^2)^c)$ . In every proper coloring of  $G \cup (G^2)^c$ , if there exist some vertices in a same class, then they will be at distance 2 from each other. Therefore, they are at distance 2 from each other in  $G$  too. If in  $G$  there exist some vertices that are mutually at distance two from each other, then they will have a common neighbor. Since the girth of  $G$  is at least 7, each cycle in  $G$  has a length at least 7. Thus any  $\chi$ -coloring of  $(G \cup (G^2)^c)$  gives us a  $\chi_{dom}$  of  $G$ . Therefore  $\chi_{dom}(G) \leq \chi(G \cup (G^2)^c)$  and the result holds (see Figure 1).  $\square$

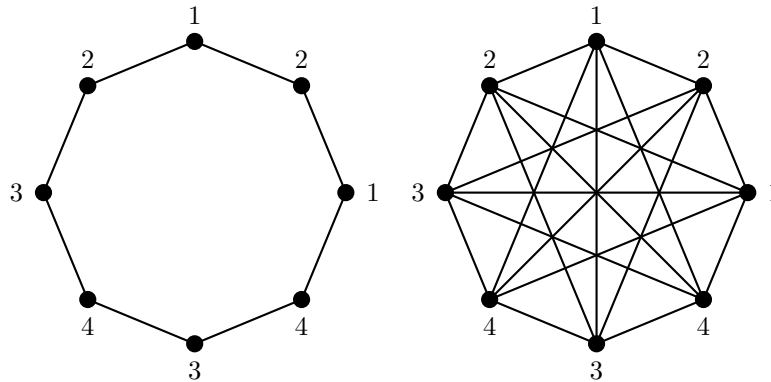


Figure 1. The dom-coloring of  $C_8$  and proper coloring of  $C_8 \cup (C_8^2)^c$ .

### 3. Dominated colorings and (open) packing number

Packing number is a parameter in graph that has been studied in the related parameters like domination theory. However, we think that the relation between packing number and dominated coloring has not been studied yet. In this section we wish to find this relation.

**Theorem 3.1** In any graph  $G$ ,  $\chi_{dom}(G) \geq \rho_0(G) \geq \rho(G)$ .

**Proof** According to Lemma 2.6,  $\omega(G \cup (G^2)^c) \leq \chi(G \cup (G^2)^c) \leq \chi_{dom}(G)$ . On the other hand, since in any open packing set in  $G$  no two vertices have a common open neighbor, in the graph  $G \cup (G^2)^c$  any two vertices of this set are adjacent and therefore  $\rho_0(G) \leq \omega(G \cup (G^2)^c)$  and this implies that  $\chi_{dom}(G) \geq \rho_0(G)$ . Moreover, by the definition, it is obvious that  $\rho_0(G) \geq \rho(G)$  and the result then holds.  $\square$

**Theorem 3.2** If  $diam(G) \geq 3$ , then  $\chi_{dom}(G) \leq n - \rho(G)$ .

**Proof** If  $diam(G) \geq 3$ , then  $\rho \geq 2$ . Consider a maximum packing set  $S$ . Since the distance between the vertices in  $S$  is more than or equal to 3, we assign to any vertex of  $V \setminus S$  a color and then we assign to the vertices in  $S$  the color of the vertex that is at distance 2 from any vertex of  $S$ . Hence, we have a dom-coloring and in fact  $\chi_{dom}(G) \leq n - \rho(G)$ .  $\square$

**Corollary 3.3** Let  $G$  be a graph of order  $n$  with  $diam(G) \geq 3$ . Then

- 1)  $\rho(G) \leq \rho_0(G) \leq \chi_{dom}(G) \leq n - \rho(G) \leq n - \rho_0(G)/2$ .
- 2) If  $\rho_0(G) = 2n/3$ , then  $\chi_{dom}(G) = \rho_0(G)$  and if  $\rho(G) = n/2$ , then  $\chi_{dom}(G) = \rho(G)$ .
- 3) In any corona of the graph  $G$  with  $n$  vertices,  $\chi_{dom}(G \circ K_1) = n$ .

**Proof**

- 1) In [6] it is proved that  $\rho_0 \leq 2\rho$ . Now using Theorem 3.2 the result holds.
- 2) According to part 1, it is obtained.
- 3) In any corona of the graph  $G$ , since a pendant vertex is added to any vertex of  $G$  and the pendant vertices in the corona of the graph have no common neighbors with each other, then  $\rho(G \circ K_1) = n$  and in fact  $\rho(G \circ K_1) = |V(G \circ K_1)|/2$ . Therefore, according to part 2,  $\chi_{dom}(G \circ K_1) = n$ .

□

**4. Dominated coloring of graphs obtained from two graphs**

Dominated coloring has not been obtained for many graphs. In this section we try to obtain a graph with combination of some graphs and then study its dominated coloring in terms of dominated colorings of subgraphs. We start with some results from [3, 6].

$$\text{For } n \geq 3, \gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise} \end{cases}.$$

Using Proposition 1.4 and the above results, the following theorem holds:

**Theorem 4.1** For  $n \geq 3$ ,  $\chi_{dom}(P_n) = \chi_{dom}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise} \end{cases}.$

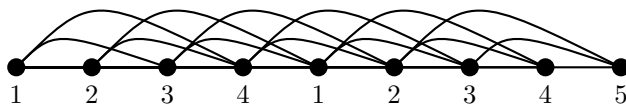
The next theorem is the generalization of Theorem 4.1.

**Theorem 4.2** Let  $i$  be a positive integer and  $\lfloor \frac{n}{2i} \rfloor = r$  and  $n \equiv l \pmod{2i}$ . Then

$$\chi_{dom}(P_n^{i-1}) = \chi_{dom}(C_n^{i-1}) = \begin{cases} ri + l & \text{if } 0 \leq l \leq i \\ i(r + 1) & \text{if } i < l < 2i \end{cases}.$$

**Proof** According to the dominated coloring of  $P_n^{i-1}$ , it is obvious that all  $2i$  adjacent vertices receive  $i$  colors. Hence,  $ri$  colors are assigned to  $2ri$  vertices and  $l$  colors are assigned to  $l$  other remaining vertices, provided that  $0 \leq l \leq i$  and  $i$  colors if  $i < l < 2i$  (see Figure 2).

With a similar argument, we obtain  $\chi_{dom}(C_n^{i-1})$ . □



**Figure 2.** Dom-coloring of  $P_9^3$ .

The Mycielsky graph  $G$ , denoted by  $\mu(G)$  is a simple graph whose vertex set is  $V(\mu(G)) = V \cup U \cup \{w\}$  in which  $V = V(G) = \{v_1, v_2, \dots, v_n\}$  and  $U = \{u_1, u_2, \dots, u_n\}$ , which is disjoint from the set  $V$  and its edge set is

$$E(\mu(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}, i = 1, 2, \dots, n\},$$

where the vertices  $v_i$  and  $u_i$  are called the *twins* of each other, and  $w$  is said to be the *root* of  $\mu(G)$ .

**Theorem 4.3** For every graph  $G$ ,  $\chi_{dom}(\mu(G)) = \chi_{dom}(G) + 1$ .

**Proof** Let  $\chi_{dom}(G) = k$ . If  $c$  is a  $k$ -coloring of  $G$ , then assigning an arbitrary color  $i$ ,  $1 \leq i \leq k$  to  $w$  and color  $k + 1$  to the vertex set  $U$  and considering the coloring  $c$  for the vertices of  $V(G)$ , we will obtain a  $k + 1$ -dom coloring for  $\mu(G)$ . Hence,  $\chi_{dom}(\mu(G)) \leq k + 1$ . Now we should prove that  $\chi_{dom}(\mu(G)) \geq k + 1$ . Assume that the graph  $\mu(G)$  is dom-colored. We consider two cases.

(1) The color class of  $w$  has only one vertex.

In this case, restricting the color function of  $\mu(G)$  to the vertices of  $V(G)$ , we obtain a  $\chi_{dom}(\mu(G)) - 1$ -dom-coloring for the graph  $G$ . Hence,  $\chi_{dom}(\mu(G)) \geq k + 1$ .

(2) The color class of  $w$  has more than one vertex. In this case,  $w$  is placed in a color class with at least one vertex of  $V(G)$  and consequently a vertex of  $U$  dominates this class. By assigning the twin color of each vertex of  $V(G)$ , which is located in the same class as  $w$ , to these vertices and also assigning the color of any other vertex of  $V(G)$  to its twin, we obtain a dom-coloring of  $\mu(G)$ . This coloring is a proper coloring since the twin of any vertex is adjacent to the neighbors of that vertex and therefore has a different color. On the other hand, it is also a dom-coloring. Restricting this coloring to the vertices of  $V(G)$ , we get a dom-coloring for  $G$  that is less than or equal to  $\chi_{dom}(\mu(G)) - 1$ . Thus  $\chi_{dom}(\mu(G)) - 1 \geq k$  and the result holds.

□

**Theorem 4.4** Given two graphs  $G$  and  $H$ ,

$$\chi_{dom}(G + H) = \chi(G) + \chi(H).$$

**Proof** It is obvious that  $\chi_{dom}(G + H) \geq \chi(G + H) = \chi(G) + \chi(H)$ . On the other hand, assigning the colors  $1 \leq k \leq \chi(G)$  to the vertices of the graph  $G$  and assigning the colors  $\chi(G) + 1 \leq k \leq \chi(G) + \chi(H)$  to the vertices of the graph  $H$ , a dom-coloring of the graph  $G + H$  will be obtained. Hence,  $\chi_{dom}(G + H) \leq \chi(G) + \chi(H)$ . □

The next theorem states the dominated chromatic number of an extended case of part 3 in Corollary 3.3.

**Theorem 4.5** Let  $G$  be a connected graph of order  $n$ . If  $G \circ H$  is the corona of two graphs  $G$  and  $H$ , then  $\chi_{dom}(G \circ H) = n\chi_H$ .

**Proof** We color  $G \circ H$  as follows. We properly color every copy of  $H$  with  $\chi_H$  distinct colors and we assign to every vertex  $a$  of  $G$  an arbitrary color of the assigned colors to the copy of  $H$ , which is connected to one of the adjacent vertices to  $a$ . Clearly, the obtained coloring is a dominated coloring. Therefore  $\chi_{dom}(G \circ H) \leq n\chi_H$ . On the other hand, since the distance between any two distinct vertices that are in two distinct copies of  $H$  is at least three, these two vertices do not receive the same color and so  $\chi_{dom}(G \circ H) \geq n\chi_H$  and the statement is proved. □

Here we show that the Vizing-type conjecture holds for dominated colorings of the direct product of two graphs.

In the next result,  $a \sim b$  means that  $a$  and  $b$  are adjacent.

**Theorem 4.6** *If  $G \times H$  is the direct product of two graphs  $G$  and  $H$ , then  $\chi_{dom}(G \times H) \leq \chi_{dom}(G)\chi_{dom}(H)$ .*

**Proof** Let  $G$  and  $H$  be dominated colored and two vertices  $b$  and  $c$  be in a same color class of  $G$  and  $b'$  and  $c'$  in a same color class of  $H$ . In this case, there is a vertex  $a$  in  $G$  and a vertex  $a'$  in  $H$  such that:

$$b \sim a, b' \sim a' \Rightarrow (b, b') \sim (a, a').$$

$$c \sim a, c' \sim a' \Rightarrow (c, c') \sim (a, a').$$

$$b \approx c, b' \approx c' \Rightarrow (b, b') \approx (c, c').$$

Then two vertices  $(b, b')$  and  $(c, c')$  of  $G \times H$  can be in a same color class of  $G \times H$  and therefore  $G \times H$  is dominated colored and then  $\chi_{dom}(G \times H) \leq \chi_{dom}(G)\chi_{dom}(H)$ .  $\square$

**Theorem 4.7** [11] *If  $T$  is any tree of order at least two and  $H$  is any graph without isolated vertices, then  $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$ .*

A classical result says that the direct product of two graphs  $G$  and  $H$  has a triangle if both  $G$  and  $H$  have triangles. Let  $(a, a'), (b, b')$ , and  $(c, c')$  be three vertices of a triangle in  $G \times H$ . In this case,  $(b, b') \sim (a, a'), (c, c') \sim (a, a')$  and  $(c, c') \sim (b, b')$ . By definition,  $a \sim b, a \sim c$  and  $b \sim c$  in  $G$  and  $a' \sim b', a' \sim c'$ , and  $b' \sim c'$  in  $H$ . Then  $a, b, c$  are three vertices of a triangle in  $G$  and  $a', b', c'$  are three vertices of a triangle in  $H$ . Therefore we have the following:

**Theorem 4.8** *If  $T$  is a tree of order at least two and  $H$  is a graph without isolated vertices and triangles, then  $\chi_{dom}(T \times H) = \chi_{dom}(T)\chi_{dom}(H)$ .*

**Proof** According to the assumptions of the theorem and the above explanation  $T \times H$  is triangle-free. By using Proposition 1.4 and Theorem 4.7, we have:  $\chi_{dom}(T \times H) = \gamma_t(T \times H) = \gamma_t(T)\gamma_t(H) = \chi_{dom}(T)\chi_{dom}(H)$ .  $\square$

### 5. Nordhaus–Gaddum-type relations for dominated chromatic number

The Nordhaus–Gaddum-type relations are upper and lower bounds for a parameter on the sum (or product) of a graph and its complement. For a comprehensive survey of Nordhaus–Gaddum-type relations, see [1], [12].

In this section we study Nordhaus–Gaddum-type relations for dominated chromatic number. First we state some results that have straightforward proof.

**Observation 5.1** *In every graph  $G$ , each edge and a nonadjacent vertex to it induce two vertices with a common neighbor in  $G^c$  and then it makes a dom-color class of order (at least) 2 in  $G^c$ .*

**Observation 5.2** *With a similar argument to Observation 5.1, if  $G$  is a graph with clique number  $\omega$  and has a vertex that is nonadjacent to this clique, then this clique makes a dom-color class of order  $\omega$  in  $G^c$  and then  $\chi_{dom}(G^c) \leq n - \omega$ .*

**Lemma 5.3** *If  $G$  is a graph with diameter  $d$ , then*

$$\chi_{dom}(G) \leq n - \lfloor (d - 2)/2 \rfloor.$$

**Proof** Let  $P$  be a diametral path with end vertices  $v_0$  and  $v_d$ . Then for this path, any dom-coloring of  $G$  has at most two vertices in a class of size 1 and other color classes of size 2. Hence, we have at least  $\lfloor (d-2)/2 \rfloor$  color class of size 2. If we assign different colors to the other vertices of the graph, then we will have a dom-coloring with  $n - \lfloor (d-2)/2 \rfloor$  colors. Thus,

$$\chi_{dom}(G) \leq n - \lfloor (d-2)/2 \rfloor.$$

□

**Lemma 5.4** *If  $G$  is a graph with diameter  $d$ , then*

$$\lceil d/2 \rceil \leq \chi_{dom}(G^c).$$

*Furthermore, if  $d \geq 5$ , then*

$$\chi_{dom}(G^c) \leq n - \lfloor d/2 \rfloor.$$

**Proof** If  $d \leq 4$ , then it is obvious. Hence, let  $d \geq 5$  and assume that  $G^c$  is dom-colored. Let  $P$  be a diametral path with end vertices  $u = v_0$  and  $v = v_d$ , that is  $P : u = u_0 - u_1 - \dots - u_d = v$ . According to Observation 5.1 both of the vertices  $u_i$  and  $u_{i+1}$ ,  $0 \leq i \leq d-1$ , in  $G^c$  can be located in a same color class but since none of the nonadjacent vertices of  $P$  are adjacent in  $G$  and so are adjacent in  $G^c$ , then no three of these vertices are in a same color class. Hence the statement is true. □

Now we state the Nordhaus–Gaddum-type for dominated chromatic number.

**Theorem 5.5** *If  $G$  is a graph with diameter  $d$ , then for the sum case*

$$\chi_{dom}(G) + \chi_{dom}(G^c) \geq \lceil d/2 \rceil + 2$$

*and if  $d \geq 5$ , then*

$$\chi_{dom}(G) + \chi_{dom}(G^c) \leq 2n - 2\lfloor d/2 \rfloor + 1.$$

*For the product case we have*

$$\chi_{dom}(G) \cdot \chi_{dom}(G^c) \geq 2\lceil d/2 \rceil$$

*and if  $d \geq 5$ , then*

$$\chi_{dom}(G) \cdot \chi_{dom}(G^c) \leq (n - \lfloor (d-2)/2 \rfloor)(n - \lfloor d/2 \rfloor).$$

**Proof** According to Lemma 5.3 and Lemma 5.4, the proof is obvious. □

**Theorem 5.6** *Let  $G$  be a triangle-free and  $C_4$ -free graph with  $n$  vertices. Then*

1)  $\chi_{dom}(G^c) \leq n - \alpha'(G).$

2)  $\chi_{dom}(G) + \chi_{dom}(G^c) \leq n + \alpha'(G).$

*Let  $G$  be a triangle-free graph. Then*

3)  $\chi_{dom}(G) \leq 2\alpha'(G).$



**Proof**

- 1) Let  $M$  be a maximum matching of the graph  $G$ . If  $|M| = \alpha'(G) \leq 1$ , the claim is obvious. If  $\alpha'(G) \geq 2$ , then each edge of  $M$  is nonadjacent to at least one vertex from other edges of  $M$ . Consequently, according to Observation 5.1 there exists a color class of order of at least 2 for every edge of  $M$ . Hence,  $\chi_{dom}(G^c) \leq n - \alpha'(G)$ .
- 2) Let  $M$  be a maximum matching of the graph  $G$ . Then, if we assign one color to each of the vertices of the edges of  $M$  then, with considering these colors, we can dom-color the other vertices of  $G$  and therefore  $\chi_{dom}(G) \leq 2\alpha'(G)$ .
- 3) According to parts 1 and 3, the proof is obvious. □

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