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Quasinilpotents in rings and their applications

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Abstract: An element a of an associative ring R is said to be quasinilpotent if $1-ax$ is invertible for every $x \in R$ with $xa = ax$. Nilpotents and elements in the Jacobson radical of a ring are well-known examples of quasinilpotents. In this paper, properties and examples of quasinilpotents in a ring are provided, and the set of quasinilpotents is applied to characterize rings with some certain properties.

Key words: Quasinilpotent, nilpotent, idempotent, local ring, Boolean ring

1. Introduction

Rings are associative with identity. Let R be a ring. The symbols $U(R)$, $Id(R)$, and R^{nil} stand for the sets of all units, all idempotents, and all nilpotents of R, respectively. The commutant of $a \in R$ is defined by $\mathrm{comm}_R(a) = \{x \in R \mid ax = xa\}$ (if there is no ambiguity, we simply use $\mathrm{comm}(a)$ for short). For an integer $n \geq 1$, we write $M_n(R)$ for the $n \times n$ matrix ring over R whose identity element we write as I_n or I.

The intersection of all maximal left (right) ideals of R is said to be the *Jacobson radical* of R, which is denoted by $J(R)$. As is well known, $J(R) = \{a \in R \mid 1 - ax \in U(R) \text{ for all } x \in R\}$. Due to Harte [10], an element $a \in R$ is called *quasinilpotent* if $1 - ax \in U(R)$ for every $x \in \mathrm{comm}(a)$; the set of all quasinilpotents of R is denoted by R^{qnil}. It is clear that both R^{nil} and $J(R)$ are contained in R^{qnil}. It is worth noting that quasinilpotents play an important role in a Banach algebra $\mathcal{A}$. According to [9], $\mathcal{A}^{\mathrm{qnil}} = \{a \in \mathcal{A} \mid \lim_{n\to\infty} \|a^n\|^{1/n} = 0\} = \{a \in \mathcal{A} \mid x - a \in U(\mathcal{A}) \text{ for all nonzero complex } x\}$. By means of quasinilpotents, some interesting concepts are introduced, such as strongly J-clean rings [3], nil clean rings [7], generalized Drazin inverses [11], quasipolar rings [16], etc. However, there were few results concerning properties of quasinilpotents in a ring. Recall that a ring R is *local* [13] if $R = U(R) \cup J(R)$, and it was shown in [4] that R is a division ring or a Boolean ring if and only if $R = U(R) \cup Id(R)$. A natural question is: What can be said about a ring R for which $R = U(R) \cup R^{\mathrm{qnil}}$ (resp., $R = U(R) \cup R^{\mathrm{nil}}$; $R = U(R) \cup R^{\mathrm{qnil}} \cup Id(R)$)?

Motivated by the above, we study properties and structures of quasinilpotents in a ring and provide several illustrative examples. Jacobson's lemma for quasinilpotents is also considered. Furthermore, the sets $Id(R)$, $U(R)$, and R^{qnil} are used to characterize rings. We prove that a ring R is local if and only if $R = U(R) \cup R^{\mathrm{qnil}}$, a ring R is local with $J(R)$ nil if and only if $R = U(R) \cup R^{\mathrm{nil}}$, and if $R = U(R) \cup R^{\mathrm{qnil}} \cup Id(R)$, then R is a local ring or a Boolean ring or a nonabelian directly finite ring with $\mathrm{char}\, R = 2$.

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2. Quasinilpotents in rings

We begin with the following examples, which will reveal that quasinilpotents in a ring R are very different from elements in $J(R)$ and nilpotents of R.

Example 2.1 (1) *Let* $R = M_2(\mathbb{Z}_{(2)})$ *where* $\mathbb{Z}_{(2)} = \{\frac{b}{a} \mid a, b \in \mathbb{Z},\ 2 \nmid a\}$. *Take* $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in R$. *Then* $A^2 \in J(R)$, *and so* $A \in R^{\mathrm{qnil}}$ (*as for any* $X \in \mathrm{comm}(A)$, $(I_2 - AX)(I_2 + AX) = I_2 - A^2X^2 \in U(R)$ *implies* $I_2 - AX \in U(R)$). *Clearly,* A *is neither nilpotent nor in* $J(R)$.

(2) *Define an operator* A *on the Banach space* l^1 *by the infinite matrix* $\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 0 & \cdots \\ 0 & 0 & 1/3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$. *In view of [15, Example 4.2],* A *is quasinilpotent in the Banach algebra* $L(l^1)$ *of all bounded linear operators on* l^1. *However,* A *is not nilpotent and* $A \notin J(L(l^1))$ *as* $J(L(l^1)) = 0$.

Example 2.2 *Let* R *be a division ring and* $S = M_n(R)$. *Then* $S^{\mathrm{qnil}} = S^{\mathrm{nil}}$.

Proof Let $A \in S$. Note that S can be viewed as the endomorphism ring of an n-dimensional vector space over R. Then S is a simple Artinian ring, and so the chain $AS \supseteq A^2S \supseteq \cdots$ must terminate. In view of [2, Lemma 1], there exist an integer $k \geq 1$ and $X \in S$ such that $A^k = A^{k+1}X$ and $AX = XA$. Assume that $A \in S^{\mathrm{qnil}}$. Then $I_n - AX \in U(S)$. From $A^k(I_n - AX) = A^k - A^{k+1}X = 0$, we have $A^k = 0$, so $S^{\mathrm{qnil}} \subseteq S^{\mathrm{nil}}$, and therefore $S^{\mathrm{qnil}} = S^{\mathrm{nil}}$. □

Lemma 2.3 *Let* $f : R \to S$ *be an isomorphism of rings. Then* $a \in R^{\mathrm{qnil}}$ *if and only if* $f(a) \in S^{\mathrm{qnil}}$. *In particular, if* $a \in R^{\mathrm{qnil}}$, *then* $u^{-1}au \in R^{\mathrm{qnil}}$ *for any* $u \in U(R)$.

Proof It suffices to show that if $a \in R^{\mathrm{qnil}}$ then $f(a) \in S^{\mathrm{qnil}}$. Let $s \in \mathrm{comm}_S(f(a))$. Then there exists $b \in R$ such that $s = f(b)$, so we have $f(ab) = f(a)s = sf(a) = f(ba)$. Since f is an isomorphism, $ab = ba$. It follows that $1 - ab \in U(R)$ as $a \in R^{\mathrm{qnil}}$. Thus, $1 - f(a)s = f(1 - ab) \in U(S)$, from which $f(a) \in S^{\mathrm{qnil}}$. □

The polynomial ring over a ring R in the indeterminate t is denoted by $R[t]$. For a monic polynomial $f(t) = t^n - a_{n-1}t^{n-1} - \cdots - a_1t - a_0 \in R[t]$, the $n \times n$ matrix $C_f = \left(\begin{smallmatrix} 0 & a_0 \\ I & \alpha \end{smallmatrix}\right)$ is called the *companion matrix* of $f(t)$, where $\alpha = (a_1, a_2, \ldots, a_{n-1})^T$. A matrix $C \in M_n(R)$ is called a companion matrix if $C = C_f$ for a monic polynomial $f(t) \in R[t]$ of degree n. The following result is due to Diesl and Dorsey.

Lemma 2.4 *Let* R *be a commutative ring and* C *be a companion matrix of a monic polynomial of degree* n. *Then* $\mathrm{comm}_{M_n(R)}(C) = \{h(C) \mid \text{for every } h(t) \in R[t]\}$.

Proof It is enough to prove that $\mathrm{comm}_{M_n(R)}(C) \subseteq \{h(C) \mid \text{for every } h(t) \in R[t]\}$. Let $C = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$.
Write $e_0 = (1, 0, \ldots, 0)^T$, $e_1 = (0, 1, \ldots, 0)^T, \ldots, e_{n-1} = (0, 0, \ldots, 1)^T$. Then $Ce_i = e_{i+1}$ for every $0 \leq i \leq n-2$ and $Ce_{n-1} = a_0e_0 + a_1e_1 + \cdots + a_{n-1}e_{n-1}$.

Suppose that $X \in M_n(R)$ and $CX = XC$. Set $X = (X_0, X_1, \ldots, X_{n-1})$ with column vectors X_i. Then, for every $0 \le i \le n-2$, we have

$$CX_i = C(Xe_i) = X(Ce_i) = Xe_{i+1} = X_{i+1}$$

and

$$\begin{aligned} CX_{n-1} &= C(Xe_{n-1}) = XCe_{n-1} \\ &= X(a_0e_0 + a_1e_1 + \cdots + a_{n-1}e_{n-1}) \\ &= a_0Xe_0 + a_1Xe_1 + \cdots + a_{n-1}Xe_{n-1} \\ &= a_0X_0 + a_1X_1 + \cdots + a_{n-1}X_{n-1}. \end{aligned}$$

Write $X_0 = (b_0, b_1, \ldots, b_{n-1})^T$. Then all X_i (and thus X) can be constructed by X_0. We claim that $X = b_0I + b_1C + \cdots + b_{n-1}C^{n-1}$. Indeed, we only need to verify that it agrees on $e_0, e_1, \ldots, e_{n-1}$. Note that

$$(b_0I + b_1C + \cdots + b_{n-1}C^{n-1})e_0 = b_0e_0 + b_1e_1 + \cdots + b_{n-1}e_{n-1} = X_0 = Xe_0,$$

and similarly, for each $1 \le i \le n-1$,

$$\begin{aligned} Xe_i &= X(C^ie_0) = C^i(Xe_0) \\ &= C^i(b_0I + b_1C + \cdots + b_{n-1}C^{n-1})e_0 \\ &= (b_0I + b_1C + \cdots + b_{n-1}C^{n-1})(C^ie_0) \\ &= (b_0I + b_1C + \cdots + b_{n-1}C^{n-1})e_i. \end{aligned}$$

This completes the proof. □

For a ring R, let $\sqrt{J(R)} = \{x \in R \mid x^n \in J(R) \text{ for some integer n} \ge 1\}$. One may easily check that $\sqrt{J(R)} \subseteq R^{\text{qnil}}$. It is shown in [15] that $S^{\text{qnil}} = \sqrt{J(S)}$ if S is a 2×2 matrix ring over a commutative ring. We have the following result.

Theorem 2.5 *If R is a commutative local ring and $S = M_n(R)$, then $S^{\text{qnil}} = \sqrt{J(S)}$.*

Proof It suffices to prove that $S^{\text{qnil}} \subseteq \sqrt{J(S)}$. Let $A \in S^{\text{qnil}}$. Then for any polynomial $f(t) \in R[t]$, $I - Af(A)$ is a unit of $M_n(R)$. Thus, $\bar{I} + \bar{A}\overline{f(A)}$ is invertible in $M_n(R/J(R)) \cong S/J(S)$. Note that $R/J(R)$ is a field. Thus, $\bar{A}$ is similar to its rational canonical form $C := \begin{pmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_l} \end{pmatrix}$ where C_{f_i} is the companion matrix over $R/J(R)$. Since $\bar{I} - \bar{A}\overline{f(A)}$ is invertible, it follows that $\bar{I} - C_{f_i}\overline{f(C_{f_i})}$ is invertible for $i = 1, 2, \ldots, l$. As $\overline{f(t)} \in \bar{R}[t]$ is arbitrary, by Lemma 2.4 all C_{f_i} are quasinilpotent. In view of Example 2.2, C_{f_i} is nilpotent where $1 \le i \le l$. By Lemma 2.3, one has $(\bar{A})^k = 0 \in S/J(S)$ for some integer k, which implies $A^k \in J(S)$. Therefore, $S^{\text{qnil}} \subseteq \sqrt{J(S)}$, as desired. □

For a ring R, the center of R is denoted by $C(R)$.

Proposition 2.6 *Let R be a ring. Then $R^{\rm qnil} = J(R)$ if one of the following holds:*
(1) $R^{\rm qnil} \subseteq C(R)$.
(2) $R^{\rm qnil}$ *is an one-sided ideal of* R.

Proof (1) Let $a \in R^{\rm qnil} \subseteq C(R)$. Then for any $x \in R$, $ax = xa$, so we have $1 - ax \in U(R)$, which implies $a \in J(R)$.

(2) Assume that $R^{\rm qnil}$ is a right ideal of R. Given any $a \in R^{\rm qnil}$, then $ax \in R^{\rm qnil}$ for any $x \in R$. Thus, $1 - ax \in U(R)$, and hence $a \in J(R)$. □

Clearly, $R^{\rm qnil}$ coincides with $J(R)$ if R is a commutative ring.

Proposition 2.7 *Let R be a ring and $a \in R$, $c \in C(R)$.*
(1) *If $a^n \in R^{\rm qnil}$ for some integer $n \geq 1$, then $a \in R^{\rm qnil}$.*
(2) *If $a \in R^{\rm qnil}$, then $ac \in R^{\rm qnil}$. The converse holds if $c \in U(R)$.*

Proof (1) Take $x \in \mathrm{comm}(a)$. Then we have $x^n a^n = a^n x^n$. Write $b = 1 + ax + (ax)^2 + \cdots + (ax)^{n-1}$. It follows that $b(1 - ax) = (1 - ax)b = 1 - (ax)^n = 1 - a^n x^n \in U(R)$ since $a^n \in R^{\rm qnil}$, so $1 - ax \in U(R)$. This proves $a \in R^{\rm qnil}$.

(2) Let $x \in R$ with $(ac)x = x(ac)$. As $c \in C(R)$, $a(cx) = (cx)a$, so $a \in R^{\rm qnil}$ implies that $1 - acx \in U(R)$, which yields $ac \in R^{\rm qnil}$. Conversely, assume that $c \in U(R)$ and $y \in \mathrm{comm}(a)$. Then $ac(yc^{-1}) = (yc^{-1})ac$. Since $ac \in R^{\rm qnil}$, $1 - ay = 1 - ac(yc^{-1}) \in U(R)$, which implies that $a \in R^{\rm qnil}$. □

Lemma 2.8 *Let $q \in R^{\rm qnil}$ and $e^2 = e \in R$. If $eq = qe$, then $eq \in R^{\rm qnil} \cap eRe = (eRe)^{\rm qnil}$.*

Proof In view of [16, Lemma 3.5], $R^{\rm qnil} \cap eRe = (eRe)^{\rm qnil}$. We only need to show that $eq \in R^{\rm qnil}$. Let $t \in R$ with $t(eq) = (eq)t$. As $eq = qe$, we have $q(qet^2e) = (qeqe)t^2e = qet^2qe = (qet^2e)q$. Since $q \in R^{\rm qnil}$, it follows that $(1 - teq)(1 + teq) = 1 - (teq)(teq) = 1 - (qet^2e)q \in U(R)$. Thus, $1 - teq \in U(R)$, and hence $eq \in R^{\rm qnil}$. □

The condition "$eq = qe$" in Lemma 2.8 is not superfluous. Let $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ be in $M_2(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ is the ring of integers $\mathbb{Z}$ modulo 2. Then $E^2 = E$ and Q is nilpotent, but $EQ = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = (EQ)^2$ is not quasinilpotent in $M_2(\mathbb{Z}_2)$.

Proposition 2.9 *Let $e^2 = e \in R$, and $a \in R$ with $ae = ea$. The following are equivalent:*
(1) *ae is quasinilpotent in R.*
(2) *For any $y \in \mathrm{comm}_R(ae)$, $Re \subseteq R(1 - ay)$ and $l(1 - ya) \subseteq l(e)$.*
(3) *For any $y \in \mathrm{comm}_R(ae)$, $Re \subseteq (1 - ya)R$ and $r(1 - ay) \subseteq r(e)$.*

Proof By Lemma 2.8, the proof can be shown in a similar manner as [6, Corollary 3.2]. □

Let $\mathcal{A}$ be a Banach algebra. It is well known that for any $a \in \mathcal{A}^{\rm qnil}$ and $b \in \mathcal{A}$, if $ab = ba$ then $ab \in \mathcal{A}^{\rm qnil}$, and in addition, $a + b \in \mathcal{A}^{\rm qnil}$ if $b \in \mathcal{A}^{\rm qnil}$ (see also [10]). However, it is still unknown whether the above results hold for a ring. For a ring R, let $Q(R) = \{q \in R \mid 1 + q \in U(R)\}$.

Proposition 2.10 *Let R be a ring with $Q(R) = R^{\rm qnil}$ or $U(R) = 1 + R^{\rm qnil}$.*

(1) *If* $a \in R^{\text{qnil}}$ *and* $b \in \text{comm}(a)$, *then* $ab \in R^{\text{qnil}}$.

(2) *If* $a, b \in R^{\text{qnil}}$ *and* $ab = ba$, *then* $a + b \in R^{\text{qnil}}$.

Proof We first show the following claim.

Claim: $Q(R) = R^{\text{qnil}}$ if and only if $U(R) = 1 + R^{\text{qnil}}$.

Proof of the Claim. Assume that $U(R) = 1 + R^{\text{qnil}}$. Clearly, $Q(R) \supseteq R^{\text{qnil}}$. Take $q \in Q(R)$. Then $1 + q \in U(R) = 1 + R^{\text{qnil}}$. Therefore, $q \in R^{\text{qnil}}$, and thus $Q(R) \subseteq R^{\text{qnil}}$. Conversely, suppose that $Q(R) = R^{\text{qnil}}$. To show that $U(R) = 1 + R^{\text{qnil}}$, it suffices to prove that $U(R) \subseteq 1 + R^{\text{qnil}}$. Let $u \in U(R)$. Since $1 + (u - 1) = u \in U(R)$, we have $u - 1 \in Q(R) = R^{\text{qnil}}$, which implies that $u \in 1 + R^{\text{qnil}}$, as desired.

(1) Since $a \in R^{\text{qnil}}$ and $ab = ba$, one has $1 + ab \in U(R)$. Thus, $ab \in Q(R) \subseteq R^{\text{qnil}}$.

(2) As $a, b \in R^{\text{qnil}}$ and $ab = ba$, $1 + a \in U(R)$ and $(1 + a)^{-1} \in \text{comm}(b)$. It follows that $1 + a + b = (1 + a)[1 + (1 + a)^{-1}b] \in U(R)$, so $a + b \in Q(R) \subseteq R^{\text{qnil}}$. □

For a ring R, Jacobson's lemma states that for any $a, b \in R$, if $1 - ab \in U(R)$ then $1 - ba \in U(R)$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$. We have the following result.

Theorem 2.11 *Let* $a, b \in R$. *If* $1 - ab \in R^{\text{qnil}}$, *then* $1 - ba \in R^{\text{qnil}}$ *if and only if* $a, b \in U(R)$.

Proof Assume that $a, b \in U(R)$. Let $x \in \text{comm}(1 - ba)$. Then multiplying $(1 - ba)x = x(1 - ba)$ by a on the left and by b on the right yields $(1 - ab)axb = axb(1 - ab)$. Thus, $(ab)axb = axb(ab)$. It follows that $[(ab)^{-1}axb](1 - ab) = (1 - ab)[(ab)^{-1}axb]$. Since $1 - ab \in R^{\text{qnil}}$, $1 - [(ab)^{-1}axb](1 - ab) = 1 - (ab)^{-1}ax(1 - ba)b = 1 - b^{-1}x(1 - ba)b \in U(R)$. By Jacobson's lemma, $1 - x(1 - ba)bb^{-1} = 1 - x(1 - ba) \in U(R)$. This proves $1 - ba \in R^{\text{qnil}}$.

Conversely, $1 - ab \in R^{\text{qnil}}$ implies $ab \in -1 + R^{\text{qnil}} \subseteq U(R)$. Similarly, we can get $ba \in U(R)$ from the assumption $1 - ba \in R^{\text{qnil}}$. Thus, $a, b \in U(R)$. □

Recall that a ring R is *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$ (equivalently, $aR = R$ implies $Ra = R$). We have the following result immediately.

Corollary 2.12 *Let* R *be a ring. Then for any* $a, b \in R$, $1 - ab \in R^{\text{qnil}}$ *implies* $1 - ba \in R^{\text{qnil}}$ *if and only if* R *is a directly finite ring.*

Corollary 2.13 *Let* $a, b \in R$. *If* $1 - ab \in R^{\text{nil}}$, *then* $1 - ba \in R^{\text{nil}}$ *if and only if* $a, b \in U(R)$.

Proof One direction follows from Theorem 2.11. Now suppose that $a, b \in U(R)$ and $(1 - ab)^k = 0$ for some integer k. Then

$$(ab)^{-1} = [1 - (1 - ab)]^{-1} = 1 + (1 - ab) + (1 - ab)^2 + \cdots + (1 - ab)^{k-1}.$$

Thus, we have

$$\begin{aligned}(1-ba)^{k+1} &= \sum_{i=0}^{k+1} C_{k+1}^{i}(-1)^{i}(ba)^{i} \\ &= 1 - C_{k+1}^{1}(ba) + C_{k+1}^{2}(ba)^{2} + \cdots + (-1)^{i}C_{k+1}^{i}(ba)^{i} + \cdots + (-1)^{k+1}(ba)^{k+1} \\ &= 1 - b[C_{k+1}^{1} - C_{k+1}^{2}(ba) + \cdots + (-1)^{i-1}C_{k+1}^{i}(ba)^{i-1} + \cdots + (-1)^{k}(ba)^{k}]a \\ &= 1 - b[1 + (1-ab) + (1-ab)^{2} + \cdots + (1-ab)^{k-1}]a \\ &= 1 - b(ab)^{-1}a = 1 - bb^{-1}a^{-1}a = 0.\end{aligned}$$

Thus, $1-ba \in R^{\text{nil}}$, and the proof is completed. □

Corollary 2.14 *Let $a, b \in R$. If $1-ab \in J(R)$, then $1-ba \in J(R)$ if and only if $a, b \in U(R)$.*

Proof By Theorem 2.11, it suffices to show that if $a, b \in U(R)$ then $1-ba \in J(R)$. For any $x \in R$, $1-(ba)^{-1}x(1-ba)ba = 1-(ba)^{-1}xb(1-ab)a \in U(R)$ since $1-ab \in J(R)$. By Jacobson's lemma, $1-x(1-ba)ba(ba)^{-1} = 1-x(1-ba) \in U(R)$, so $1-ba \in J(R)$. □

Cline proved in 1965 [5] that if ab is Drazin invertible then so is ba. Many authors generalized the above result to elements of rings with some kind of property. For example, similar results hold for strongly clean elements [8], strongly nil clean elements [12], etc.

Lemma 2.15 *[14, Lemma 2.2] Let $a, b \in R$. If ab is quasinilpotent in R, then so is ba.*

For positive integers m, n, let $R^{m\times n}$ be the set of all $m \times n$ matrices over the ring R.

Proposition 2.16 *Let $A \in R^{m\times n}$ and $B \in R^{n\times m}$. Then AB is quasinilpotent in $M_m(R)$ if and only if BA is quasinilpotent in $M_n(R)$.*

Proof If $m = n$, the result follows by Lemma 2.15. Assume that $m > n$. Let $A_1 = (A, O)$, $B_1 = \left(\begin{smallmatrix} B \\ O \end{smallmatrix}\right) \in M_m(R)$ where O is a matrix with all entries zeros. Clearly, $A_1B_1 = AB$ and $B_1A_1 = \left(\begin{smallmatrix} BA & O \\ O & O \end{smallmatrix}\right) \in M_m(R)$. Since $AB \in (M_m(R))^{\text{qnil}}$, Lemma 2.15 implies that $\left(\begin{smallmatrix} BA & O \\ O & O \end{smallmatrix}\right) \in (M_m(R))^{\text{qnil}}$. Clearly, $\left(\begin{smallmatrix} BA & O \\ O & O \end{smallmatrix}\right)$ is also quasinilpotent in $\left(\begin{smallmatrix} M_n(R) & O \\ O & O \end{smallmatrix}\right)$. We note that, as a subring of $M_m(R)$, $\left(\begin{smallmatrix} M_n(R) & O \\ O & O \end{smallmatrix}\right)$ is isomorphic to $M_n(R)$. By Lemma 2.3, $BA \in (M_n(R))^{\text{qnil}}$. If $m < n$, the result can be proved by a similar manner as above. □

3. Applications

This section focuses on the study of rings with some certain properties by means of $U(R), Id(R)$, and R^{qnil}. We first give the following lemma, which will be used freely.

Lemma 3.1 *Let R be a ring. Then $R^{\text{qnil}} \cap U(R) = \emptyset$ and $R^{\text{qnil}} \cap Id(R) = 0$.*

Theorem 3.2 *Let R be a ring. The following are equivalent:*

(1) *R is a local ring.*

(2) *For every $a \in R$, a is invertible or a is quasinilpotent.*

(3) *$R = U(R) \cup R^{\text{qnil}}$.*

(4) *$R = U(R) \cup J(R)$.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are clear.

(3) $\Rightarrow$ (4). For any $a \in R^{\text{qnil}}$, we first show that $ax \notin U(R)$ for all $x \in R$. Suppose that there exists $y \in R$ such that $ay \in U(R)$. Then $(ay)z = a(yz) = 1$ for some $z \in U(R)$. Clearly, $((yz)a)^2 = (yz)a$. As $R = U(R) \cup R^{\text{qnil}}$, we have $(yz)a = 0$ or $(yz)a = 1$. If $(yz)a = 0$ then $yz = (yz)a(yz) = 0$, which contradicts $a(yz) = 1$. Therefore, $(yz)a = 1$, and so $a \in U(R)$, which is impossible by Lemma 3.1. Thus, $ax \in R^{\text{qnil}}$ for all $x \in R$. It follows that $1 - ax \in U(R)$. Therefore, $a \in J(R)$. □

Corollary 3.3 *Let R be a ring. Then $R = U(R) \cup R^{\text{nil}}$ if and only if R is a local ring and $J(R)$ is nil.*

Proof Suppose that $R = U(R) \cup R^{\text{nil}}$. Since $R^{\text{nil}} \subseteq R^{\text{qnil}}$, we have $R = (U(R) \cup R^{\text{nil}}) \subseteq (U(R) \cup R^{\text{qnil}}) \subseteq R$. Thus, $R = U(R) \cup R^{\text{qnil}}$ and $R^{\text{qnil}} = R^{\text{nil}}$. By Theorem 3.2, R is a local ring, and hence $R^{\text{qnil}} = J(R)$. It follows that $J(R) = R^{\text{nil}}$. This proves that $J(R)$ is nil.

Conversely, assume that R is a local ring and $J(R)$ is nil. In view of Theorem 3.2, $R = U(R) \cup R^{\text{qnil}}$ and $R^{\text{qnil}} = J(R)$. As $J(R)$ is nil, we have $R^{\text{qnil}} = J(R) \subseteq R^{\text{nil}}$. Thus, $R^{\text{qnil}} = R^{\text{nil}}$ and $R = U(R) \cup R^{\text{nil}}$. □

Recall that a ring R is called *reduced* if it has no nonzero nilpotents, and R is said to be *abelian* if all idempotents of R are central. It is well known that reduced rings are abelian, and abelian rings are directly finite.

In [1, Theorem 14], the authors proved that a commutative ring $R = U(R) \cup Id(R)$ if and only if R is a field or a Boolean ring. Chen and Cui [4] extended the above result to the case of noncommutative rings. We give a simpler proof here.

Proposition 3.4 *Let R be a ring. Then $R = U(R) \cup Id(R)$ if and only if R is a division ring or a Boolean ring.*

Proof One direction is obvious. Suppose that $R = U(R) \cup Id(R)$. If $2 \in U(R)$, then for any $e \in Id(R)\backslash\{1\}$, we have $-e \in Id(R)$, so $(-e)^2 = e^2 = e = -e$, and then $2e = 0$. Thus, $e = 0$, and this proves that R is a division ring with $Id(R) = \{0, 1\}$. If $2 \in Id(R)$, then $2 = 0$ as $2 = 2^2$. Note that R is reduced and thus abelian. We may choose $e \in Id(R)\backslash\{0, 1\}$. For any $u \in U(R)$, then either $(u + e)^2 = u + e$ or $(u + e)v = 1$ for some $v \in U(R)$. As $2 = 0$, $(u + e)^2 = u + e$ yields $u = 1$. If $(u + e)v = 1$ then $uv = 1 - ev$. Since $e \notin U(R)$, we have $(ev)^2 = ev$, which implies $uv = 1 - ev \in U(R) \cap Id(R) = \{1\}$, so $e = 0$, a contradiction. Thus, the only unit of R is 1, and so R is a Boolean ring. □

For a ring R, we say that $Id(R)$ (resp., $U(R)$; R^{qnil}) is trivial whenever $Id(R) = \{0, 1\}$ (resp., $U(R) = \{1\}$; $R^{\text{qnil}} = 0$).

Theorem 3.5 *Let R be a ring. If $R = U(R) \cup R^{\text{qnil}} \cup Id(R)$, then exactly one of the following holds:*
(1) *R is a local ring.*
(2) *R is a Boolean ring.*
(3) *R is a nonabelian directly finite ring and $\text{char}R = 2$.*

Proof The proof is divided into the following cases.

Case 1. If R^{qnil} is trivial, then $R = U(R) \cup Id(R)$. By Proposition 3.4, R is local as a division ring or R is a Boolean ring.

Case 2. Assume that $R^{\rm qnil} \neq 0$. Then choose $q \in R^{\rm qnil}\backslash\{0\}$. Thus, $U(R)$ is nontrivial as $1+q \in U(R)\backslash\{1\}$.

Subcase 1. If $Id(R)$ is trivial, then by Theorem 3.2, $R = U(R) \cup R^{\rm qnil}$ is a local ring.

Subcase 2. Suppose that $U(R)$, $R^{\rm qnil}$, and $Id(R)$ are all nontrivial. Set fixed elements $e \in Id(R)\backslash\{0,1\}$ and $u = 1+q \in U(R)\backslash\{1\}$. We conclude that $2 \notin U(R)$. Otherwise, $2 \in U(R)$. Clearly, $2e \notin U(R)$. If $2e \in Id(R)$, then $(2e)^2 = 2e$ implies $e = 0$, which contradicts the assumption $e \in Id(R)\backslash\{0,1\}$. Thus, $2e \in R^{\rm qnil}$. Notice that $2 \in C(R)$. By Proposition 2.7(2), $e \in R^{\rm qnil}$, from which $e = 0$, and this causes the same contradiction as above. Hence, $2 \in Id(R)$ or $2 \in R^{\rm qnil}$.

If $2^2 = 2$ then $2 = 0$. If $2 \in R^{\rm qnil}$, then $3 \in U(R)$. Clearly, $3e \notin U(R)$ as $e \neq 1$. If $3e \in R^{\rm qnil}$, then by Proposition 2.7(2), $e \in R^{\rm qnil} \cap Id(R) = 0$. Thus, $3e \in Id(R)$, and $(3e)^2 = 3e$ yields $6e = 0$. One thus gets $2e = 0$ as $3 \in U(R)$. Now replacing e by $1-e$, a similar argument will reveal that $2(1-e) = 0$. Therefore, $2 = 2e + 2(1-e) = 0$. Thus, $\operatorname{char} R = 2$. We now show that R is nonabelian. Assuming the contrary, then $eu = ue$. Clearly, $ue \notin U(R)$ and $ue \notin R^{\rm qnil}$ (since $1 - u^{-1}(ue) = 1-(ue)u^{-1}$ is not a unit), so $(ue)^2 = ue \in Id(R)$, which gives $ue = e$. Similarly, we can obtain $u(1-e) = 1-e$. Combining $ue = e$ with $u(1-e) = 1-e$, we have $u = 1$, a contradiction. Thus, R is a nonabelian ring and $\operatorname{char} R = 2$.

We finish the proof by showing that R is directly finite. Let $a, b \in R$ with $ab = 1$. Suppose that $a \in R^{\rm qnil}$. Then $b \in R^{\rm qnil}$ (indeed, $ab = 1$ implies $b \notin U(R)$, and if $b \in Id(R)$ then $1-b = ab(1-b) = 0$, so $b = 1$ and $a = 1 \in U(R)$). Since $\operatorname{char} R = 2$, it follows that $a + b = (1+a)(1+b) := v \in U(R)$. Multiplying the equation $a+b = v$ by b on the left, we have $b(a+b) = bv$. Clearly, $bv \notin U(R)$. If $bv \in Id(R)$, then $0 = (bv)^2 - bv = (bvb - b)v = b^3v$, so $b^3 = 0$, but this contradicts $1 = ab = a^3b^3$. Thus, $bv \in R^{\rm qnil}$. In view of Lemma 2.15, we have $vb \in R^{\rm qnil}$. However, $vb = (a+b)b = 1 + b^2 \in U(R)$ as $b \in R^{\rm qnil}$. Therefore, $a \notin R^{\rm qnil}$. If $a \in U(R)$, then we are done. If $a \in Id(R)$, then $1 - a = (1-a)(ab) = (a-a^2)b = 0$, so $a = 1$ and $ba = 1$. Hence, R is directly finite. □

There are plenty of rings that satisfy $R = U(R) \cup R^{\rm qnil} \cup Id(R)$. For instance, let $R = T_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring over $\mathbb{Z}_2$. Then $U(R) = \{I_2, \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\}$, $Id(R) = \{O, I_2, \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)\}$, and $R^{\rm qnil} = \{O, \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\}$. Clearly, $R = U(R) \cup R^{\rm qnil} \cup Id(R)$. One may check that $R = U(R) \cup R^{\rm qnil} \cup Id(R)$ does also hold if $R = M_2(\mathbb{Z}_2)$.

Remark 3.6 *There exists a nonabelian directly finite ring R with $\operatorname{char} R = 2$ but $R \neq U(R) \cup R^{\rm qnil} \cup Id(R)$. Let $R = M_2(\mathbb{Z}_2[[x]])$ where $\mathbb{Z}_2[[x]]$ is the power series ring over $\mathbb{Z}_2$. Clearly, R is a nonabelian ring and $\operatorname{char} R = 2$. As $\mathbb{Z}_2[[x]]$ is commutative, R is directly finite. Take $A = \left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right) \in R$. Then $A \notin U(R)$ and $A \notin Id(R)$. Notice that $I_2 - A \notin U(R)$, so $A \notin U(R) \cup R^{\rm qnil} \cup Id(R)$.*

Note that local rings and Boolean rings are abelian (and thus directly finite). We thus have the following result immediately.

Corollary 3.7 *If $R = U(R) \cup R^{\rm qnil} \cup Id(R)$, then R is a directly finite ring.*

Corollary 3.8 *Let R be a commutative ring. The following are equivalent*:
(1) $R = U(R) \cup J(R) \cup Id(R)$.

(2) *R is a local ring or a Boolean ring.*

(3) $R = U(R) \cup J(R)$ *or* $R = U(R) \cup Id(R)$.

Proof (1) $\Rightarrow$ (2). Since R is a commutative ring, $R^{\text{qnil}} = J(R)$. Note that R is abelian. The result follows from Theorem 3.5.

(2) $\Rightarrow$ (3) follows from Theorem 3.2 and Proposition 3.4.

(3) $\Rightarrow$ (1). If $R = U(R) \cup J(R)$, take $Id(R) = \{0, 1\}$ and then $R = U(R) \cup J(R) \cup Id(R)$. If $R = U(R) \cup Id(R)$, then the result follows by taking $J(R) = 0$. □

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