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Compactness and Duality on Poletsky–Stessin Hardy Spaces of Complex Ellipsoids

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Abstract: In the first part of this study, we characterize the compact subspaces of $H_u^p(\mathbb{B}^{\mathbf{p}})$ and their relation to the vanishing Carleson measures. In the second part, we discuss the dual complement of the complex ellipsoid and give a duality result for $H_u^p(\mathbb{B}^{\mathbf{p}})$ spaces in the sense of Grothendieck–Köthe–da Silva.

Key words: Duality, dual complement, Poletsky–Stessin Hardy space, compactness, complex ellipsoid

1. Introduction

In their seminal work [4], Poletsky and Stessin showed that it is possible to generalize the whole idea of Hardy and Bergman spaces in the general context of hyperconvex domains in higher dimensions. After this leading work, in [5, 6] we concentrated on these generalized spaces in various domains but especially complex ellipsoids and in the present work of continuation, we consider the compactness and Grothendieck–Köthe–da Silva duality properties of these spaces. Firstly, we try to identify the characteristics of compact subspaces of these generalized Hardy spaces, and then relate these properties with vanishing Carleson measures and compact linear operators. Secondly, we describe the dual complements of complex ellipsoids and give a duality result analogous to [2] in the sense of Grothendieck–Köthe–da Silva.

The organization of the present paper is as follows: In Section 2, we recall the Poletsky–Stessin Hardy spaces, $H_u^p(\mathbb{B}^{\mathbf{p}})$, on the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$ and we introduce the Cauchy–Fantappie integral associated with the Monge–Ampère measure μ_u, together with an integral representation for $H_u^p(\mathbb{B}^{\mathbf{p}})$. The main results of the present study are given in the following sections: In Section 3, compact subspaces of $H_u^p(\mathbb{B}^{\mathbf{p}})$ are considered and through their characterization we can see the relation between the vanishing Carleson measures and compact operators on $H_u^p(\mathbb{B}^{\mathbf{p}})$. Finally, in Section 4, we first give a brief introduction about Grothendieck-Köthe-da Silva duality for the spaces of holomorphic functions defined in a convex domain, and then using a general characterization of dual complements of Reinhardt domains, [2] we give the dual complement of some special type of complex ellipsoids. Finally, we prove a duality result for Poletsky–Stessin Hardy space of complex ellipsoid.

2. Preliminaries

In this section, we give the preliminary definitions and some important results that we use throughout the present study, where we focus on Poletsky–Stessin Hardy spaces on the complex ellipsoids in $\mathbb{C}^n$, which are the

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basic examples of domains of finite type. Due to the Levi flat points at the boundary of these domains, they are pseudoconvex but not strictly pseudoconvex, which is actually the reason behind the fact that PS-Hardy spaces of the complex ellipsoid are a much richer class of holomorphic functions than the usual Hardy spaces of the complex ellipsoids. The complex ellipsoid $\mathbb{B}^{\mathbf{p}} \in \mathbb{C}^n$ is given as

$$\mathbb{B}^{\mathbf{p}} = \{z \in \mathbb{C}^n, \rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1 < 0\},$$

where $\mathbf{p} = (p_1, p_2, ..., p_n) \in \mathbb{Z}^n$. Clearly $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + ... + |z_n|^{2p_n})$ is a continuous, plurisubharmonic exhaustion function for $\mathbb{B}^{\mathbf{p}}$, so we can consider the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ associated with this exhaustion function. For the most general case of PS-Hardy spaces on bounded hyperconvex domains, one may refer to [4, 5]. For the given exhaustion function u, the corresponding Monge–Ampère measures supported on the sublevel sets $S_u(r) = \{u = r\}$, $r < 0$ are defined as $\mu_{u,r} = dd^c(\max\{u, r\})^n - \chi_{\mathbb{B}^{\mathbf{p}} \setminus \{u<r\}}(dd^c u)^n$. Once we have the well-defined sublevel sets and corresponding measures, practically we have all the necessary tools to define the PS–Hardy spaces as follows:

$$H^p_u(\mathbb{B}^{\mathbf{p}}) = \{f \in \mathcal{O}(\mathbb{B}^{\mathbf{p}}) | \sup_{r<0} \int_{u=r} |f|^p d\mu_{u,r} < \infty\}.$$

Let $d(\xi, z) \doteq |v(\xi, z)| + |v(z, \xi)|$ be the quasi-metric defined on $\overline{\mathbb{B}^{\mathbf{p}}}$, where $v(\xi, z) = \langle \partial\rho(\xi), \xi - z\rangle$. Then explicitly $v(\xi, z) = \sum_{j=1}^{n} p_j |\xi_j|^{2(p_j-1)} \bar{\xi}_j (\xi_j - z_j)$. It is shown that $(\partial\mathbb{B}^{\mathbf{p}}, d, d\mu_u)$ is a space of homogeneous type ([3],pg:1483) and $\dfrac{1}{(v(\xi, z))^n}$ is a standard kernel.

The Cauchy–Fantappie integral (which will be referred to as the CF integral throughout the present study) of an $L^p(d\mu_u)$ function f^* is defined as

$$Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\mathbb{B}^{\mathbf{p}}} \frac{f^*(\xi) d\mu_u(\xi)}{(v(\xi, z))^n}.$$

In [3], Hansson showed that the CF integral is a bounded operator on the boundary values of the classical Hardy spaces defined with respect to the boundary measure $\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}$, where the function ρ is defined as $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$. One may easily show that $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$ is exactly the boundary Monge–Ampère measure associated with the exhaustion function $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + ... + |z_n|^{2p_n})$, $\mathbf{p} = (p_1, p_2, ..., p_n) \in \mathbb{Z}^n$ of the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$, which makes us deduce that the Hardy spaces that are examined in [3] are merely the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ that are generated by the exhaustion function u. In [5, 6], it is shown that for the holomorphic functions $f \in H^p_u(\mathbb{B}^{\mathbf{p}})$, the boundary value function $f^* \in L^p(d\mu_u)$ exists, so the CF integral of f^* is well-defined. In [5], we showed that the CF integral has reproducing property for the functions in $H^p_u(\mathbb{B}^{\mathbf{p}})$:

Proposition 2.1 *Let $f \in H^p_u(\mathbb{B}^p)$ be a holomorphic function, then*

$$f(z) = Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\mathbb{B}^p} \frac{f^*(\xi) d\mu_u(\xi)}{(v(\xi, z))^n}$$

3. Compactness

In this section we will give the compactness properties for the subsets of Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ and analytic characterization of the vanishing Carleson measures on $H^p_u(\mathbb{B}^{\mathbf{p}})$, but before that we need to recall the following from the leading work of Poletsky and Stessin:

Definition 1 *Let D be a hyperconvex domain in $\mathbb{C}^n$ and v be a continuous, plurisubharmonic exhaustion function. For ϕ being a nonnegative plurisubharmonic function on D, we have*

$$\|\phi\|_v = \lim_{r\to 0^-} \int_{S_v(r)} \phi d\mu_{v,r}$$

Theorem 3.1 ([4],Theorem 3.6) *Let v be a continuous, plurisubharmonic exhaustion function on a hyperconvex domain D. Then for any compact set $K \subset D$, there is a constant C such that for all $w \in K$, and all nonnegative plurisubharmonic functions φ on D, we have*

$$\varphi(w) \le C\|\varphi\|_v$$

As an immediate consequence of the above result, we have the following:

Corollary 3.1 *Let $1 \le p \le \infty$. Then for every relatively compact subdomain $D_0 \subset\subset \mathbb{B}^{\mathbf{p}}$, we can find a constant $C = C(D_0, p) > 0$ such that*

$$\sup_{z\in D_0} |f(z)| \le C\|f\|_{H^p_u(\mathbb{B}^{\mathbf{p}})}$$

for all $f \in H^p_u(\mathbb{B}^{\mathbf{p}})$

Now using this, we give a basic compactness property for the subsets of Poletsky–Stessin Hardy space $H^p_u(\mathbb{B}^{\mathbf{p}})$ on complex ellipsoid $\mathbb{B}^{\mathbf{p}}$:

Lemma 3.1 *Let $\mathbb{B}^{\mathbf{p}}$ be the complex ellipsoid, $1 \le p < \infty$. Then:*

(i) If $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ is a norm-bounded sequence converging uniformly on compact subsets to $h \in \mathcal{O}(\mathbb{B}^{\mathbf{p}})$, then $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$.

(ii) The inclusion $H^p_u(\mathbb{B}^{\mathbf{p}}) \hookrightarrow \mathcal{O}(\mathbb{B}^{\mathbf{p}})$ is compact, that is any norm-bounded subset of $H^p_u(\mathbb{B}^{\mathbf{p}})$ is relatively compact in $\mathcal{O}(\mathbb{B}^{\mathbf{p}})$.

Proof

(i) Assume that $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ is a norm-bounded sequence converging uniformly on compact subsets to $h \in \mathcal{O}(\mathbb{B}^{\mathbf{p}})$. Then,

$$\int_{S_u(r)} |h|^p d\mu_{u,r} = \int_{S_u(r)} \lim_{k\to\infty} |f_k|^p d\mu_{u,r}$$

$$\le \lim_{k\to\infty} \int_{S_u(r)} |f_k|^p d\mu_{u,r} \le \sup_k \|f_k\|_{H^p_u(\mathbb{B}^{\mathbf{p}})}$$

by Fatou's lemma and thus as $r \to 0$, we have $\|h\|_{H^p_u(\mathbb{B}^{\mathbf{p}})} \le \sup_k \|f_k\|_{H^p_u(\mathbb{B}^{\mathbf{p}})} < \infty$ and $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$ as claimed.

(ii) We have to prove that any norm-bounded sequence in $H^p_u(\mathbb{B}^{\mathbf{p}})$ admits a subsequence converging uniformly on compact subsets. However, indeed the previous theorem says that sup-norm on a relatively compact subset $D_0 \subset\subset \mathbb{B}^{\mathbf{p}}$ of any $f \in H^p_u(\mathbb{B}^{\mathbf{p}})$ is bounded by a constant times its $H^p_u(\mathbb{B}^{\mathbf{p}})$-norm. Therefore, if $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ is norm-bounded, by considering $B_u(r)$s as an increasing exhaustion and applying Montel's theorem to each $B_u(r)$, we obtain a subsequence $\{f_{k_j}\}$ converging uniformly on compact subsets to a holomorphic $h \in \mathcal{O}(\mathbb{B}^{\mathbf{p}})$, and moreover, $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$.

□

Now we will see the relation between compactness and vanishing Carleson measures, but first let us give the definition of Carleson measures in the most general setting (for more details see [1]):

Definition 2 *Let A be a Banach space of holomorphic functions on a domain $D \subset \mathbb{C}^n$; given $p \geq 1$, a finite positive Borel measure μ on D is a* Carleson measure *of A if there is a continuous inclusion $A \hookrightarrow L^p(\mu)$. Furthermore, μ is called a* vanishing Carleson measure *of A if the inclusion $A \hookrightarrow L^p(\mu)$ is compact.*

Proposition 3.1 *Let μ be a finite, positive Borel measure on $\overline{\mathbb{B}^{\mathbf{p}}}$ and $1 < p < \infty$. Then μ is a vanishing Carleson measure of $H^p_u(\mathbb{B}^{\mathbf{p}})$ if and only if $\|f_k\|_{L^p(\mu)} \to 0$ for all norm-bounded sequences $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ converging to 0 uniformly on compact subsets.*

Proof Assume that μ is a vanishing Carleson measure of $H^p_u(\mathbb{B}^{\mathbf{p}})$. Then, $H^p_u(\mathbb{B}^{\mathbf{p}}) \hookrightarrow L^p(\mu)$ is compact by the definition of vanishing Carleson measure and take $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ norm bounded and converging to 0 uniformly on compacta. In particular, $\{f_k\}$ is relatively compact in $L^p(\mu)$; we must prove that $f_k \to 0$ in $L^p(\mu)$. Now, for $0 < h_0 < 1$,

$$\int_{\overline{\mathbb{B}^{\mathbf{p}}}} |f_k|^p d\mu = \int_{\overline{\mathbb{B}^{\mathbf{p}}} \setminus (1-h_0)\mathbb{B}^{\mathbf{p}}} |f_k|^p d\mu + \int_{(1-h_0)\mathbb{B}^{\mathbf{p}}} |f_k|^p d\mu$$

and the second integral on the right can be made arbitrarily small since $f_k \to 0$ uniformly on compacta. For the first integral by ([6], proof of Theorem 3.3) we know that by choosing appropriate h_0 we have,

$$\int_{\overline{\mathbb{B}^{\mathbf{p}}} \setminus (1-h_0)\mathbb{B}^{\mathbf{p}}} |f_k|^p d\mu \leq C\varepsilon \|f_k\|_{H^p_u(\mathbb{B}^{\mathbf{p}})}$$

for arbitrary $\varepsilon > 0$ since μ is a vanishing Carleson measure of $H^p_u(\mathbb{B}^{\mathbf{p}})$. Therefore,

$$\int_{\overline{\mathbb{B}^{\mathbf{p}}}} |f_k|^p d\mu \to 0$$

as claimed.

Conversely assume that all norm-bounded sequences in $H^p_u(\mathbb{B}^{\mathbf{p}})$ converging to 0 uniformly on compacta converge to $0 \in L^p(\mu)$. To prove that the inclusion $H^p_u(\mathbb{B}^{\mathbf{p}}) \hookrightarrow L^p(\mu)$ is compact, it suffices to show that if $\{f_k\}$ is norm-bounded in $H^p_u(\mathbb{B}^{\mathbf{p}})$ then it admits a subsequence converging in $L^p(\mu)$. Lemma 2.1 yields a subsequence $\{f_{k_j}\}$ converging uniformly on compacta to $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$. Then $\{f_{k_j} - h\}$ converges to 0 uniformly on compacta, by assumption this yields $\|f_{k_j} - h\|_{L^p(\mu)} \to 0$, and thus $\{f_{k_j}\} \to h$ in $L^p(\mu)$ as desired. □

Recall that a sequence $\{x_k\}$ in a normed space X is called *weakly convergent* if there is an $x \in X$ such that for every $\phi \in X^*$,

$$\lim_{k\to\infty} \phi(x_k) = \phi(x).$$

Now we will continue with a characterization of weakly convergent sequences in $H^p_u(\mathbb{B}^{\mathbf{p}})$ for $1 < p < \infty$ but before that we need the following lemma:

Lemma 3.2 *Let $1 < p < \infty$. $H^p_u(\mathbb{B}^{\mathbf{p}})$ is reflexive, and thus the unit ball of $H^p_u(\mathbb{B}^{\mathbf{p}})$ is weakly compact.*

Proof In the proof of Theorem 2.1 in [6], we have showed that $H^p_u(\mathbb{B}^{\mathbf{p}})$ is a closed subspace of the Lebesgue space $L^p_u(\partial\mathbb{B}^{\mathbf{p}})$ so $H^p_u(\mathbb{B}^{\mathbf{p}})$ is also reflexive. Hence the closed unit ball of $H^p_u(\mathbb{B}^{\mathbf{p}})$ is weakly compact. □

Lemma 3.3 *Let $1 < p < \infty$. Then a sequence $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ is norm-bounded and converges uniformly on compacta to $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$ if and only if it converges weakly to h.*

Proof Let $\{f_k\}$ be a norm-bounded sequence in $H^p_u(\mathbb{B}^{\mathbf{p}})$ and converges uniformly to $h \in H^p_u(\mathbb{B}^{\mathbf{p}})$ on compact subsets. We need to show that $\Phi(f_k)$ converges to $\Phi(h)$ for all $\Phi \in (H^p_u(\mathbb{B}^{\mathbf{p}}))^*$. Take an arbitrary subsequence $\Phi(f_{k_j})$ and by the previous lemma we know that the unit ball of $H^p_u(\mathbb{B}^{\mathbf{p}})$ is weakly compact and by the Eberlein–Šmulian theorem we can characterize this by sequential compactness (although the weak topology is not metrizable) so we have that there exists a subsequence $f_{k_{j_l}}$ such that $\Phi(f_{k_{j_l}}) \to \Phi(\gamma)$ for all $\Phi \in (H^p_u(\mathbb{B}^{\mathbf{p}}))^*$. Since this is true for all $\Phi \in (H^p_u(\mathbb{B}^{\mathbf{p}}))^*$, it is also true for point evaluations and $f_{k_{j_l}}(x) = \gamma(x) = h(x)$. The last part is due to f_k converging to h uniformly on compacta and consequently it being convergent pointwise. Hence $\Phi(f_{k_{j_l}})$ converges $\Phi(h)$ for all $\Phi \in (H^p_u(\mathbb{B}^{\mathbf{p}}))^*$. Therefore, every subsequence of $\Phi(f_k)$ has a subsequence converging to $\Phi(h)$, hence $\Phi(f_k) \to \Phi(h)$.

Conversely, assume that a sequence $f_k \to 0$ weakly in $H^p_u(\mathbb{B}^{\mathbf{p}})$, in particular, is norm bounded in $H^p_u(\mathbb{B}^{\mathbf{p}})$. Therefore by Lemma 3.1 (ii) to prove that $f_k \to 0$ uniformly on compacta it is sufficient to show that any converging (uniformly on compacta) subsequence must converge to 0. But if $\{f_{k_j}\} \to h \in H^p_u(\mathbb{B}^{\mathbf{p}})$ uniformly on compacta, the previous argument shows that f_{k_j} converges weakly to h, the uniqueness of weak limit then gives $h \equiv 0$ and we are done. □

Thus for $1 < p < \infty$, Proposition 3.1. is a particular case of the following well-known result; [1], Proposition 4.7.)

Theorem 3.2 *Let $T : X \to Y$ be a linear operator between Banach spaces. Then:*

(i) If T is compact then for any sequence $\{x_k\} \subset X$ weakly converging to 0, the sequence $\{Tx_k\}$ strongly converges to 0 in Y.

(ii) Suppose that the unit ball of X is weakly compact. Then T is compact if for any sequence $\{x_k\} \subset X$ weakly converging to 0 the sequence $\{Tx_k\}$ strongly converges to 0 in Y.

Now as an immediate consequence of this, we have the following:

Corollary 3.2 *Let $1 < p < \infty$. Then a linear operator $T : H^p_u(\mathbb{B}^{\mathbf{p}}) \to X$ taking values in a Banach space X is compact if and only if for any norm-bounded sequence $\{f_k\} \subset H^p_u(\mathbb{B}^{\mathbf{p}})$ converging uniformly on compacta to 0, the sequence $\{Tf_k\}$ converges to 0 in X.*

4. Duality

Aizenberg et al. [2] considered Grothendieck–Köthe–da Silva duality for the classical Hardy spaces of a convex domain and in this section, we will give analogous results for Poletsky–Stessin Hardy spaces following their general idea. Before proceeding with the duality arguments, we will first consider the dual complement $\widetilde{\mathbb{B}^{\mathbf{p}}}$ of the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$, and then we will prove the duality relation for the Poletsky–Stessin Hardy spaces of complex ellipsoids. Now let us first give some basic facts about the dual complements following [2] :

Definition 3 *A domain $\Omega \subset \mathbb{C}^n$ is called linearly convex if for every $\xi \in \partial\Omega$, there exists a complex hyperplane*

$$\alpha = \{z \in \mathbb{C}^n : \quad \alpha_1 z_1 + ... + \alpha_n z_n + \beta = 0\}$$

through ξ that does not intersect Ω.

Let Ω be a linearly convex domain. If $0 \in \Omega$, then its dual complement

$$\widetilde{\Omega} = \{w \in \mathbb{C}^n : \quad w_1 z_1 + ... + w_n z_n \neq 1, \quad z \in \Omega\}$$

is the set of hyperplanes that do not intersect the domain Ω. Now let us continue with the main result given in [2] considering the duality of the classical Hardy spaces on linearly convex domains. The classical Hardy space on the dual complement of the domain Ω is defined as follows:

Definition 4 *Let $0 \in \Omega$ be a linearly convex domain with $\mathcal{C}^2$ boundary. By Hardy space for $q \geq 1$ on the dual complement $\tilde{\Omega}$, we mean the space of functions g, holomorphic in the open domain $int(\tilde{\Omega})$ so that*

$$\limsup_{\epsilon \to 0} \int_{\partial\tilde{\Omega}} |g(\xi - \epsilon\nu_\xi)|^q d\sigma(\xi) < \infty,$$

where the vector ν_ξ is the exterior normal unit vector at $\xi \in \tilde{\Omega}$. Since $\partial\tilde{\Omega} = \partial int(\tilde{\Omega})$, this definition is meaningful and this space is denoted by $H^q(\tilde{\Omega})$.

The duality result for the classical Hardy spaces is the following ([2], Theorem 3.1, pp:1354):

Theorem 4.1 *Let $\Omega = \{z \in \mathbb{C}^n \quad : \quad \varrho(z, \bar{z}) < 0\}$, where $\varrho \in \mathcal{C}^3(\overline{\Omega})$ is its defining function, be a bounded, strictly convex domain. If $0 \in \Omega$, then*

$$(H^p(\Omega))^{'} = H^q(\tilde{\Omega}),$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p > 1$. Furthermore, the isomorphism is realized:

$$F(f) = F_\phi(f) = \int_{\partial\Omega} \phi(w) f(z) \omega(z, w),$$

where $\phi \in H^q(\tilde{\Omega})$ and $f \in H^p(\Omega)$.

As it can be seen, this theorem is valid on a strictly convex domain, and now combining our work in [5] and [6] with the idea given in [2], we can extend this result to the Poletsky–Stessin Hardy spaces of the complex ellipsoids and this result is important in two different aspects: first, Poletsky–Stessin Hardy classes are much more general than the classical Hardy spaces and second, the complex ellipsoids are not strictly convex domains. They are the model domains for pseudoconvex domains of finite type, which is more general than the strictly convex domains.

Now let us first give the setting for this generalization:

As it is pointed out in [2], in general it is quite complicated to describe the dual complement of a domain Ω; however, for the case of Reinhardt domains with center at the origin there are precise results. If Ω is a Reinhardt domain centered at the origin then $F(\Omega) \subset \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \quad x_i \geq 0\}$ and $F(z_1, z_2, ..., z_n) = (|z_1|, |z_2|, ..., |z_n|)$. For any $B \subset \mathbb{R}^n_+$, its inverse image by F^{-1} is defined to be the set $F^{-1}(B) = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : \quad F(z_1, z_2, ..., z_n) \in B\}$. Then one can verify that the domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if and only if $\Omega = F^{-1}(F(\Omega))$. Hence, any Reinhardt domain Ω is determined completely by its absolute image $F(\Omega)$. Thus, we have the following definition:

Definition 5 *Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain centered at the origin $0 \in \mathbb{C}^n$. We say that the point $(y_1, ..., y_n) \in \widetilde{F(\Omega) \in \mathbb{R}^n_+}$ if and only if $\sum_{i=1}^n x_i y_i < 1$ for every $(x_1, ..., x_n) \in F(\Omega)$. Then the dual complement of Ω is the set $\widetilde{\Omega} = F^{-1}(\widetilde{F(\Omega)})$.*

From [2], we have the following characterization of the dual complement of a Reinhardt domain centered at the origin $0 \in \mathbb{C}^n$, ([2], p. 1342).

Lemma 4.1 *For $r > 0$, $p > 1$ and $k_i \in \mathbb{R}^n_+ \setminus \{0\}$ fixed numbers, let*

$$\Omega = \{z \in \mathbb{C}^n | \quad \sum_{i=1}^n k_i |z_i|^p < r^p\}$$

be a Reinhardt domain centered at the origin. Then for $q = \frac{p}{p-1}$, the dual complement is

$$\widetilde{\Omega} = \{\xi \in \mathbb{C}^n | \quad \sum_{i=1}^n (k_i)^{\frac{1}{1-p}} |\xi_i|^q < \frac{1}{r^q}\}.$$

Now since the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$ is a Reinhardt domain centered at the origin, the above lemma allows us to deduce the following:

Corollary 4.1 *Let $\mathbb{B}^p = \{z \in \mathbb{C}^n, \quad \sum_{i=1}^n |z_i|^{2p} - 1 < 0\}$, $p \in \mathbb{Z}_+$ be the complex ellipsoid. Then for $q \in \mathbb{R}_+$ such that $q = \frac{p}{2p-1}$, the dual complement of $\mathbb{B}^p$ is*

$$\widetilde{\mathbb{B}^p} = \{\xi \in \mathbb{C}^n, \quad \sum_{i=1}^n |\xi_i|^{2q} - 1 \leq 0\}. \tag{1}$$

For $\mathbb{B}^{\mathbf{p}}$ and dual complement $\widetilde{\mathbb{B}^{\mathbf{p}}}$, choose the exhaustion functions u and $\tilde{u}$ respectively as follows:

$$u(z) = \ln(|z_1|^{2p} + |z_2|^{2p} + ... + |z_n|^{2p})$$

$$\tilde{u}(z) = \ln(|z_1|^{2q} + |z_2|^{2q} + ... + |z_n|^{2q}),$$

where p and q are given as in the previous corollary.

Now define the Poletsky–Stessin Hardy space on the dual complement of a linearly convex domain following the classical definition given in [2]:

Definition 6 *Let $0 \in \Omega$ be a linearly convex domain with $\mathcal{C}^2$ boundary and $\tilde{u}$ be a continuous, negative, plurisubharmonic exhaustion function for $\tilde{\Omega}$. For $1 < p < \infty$, the Poletsky–Stessin Hardy space on the dual complement $\tilde{\Omega}$ is the space of functions f holomorphic in the open domain $int(\tilde{\Omega})$ so that*

$$\lim_{r \to 0^-} \int_{S_{\tilde{u}}(r)} |f|^p d\mu_{\tilde{u},r} < \infty.$$

We will continue with the following duality argument for the Poletsky–Stessin Hardy spaces of the complex ellipsoids:

Theorem 4.2 $(H^r_u(\mathbb{B}^{\boldsymbol{p}}))' = (H^s_{\tilde{u}}(\widetilde{\mathbb{B}^{\boldsymbol{p}}}))$, $r > 1$, $\dfrac{1}{r} + \dfrac{1}{s} = 1$. *Furthermore, the following isomorphism is realized:*

$$F(f) = F_\phi(f) = \int_{\partial \mathbb{B}^{\boldsymbol{p}}} \phi f d\mu_u,$$

where $\phi \in H^s_{\tilde{u}}(\widetilde{\mathbb{B}^{\boldsymbol{p}}})$ *and* $f \in H^r_u(\mathbb{B}^{\boldsymbol{p}})$.

Proof Consider the space $L^r_u(\partial \mathbb{B}^{\mathbf{p}})$. Then the space $H^r_u(\mathbb{B}^{\mathbf{p}})$ is a closed subspace of $L^r_u(\partial \mathbb{B}^{\mathbf{p}})$ with respect to the L^r_u-norm. Thus for every element $F \in (H^r_u(\mathbb{B}^{\mathbf{p}}))'$, there exists a function $g \in L^s_u(\partial \mathbb{B}^{\mathbf{p}})$ such that

$$F(f) = \int_{\partial \mathbb{B}^{\mathbf{p}}} f(z)\overline{g(z)} d\mu_u(z).$$

Now using the Cauhcy-Fantappie representation of $H^r_u(\mathbb{B}^{\mathbf{p}})$ functions, we write [4] again

$$F(f) = \int_{\partial \mathbb{B}^{\mathbf{p}}} f(z)\overline{g(z)} d\mu_u(z) = \int_{\partial \mathbb{B}^{\mathbf{p}}} \overline{g(z)} \left(\lim_{t \to 1} \int_{\partial \mathbb{B}^{\mathbf{p}}_t} \frac{f(\xi) d\mu_u(\xi)}{(v(z,\xi)^n)} \right) d\mu_u(z).$$

Taking the limit outside the integral and changing the order of integration leads to

$$F(f) = \lim_{t \to 1} \int_{\partial \mathbb{B}^{\mathbf{p}}_t} f(\xi) \left(\int_{\partial \mathbb{B}^{\mathbf{p}}} \frac{\overline{g(z)} d\mu_u(z)}{(v(z,\xi))^n} \right) d\mu_u(\xi)$$

and the convexity of the ellipsoid implies that $\widetilde{(\widetilde{\mathbb{B}^{\mathbf{p}}})} = \mathbb{B}^{\mathbf{p}}$. Thus in the inner integral, we make a change of variables

$$w : \xi \in \mathbb{B}^{\mathbf{p}} \to w(\xi) \in (\widetilde{\mathbb{B}^{\mathbf{p}}})$$

$$(\xi_1, \xi_2, ..., \xi_n) \to (\xi_1^{\frac{p}{q}}, \xi_2^{\frac{p}{q}}, ..., \xi_n^{\frac{p}{q}}) = w$$

and deduce that

$$F(f) = \lim_{t\to 1} \int_{\partial\mathbb{B}_t^{\mathbf{p}}} f(\xi) \left(\int_{\partial\widetilde{\mathbb{B}^{\mathbf{p}}}} \frac{G(w)}{(v(w,\xi))^n} d\mu_{\tilde{u}}(w) \right) d\mu_u(\xi).$$

Now by using the boundary value characterization of Poletsky–Stessin Hardy spaces of complex ellipsoids [5, 6] and the fact that the CF integral operator is bounded on L_u^r to H_u^r ([3], Theorem 1), the inner integral is a function from $H_{\tilde{u}}^s(\widetilde{\mathbb{B}^{\mathbf{p}}})$. Now as $t \to 1$, we have

$$F(f) = F_\phi(f) = \int_{\partial\mathbb{B}^{\mathbf{p}}} \phi f d\mu_u,$$

where $\phi \in H_{\tilde{u}}^s(\widetilde{\mathbb{B}^{\mathbf{p}}})$. Thus $(H_u^r(\mathbb{B}^{\mathbf{p}}))' = (H_{\tilde{u}}^s(\widetilde{\mathbb{B}^{\mathbf{p}}}))$. □

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