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Solutions of the Björling problem for timelike surfaces in the Lorentz-Minkowski space

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Abstract: We give a number of new examples of timelike minimal surfaces in the Lorentz–Minkowski space. Our method consists of solving the Björling problem by prescribing a circle or a helix as the core curve α and rotating with constant angular speed the unit normal vector field in the normal plane to α. As particular cases, we exhibit new examples of timelike minimal surfaces invariant by a uniparametric group of helicoidal motions.

Key words: Timelike minimal surface, Björling problem, circle, helix

1. Introduction

The Björling problem in Euclidean space asks for the existence and uniqueness of a minimal surface (a surface with zero mean curvature everywhere) that contains a given real analytic curve and a prescribed analytic unit normal along this curve. The solution to the Björling problem was obtained by Schwarz [11]. When the ambient space is the 3-dimensional Lorentz–Minkowski space $\mathbb{L}^3$, the Björling problem was solved for spacelike surfaces in [1] and for timelike surfaces in [2]. In the present paper we are interested in the solutions of the Björling problem for timelike surfaces in $\mathbb{L}^3$, which can be formulated as follows. Let $\alpha : I \subset \mathbb{R} \to \mathbb{L}^3$ be a regular analytic timelike (resp. spacelike) curve and let $V : I \to \mathbb{L}^3$ be a given unit analytic spacelike vector field along α such that $\langle \alpha', V \rangle = 0$. The Björling problem consists of determining a timelike minimal surface $X : \Omega \subset \mathbb{R}^2 \to \mathbb{L}^3$, with $I \times \{0\} \subset \Omega$, such that $X(t,0) = \alpha(t)$ and $N(t,0) = V(t)$ for all $t \in I$ (resp. $X(0,s) = \alpha(s)$ and $N(0,s) = V(s)$ for all $s \in I$) where $N : \Omega \to \mathbb{L}^3$ is the Gauss map of the surface. When α is timelike (resp. spacelike) this problem is called the *timelike Björling problem* (resp. the *spacelike Björling problem*). In order to solve the Björling problem, we identify $\mathbb{R}^2$ with the ring $\mathbb{C}'$ of split-complex numbers, which plays a role similar to the ordinary complex numbers $\mathbb{C}$ in the Riemannian case. Then the solution to the Björling problem is the parametrized surface

$$X(t,s) = \mathfrak{Re}\left(\alpha(z) + k' \int_{z_0}^{z} V(w) \times \alpha'(w)\, dw\right), \tag{1}$$

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where $z_0 \in \Omega$ is fixed and $\times$ is the Lorentzian cross-product defined as $\langle u \times v, w\rangle = \det(u, v, w)$: see [2]. Here the domain Ω is simply connected so the integral in (1), which is computed along any path between z_0 and z, is independent of any path. Although formula (1) gives a method to construct timelike minimal surfaces, only a few examples of explicit parametrizations are known. This is due to the difficulty to compute the integral in (1) as well as the real parts in the parentheses.

The aim of this paper is to provide many solutions of (1) by choosing suitable data α and V in order to integrate (1) explicitly as well as easily taking the real parts in the above expression. The general strategy is the following. Consider that the core curve α is a circle or a helix about the axis L, and denote $\{\alpha'(s), \mathbf{n}(s), \mathbf{b}(s)\}$ its Frenet frame, where $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the principal normal vector and the binormal vector, respectively. It is important to point out that the expressions of the three vectors of the Frenet frame are given by trigonometric, hyperbolic, or polynomial functions depending on the causal character of α. Since the unit spacelike vector $V(s)$ lies in the normal plane at each s, then $V(s)$ is a linear combination of $\mathbf{n}(s)$ and $\mathbf{b}(s)$. In order to expose our idea, suppose for example that $\mathbf{n}(s)$ is a spacelike vector and $\mathbf{b}(s)$ is a timelike vector. As $V(s)$ is a unit spacelike vector, then $V(s) = \cosh\varphi(s)\mathbf{n}(s) + \sinh\varphi(s)\mathbf{b}(s)$ for some function φ. We may think of $\varphi'(s)$ as the angular speed that makes $V(s)$ when it 'rotates' in each normal plane to $\alpha(s)$. The crucial fact is to choose φ as a linear function $\varphi(s) = as + b$, for two real numbers $a, b \in \mathbb{R}$; that is, V rotates with constant angular speed. In such a case, all functions involved in our initial data are trigonometric, hyperbolic functions or polynomial, so we can find the integral and the real parts in (1), obtaining the explicit parametrization of the solution of the Björling problem. This idea has been exploited by the second author and Weber in Euclidean space [9] and in the Riemannian case of $\mathbb{L}^3$ by the authors [8].

Our main contribution is the construction of a great number of explicit examples of timelike minimal surfaces. Here, by explicit examples we mean that we find the parametrization $X : \Omega \to \mathbb{L}^3$, $X(u, v) = (x(u, v), y(u, v), z(u, v))$, of the timelike minimal surfaces giving the expressions of $x(u, v)$, $y(u, v)$, and $z(u, v)$ in terms of the parameter (u, v). In many cases the expressions of the parametrization X may be lengthy and tedious because of the integral in (1), but in this point we have used a symbolic software (Mathematica) to solve this integral by quadratures. We rediscover the timelike minimal rotational surfaces when the core curve α is a circle and we choose $a = 0$ in the function $\varphi(s) = as + b$ (see also [2]). The classification of the timelike minimal surfaces of rotational type is known in the literature: see, for example, [5, 6, 12].

A second contribution of this paper is that we exhibit many examples of explicit timelike minimal surfaces invariant by a group of helicoidal motions when the axis is timelike and spacelike. These examples are different from the (timelike) helicoids: a helicoid is, by definition, a timelike minimal surface that is ruled and invariant by a uniparametric group of helicoidal motions. Helicoids are obtained as solutions of suitable Björling problems; indeed, the core curve is a straight line and V rotates with constant angular speed in the normal plane: see Examples 3.3 and 4.3 in [2]. In contrast, our examples have as the core curve a helix: see parametrizations (16) and (19) when the axis is timelike and (22), (25), and (27) when the axis is spacelike. References for the description of the timelike minimal helicoids are [4, 10].

This paper is organized as follows. After a section of preliminaries where we recall the $\mathbb{C}'$-conformal parametrization of a timelike minimal surface, we separate our examples depending on whether the core curve α is a circle (Section 3) or a helix (Section 4). In each case, we will distinguish the types of circles and helices depending on the rotational axis and the causal character of the curve α.

The examples that we obtain are summarized in the following tables, where we indicate the reference number of the parametrization in the present paper. Here we exclude the rotational timelike minimal surfaces (4), (8), (11), and (14) and those surfaces invariant by a uniparametric group of helicoidal motions mentioned previously.

Table 1. Reference equation number of examples of timelike minimal surfaces based on a circle.

	Timelike axis	Spacelike axis	Lightlike axis
Spacelike circle	(3)	(6), (7)	(13)
Timelike circle	–	(10)	–

Table 2. Reference equation number of examples of timelike minimal surfaces based on a helix.

	Timelike axis	Spacelike axis
Spacelike helix	(15)	(20), (21), (23), (24)
Timelike helix	(17), (18)	(26)

2. Preliminaries

The Lorentz–Minkowski space is the vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle , \rangle = dx^2 + dy^2 - dz^2$ where (x, y, z) are the canonical coordinates of $\mathbb{R}^3$. An immersion $X : M \to \mathbb{L}^3$ of a surface M is called timelike if the induced metric at every tangent plane is Lorentzian, that is, a metric of signature $(1, 1)$. The surface is called minimal if the mean curvature is zero at every point of M. Under suitable coordinates, the minimality property is expressed in terms of split-complex notation as follows (see [3] for details). We identify $\mathbb{R}^2$ with the split-complex number $\mathbb{C}' = \{u + k'v : u, v \in \mathbb{R}, k'^2 = 1, 1k' = k'1\}$, which is a commutative algebra over $\mathbb{R}$. If $z = u + k'v$, then $\mathfrak{Re}(z) = u$, $\mathfrak{Im}(z) = v$ and $\bar{z} = u - k'v$. Similarly, we can define the notions of split-holomorphic functions. Let M be a surface and $X : M \to \mathbb{L}^3$ be a nonconstant map. Consider on M a coordinates system (u, v). If $z = u + k'v$, define

$$\phi = \frac{\partial X}{\partial z} = \frac{1}{2}\left(\frac{\partial X}{\partial u} + k'\frac{\partial X}{\partial v}\right)$$

and let $\phi = (\phi_1, \phi_2, \phi_3)$ be the components of ϕ. We have the following properties:

1. X is an immersion if and only if

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = -\frac{1}{4}\left(\langle X_u, X_u\rangle - \langle X_v, X_v\rangle\right) > 0.$$

2. If X is an immersion, then X is conformal if and only if

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = \frac{1}{4}\left(\langle X_u, X_u\rangle + \langle X_v, X_v\rangle + 2k'\langle X_u, X_v\rangle\right) = 0.$$

3. If X is a conformal immersion, then the mean curvature is zero on M if and only if ϕ is a split-holomorphic function, that is,

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \phi}{\partial u} - k'\frac{\partial \phi}{\partial v}\right) = \frac{1}{4}(X_{uu} - X_{vv}) = 0.$$

This identity is equivalent to saying that the Laplacian of X, namely $\Delta X = X_{uu} - X_{vv}$, vanishes identically on M.

Once we have defined the function ϕ, we may express the parametrization X in terms of ϕ as

$$X(z) = 2\,\mathfrak{Re}\int_{z_0}^{z}\phi(w)dw,$$

where z_0 is a fixed base point and the integral is computed along any path from z_0 to z. The converse of the above process holds immediately: see [2, Th. 2.1]. In the case of the Björling problem for timelike minimal surfaces, the solution is expressed by parametrization (1). In this paper we will unify both Björling problems independently if the core curve α is spacelike or timelike in the sense that the curve α will always be the parameter curve $v = 0$, that is, $\alpha(s) = X(s, 0)$. This is clear for timelike curves according to (1). When α is a spacelike curve, let $X = X(u, v)$ be the solution given by (1), where we know that $\alpha(s) = X(0, s)$. We make the change of variables in the $\mathbb{C}'$-domain Ω by doing the rotation $J(z) = k'z$, $z \in \Omega$. If $\Omega' = J(\Omega)$, define $Y : \Omega' \to \mathbb{L}^3$ by $Y(z) = X(J(z))$, or equivalently $(u, v) \to (v, u)$. Then Y is a conformal immersion with $\Delta Y = 0$, that is, Y is a timelike minimal immersion and clearly $Y(s, 0) = \alpha(s)$.

As we have pointed out, we use Mathematica to solve by quadratures the integral in (1) as well as the real parts. Here we recall the identities $\cosh(k't) = \cosh(t)$, $\cos(k't) = \cos(t)$, $\sinh(k't) = k'\sinh(t)$, and $\sin(k't) = \sin(t)$ when $t \in \mathbb{R}$. It is also possible to define null coordinates in a generalized timelike minimal immersion as the sum of two null curves [4] by using D'Alembert's solutions with the change of variables $(u, v) = (t + s, t - s)$ and obtaining

$$X(t, s) = \frac{1}{2}\left(\alpha(t + s) + \alpha(t - s) + \int_{t-s}^{t+s} V(\tau) \times \alpha'(\tau)\, d\tau\right).$$

3. Timelike minimal surfaces based on a circle

In this section we solve the Björling problem for timelike surfaces when the core curve α is a circle. A circle of $\mathbb{L}^3$ is the orbit of a point under a uniparametric group of rotational motions when this orbit is not a straight line. A circle is determined by a point and the axis L of the group of rotational motions. In $\mathbb{L}^3$ the family of circles is richer than in Euclidean space because there exist three types of circles according to the causal character of the rotational axis. We separate case by case the solutions of the Björling problem according to the types of the circle. After a change of coordinates, we assume the axis is $(0, 0, 1)$ (timelike), $(1, 0, 0)$ (spacelike), or $(1, 0, 1)$ (lightlike).

3.1. The rotational axis is timelike

When the rotational axis is timelike, the circle is necessarily a spacelike curve and thus we are solving the spacelike Björling problem. After a homothety of $\mathbb{L}^3$, a spacelike circle with timelike axis $(0, 0, 1)$ parametrizes as $\alpha(s) = (\cos(s), \sin(s), 0)$, $s \in \mathbb{R}$. The normal and binormal vectors of α are

$$\mathbf{n}(s) = -(\cos(s), \sin(s), 0), \qquad \mathbf{b}(s) = (0, 0, -1).$$

Because $\{\mathbf{n}(s), \mathbf{b}(s)\}$ is an orthonormal basis of $\text{span}(\alpha'(t))^{\perp}$, any unit spacelike vector field $V(s)$ can be expressed as

$$V(s) = \cosh\varphi(s)\mathbf{n}(s) + \sinh\varphi(s)\mathbf{b}(s) \tag{2}$$

for some function φ. Let us choose $\varphi(s) = as + b$, where $a, b \in \mathbb{R}$. Then

$$V(s) \times \alpha'(s) = (\cos(s)\sinh(as+b), \sin(s)\sinh(as+b), \cosh(as+b)).$$

We distinguish cases depending on the value of a.

1. Case $a \neq 0$. From (1), the parametrization of the timelike Björling surface is

$$X(u,v) = \begin{pmatrix} \cos(u)\cos(v) - \dfrac{\sin(u)\cosh(au+b)A(v) - \cos(u)\sinh(au+b)B(v)}{a^2+1} \\ \sin(u)\cos(v) + \dfrac{\cos(u)\cosh(au+b)A(v) + \sin(u)\sinh(au+b)B(v)}{a^2+1} \\ \dfrac{\cosh(au+b)\sinh(av)}{a} \end{pmatrix}, \tag{3}$$

   where

$$\begin{aligned} A(v) &= a\cosh(av)\sin(v) - \cos(v)\sinh(av), \\ B(v) &= a\sinh(av)\cos(v) + \cosh(av)\sin(v). \end{aligned}$$

2. Case $a = 0$. Then $\varphi(s) = b$ is a constant function and we find

$$V(s) \times \alpha'(s) = (\cos(s)\sinh(b), \sin(s)\sinh(b), \cosh(b)).$$

   Parametrization (1) is

$$\begin{aligned} X(u,v) &= \begin{pmatrix} \cos(u)(\cos(v) + \sinh(b)\sin(v)) \\ \sin(u)(\cos(v) + \sinh(b)\sin(v)) \\ \cosh(b)v \end{pmatrix} \\ &= \begin{pmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(v) + \sinh(b)\sin(v) \\ 0 \\ \cosh(b)v \end{pmatrix}. \end{aligned} \tag{4}$$

   It is important to point out that parametrization (4) also appears as the limit case of expression (3) by letting $a \to 0$: we find $A(v) = 0$, $B(v) = \sin(v)$ and

$$\lim_{a\to 0} \frac{\cosh(au+b)\sinh(av)}{a} = \cosh(b)v.$$

   We prove that this surface is, indeed, a surface of revolution about the z-axis. Denote by

$$\Psi(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the rotation of the θ-angle about the z-axis [7, p. 52]. Then (4) is written as $X(u,v)=\Psi(u)\gamma(v)$, where $\gamma(v)=X(0,v)$. Thus, it is immediate that $X(u,v)$ satisfies

$$\Psi(\theta)\cdot X(u,v)=\Psi(\theta)\cdot\Psi(u)\gamma(v)=\Psi(u+\theta)\gamma(v)=X(u+\theta,v)$$

for all $\theta\in\mathbb{R}$, proving that $X(u,v)$ is invariant by the uniparametric group $\{\Psi(\theta):\theta\in\mathbb{R}\}$ of rotations about the z-axis. As a consequence, the surface is the timelike elliptic catenoid [2]. Usually, in the literature the timelike elliptic catenoid appears as the surface obtained by rotating about the z-axis the graphic of the cosine function $x=\cos(z)$. Then $X(u,v)=(\cos(u)\cos(v),\sin(u)\cos(v),v)$, noting that $b=0$ in (4).

3.2. The rotational axis is spacelike

Assume the rotational axis is the x-axis. There exist two types of nondegenerate circles with rotational axis $(1,0,0)$ depending on the causal character of the circle. We distinguish both cases.

3.2.1. Spacelike circles

In this case we have the spacelike Björling problem. After a homothety of $\mathbb{L}^3$, a spacelike circle with spacelike axis $(1,0,0)$ parametrizes as $\alpha(s)=(0,\sinh(s),\cosh(s))$, $s\in\mathbb{R}$. The normal and binormal vectors of α are

$$\mathbf{n}(s)=(0,\sinh(s),\cosh(s)),\quad \mathbf{b}(s)=(1,0,0).$$

Then a unit spacelike vector field along α orthogonal to α' is written as

$$V(s)=\sinh\varphi(s)\mathbf{n}(s)+\cosh\varphi(s)\mathbf{b}(s) \tag{5}$$

for some function $\varphi(s)$. The vector product $V\times\alpha'$ is

$$V(s)\times\alpha'(s)=(-\sinh\varphi(s),-\sinh(s)\cosh\varphi(s),-\cosh(s)\cosh\varphi(s)).$$

Let us take $\varphi(s)=as+b$, $a,b\in\mathbb{R}$. Depending on the value of a, the solutions of (1) are:

1. Case $a\neq 0$ and $a\neq\pm1$. The parametrization of the timelike Björling surface is

$$X(u,v)=\begin{pmatrix} -\dfrac{\sinh(au+b)\sinh(av)}{a} \\ \sinh(u)\cosh(v)+\dfrac{\cosh(u)\sinh(au+b)A(v)+\sinh(u)\cosh(au+b)B(v)}{a^2-1} \\ \cosh(u)\cosh(v)+\dfrac{\sinh(u)\sinh(au+b)A(v)+\cosh(u)\cosh(au+b)B(v)}{a^2-1}\end{pmatrix}, \tag{6}$$

where

$$\begin{aligned} A(v) &= -a\cosh(av)\sinh(v)+\cosh(v)\sinh(av),\\ B(v) &= -a\sinh(av)\cosh(v)+\cosh(av)\sinh(v).\end{aligned}$$

2. Case $a = \pm 1$. The parametrization of the surface depends on whether $a = 1$ or $a = -1$, obtaining

$$X(u,v) = \begin{pmatrix} -\sinh(au+b)\sinh(v) \\ \cosh(v)\sinh(u) + \frac{1}{2}a(v\sinh(b) - \frac{1}{2}\sinh(2au+b)\sinh(2v)) \\ \cosh(u)\cosh(v) - \frac{1}{2}(v\cosh(b) + \frac{1}{2}\cosh(2au+b)\sinh(2v)) \end{pmatrix}. \tag{7}$$

3. Case $a = 0$. We prove that the Björling surface is a rotational surface. Indeed, let $\varphi(s) = b$. If follows from (1) that

$$V(s) \times \alpha'(s) = -(\sinh(b), \cosh(b)\sinh(s), \cosh(b)\cosh(s)).$$

Then solution (1) is

$$\begin{aligned} X(u,v) &= \begin{pmatrix} -v\sinh(b) \\ \sinh(u)(\cosh(v) - \cosh(b)\sinh(v)) \\ \cosh(u)(\cosh(v) - \cosh(b)\sinh(v)) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(u) & \sinh(u) \\ 0 & \sinh(u) & \cosh(u) \end{pmatrix} \begin{pmatrix} -\sinh(b)v \\ 0 \\ \cosh(v) - \cosh(b)\sinh(v) \end{pmatrix}. \end{aligned} \tag{8}$$

Again, let us observe that expression (8) is the limit case of parametrization (6) by letting $a \to 0$: now $A(v) = 0$, $B(v) = \sinh(v)$ and

$$\lim_{a\to 0} -\frac{\sinh(au+b)\sinh(av)}{a} = -\sinh(b)v.$$

Consider the rotation $\Theta(\theta)$ about the axis $(1,0,0)$ given as

$$\Theta(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\theta) & \sinh(\theta) \\ 0 & \sinh(\theta) & \cosh(\theta) \end{pmatrix}.$$

See [7, p. 52]. By defining $\gamma(v) = X(0,v)$, we have $X(u,v) = \Theta(u)\cdot\gamma(v)$. It is immediate that $\Theta(\theta)\cdot X(u,v) = \Theta(\theta)\cdot\Theta(u)\gamma(v) = \Theta(u+\theta)\gamma(v) = X(u+\theta,v)$ for all $\theta \in \mathbb{R}$, concluding that $X(u,v)$ is surface invariant by the uniparametric group $\{\Theta(\theta) : \theta \in \mathbb{R}\}$ of rotations about the x-axis. This surface is called the timelike hyperbolic catenoid, which is the timelike surface obtained by rotating the curve $y = \sinh(x)$. If we parametrize this surface by $Y(u,v) = (v, \sinh(v)\sinh(u), \sinh(v)\cosh(u))$, then

$$X(u,v) = \frac{1}{\sinh(\lambda)} Y(u, v+\lambda) + (\lambda\sinh(b), 0, 0),$$

where $\lambda \in \mathbb{R}$ is chosen to be $\sinh(\lambda) = -1/\sinh(b)$. This proves that $X(u,v)$ is $Y(u,v)$ up to a dilatation and a translation.

3.2.2. Timelike circles

After a homothety of $\mathbb{L}^3$, a timelike circle with spacelike axis $(1,0,0)$ parametrizes as $\alpha(s)=(0,\cosh(s),\sinh(s))$, $s\in\mathbb{R}$. The normal and binormal vectors of α are

$$\mathbf{n}(s)=(0,\cosh(s),\sinh(s)),\quad \mathbf{b}(s)=(-1,0,0),$$

and the unit spacelike vector field V is

$$V(s)=\cos\varphi(s)\mathbf{n}(s)+\sin\varphi(s)\mathbf{b}(s) \tag{9}$$

for some function $\varphi(s)$. Then the vector product $V\times\alpha'$ is

$$V(s)\times\alpha'(s)=(\cos\varphi(s),\sin\varphi(s)\cosh(s),\sin\varphi(s)\sinh(s)).$$

Take $\varphi(s)=as+b$, with $a,b\in\mathbb{R}$ and we solve (1) obtaining the next two cases:

1. Case $a\neq 0$. The surface is

$$X(u,v)=\begin{pmatrix}\dfrac{\cos(au+b)\sin(av)}{a}\\ \cosh(u)\cosh(v)+\dfrac{\cosh(u)\sin(au+b)A(v)+\sinh(u)\cos(au+b)B(v)}{a^2+1}\\ \sinh(u)\cosh(v)+\dfrac{\sin(au+b)\sinh(u)A(v)+\cos(au+b)\cosh(u)B(v)}{a^2+1}\end{pmatrix}, \tag{10}$$

   where

$$\begin{aligned}A(v)&=a\cosh(v)\sin(av)+\cos(av)\sinh(v),\\ B(v)&=-a\cos(av)\sinh(v)+\cosh(v)\sin(av).\end{aligned}$$

2. Case $a=0$. As we expect, in this situation we obtain a rotational timelike minimal surface. Let $\varphi(s)=b$. The solution to the Björling problem according to (1) is

$$\begin{aligned}X(u,v)&=\begin{pmatrix}v\cos(b)\\ \cosh(u)(\cosh(v)+\sin(b)\sinh(v))\\ \sinh(u)(\cosh(v)+\sin(b)\sinh(v))\end{pmatrix}\\ &=\begin{pmatrix}1&0&0\\0&\cosh(u)&\sinh(u)\\0&\sinh(u)&\cosh(u)\end{pmatrix}\begin{pmatrix}\cos(b)v\\ \cosh(v)+\sin(b)\sinh(v)\\ 0\end{pmatrix}.\end{aligned} \tag{11}$$

   It follows easily that (11) is the limit case $a\to 0$ of the parametrization $X(u,v)$ in (10) because when $a=0$, we find $A(v)=\sinh(v)$, $B(v)=0$, and

$$\lim_{a\to 0}\frac{\cos(au+b)\sin(av)}{a}=v\cos(b).$$

   By using the notation as in the preceding case of spacelike circles, we have $\Theta(\theta)\cdot X(u,v)=X(u+\theta,v)$ for all $\theta\in\mathbb{R}$, proving that $X(u,v)$ is a rotational timelike minimal surface about the axis $(1,0,0)$. In the case $b=0$, then the surface is obtained by rotating about the x-axis the curve $\alpha(s)=(s,\cosh(s),0)$. This surface is called the timelike hyperbolic catenoid with spacelike profile curve [2, 12].

3.3. The rotational axis is lightlike

When the rotational axis is lightlike, the parametrization of a circle is given in terms of polynomial functions. Indeed, assume the rotational axis L is spanned by $(1,0,1)$. After a homothety, a circle with axis L parametrizes as $\alpha(s)=(-1+s^2/2, s, s^2/2)$. Then $\alpha'(s)=(s,1,s)$ and $\alpha''(s)=(1,0,1)$. Now $\alpha''(s)$ is a lightlike vector for all $s\in\mathbb{R}$. Under this setting, there is not assigned a Frenet frame of α formed by an orthonormal basis of $\mathbb{L}^3$. Instead, one can define $\mathbf{n}(s)=\alpha''(s)/2$ and $\mathbf{b}(s)$ the unique lightlike vector orthogonal to $\alpha'(s)$ such that $\langle\mathbf{n}(s),\mathbf{b}(s)\rangle=-1/2$ [7]. In the case of the curve α we find

$$\mathbf{n}(s)=\left(\frac{1}{2},0,\frac{1}{2}\right),\qquad \mathbf{b}(s)=\left(\frac{s^2-1}{2},s,\frac{s^2+1}{2}\right). \tag{12}$$

Then the vectors $e_2(s)=\mathbf{n}(s)-\mathbf{b}(s)$ and $e_3(s)=\mathbf{n}(s)+\mathbf{b}(s)$, together with $\alpha'(s)$, form an orthonormal basis $\{\alpha'(s),e_2(s),e_3(s)\}$ where $e_3(s)$ is a timelike vector, namely,

$$e_2(s)=\left(\frac{2-s^2}{2},-s,-\frac{s^2}{2}\right),\qquad e_3(s)=\left(\frac{s^2}{2},s,\frac{s^2+2}{2}\right).$$

The vector field $V(s)$ in the Björling problem is written as

$$V(s)=\cosh\varphi(s)e_2(s)+\sinh\varphi(s)e_3(s).$$

Again we take $\varphi(s)=as+b$, $a,b\in\mathbb{R}$. Then

$$V(s)\times\alpha'(s)=\begin{pmatrix}-\cosh(as+b)\frac{s^2}{2}+\sinh(as+b)\frac{s^2-2}{2}\\ s(\sinh(as+b)-\cosh(as+b))\\ -\cosh(as+b)\frac{s^2+2}{2}+\sinh(as+b)\frac{s^2}{2}\end{pmatrix}.$$

The solutions of the spacelike Björling problem are separated in two cases:

1. Case $a\neq 0$. Then the parametrization of the surface is

$$X(u,v)=\begin{pmatrix}\frac{u^2+v^2-2}{2}+e^{-au-b}A(u,v)\dfrac{a^2\left(e^{2(au+b)}+u^2+v^2-1\right)+2au+2}{2a^3}\\ u+e^{-au-b}\dfrac{av\cosh(av)-(au+1)\sinh(av)}{a^2}\\ \frac{u^2+v^2}{2}+e^{-au-b}A(u,v)\dfrac{a^2\left(e^{2(au+b)}+u^2+v^2+1\right)+2au+2}{2a^3}\end{pmatrix}, \tag{13}$$

   where $A(u,v)=2au(av+1)\cosh(au)-\sinh(au)$.

2. Case $a=0$. If $\varphi(s)=b$, then

$$X(u,v)=\begin{pmatrix}\frac{1}{6}\left(v\sinh(b)\left(3u^2+v^2-6\right)-v\cosh(b)\left(3u^2+v^2\right)+3\left(u^2+v^2-2\right)\right)\\ u+e^{-b}uv\\ \frac{1}{6}\left(v\sinh(b)\left(3u^2+v^2\right)-v\cosh(b)\left(3u^2+v^2+6\right)+3\left(u^2+v^2\right)\right)\end{pmatrix}. \tag{14}$$

We prove that this surface is invariant by the uniparametric group of rotations about the axis L. Recall that this group of motions is $\{\Omega(\theta) : \theta \in \mathbb{R}\}$, where

$$\Omega(\theta) = \begin{pmatrix} 1-\frac{\theta^2}{2} & \theta & \frac{\theta^2}{2} \\ -\theta & 1 & \theta \\ -\frac{\theta^2}{2} & \theta & \frac{\theta^2}{2}+1 \end{pmatrix}.$$

See [7, p. 52]. By defining the curve $\gamma(v) = X(0,v)$, then it is immediate that $X(u,v) = \Omega(u)\cdot\gamma(v)$. Thus, the surface satisfies $X(u+\theta, v) = X(u,v)$ for any $\theta \in \mathbb{R}$, proving that this surface is rotational about L. This surface is called the timelike parabolic catenoid in the literature.

4. Timelike minimal surfaces based on a helix

This section is devoted to the construction of Björling surfaces when the core curve α is a helix. The discussion will be done according to the causal character of the axis. For convenience, due to the lengthy parametrizations, we only consider the case where the axis is timelike and spacelike. As stated in the introduction, and in contrast to the rotational timelike minimal surfaces of Section 3, we will obtain new examples of helicoidal surfaces for the choice $\varphi(s) = b$.

4.1. The axis is timelike

Up to a homothety, a nondegenerate helix with axis $(0,0,1)$ parametrizes as $\alpha(s) = (\cos(s), \sin(s), \lambda s)$, with $\lambda \in (0,1)\cup(1,\infty)$: if $\lambda = 0$, the helix degenerates in a circle and this case has been studied in Section 3. In case $\lambda \in (0,1)$, the curve α is spacelike, and if $\lambda \in (1,\infty)$, then α is timelike. We distinguish both cases. Let us observe that curve α is not parametrized by the arc-length, but recall that formula (1) holds provided the vector field V is unitary: the argument is the same in the spacelike case [1].

4.1.1. Spacelike helix

The normal vector $\mathbf{n}(s)$ and binormal vector $\mathbf{b}(s)$ of α are

$$\mathbf{n}(s) = -(\cos(s), \sin(s), 0), \quad \mathbf{b}(s) = \frac{1}{\mu}(\lambda\sin(s), -\lambda\cos(s), -1),$$

with $\mu = \sqrt{1-\lambda^2}$. A unit spacelike vector field $V(s)$ is expressed as $V(s) = \cosh\varphi(s)\mathbf{n}(s) + \sinh\varphi(s)\mathbf{b}(s)$. Consider $\varphi(s) = as + b$, $a, b \in \mathbb{R}$.

1. Case $a \neq 0$. Then the solution of (1) is

$$X(u,v) = \begin{pmatrix} \dfrac{\cos(u)A(u,v) - \sin(u)B(u,v)}{a^2+1} \\ \dfrac{\sin(u)A(u,v) + \cos(u)B(u,v)}{a^2+1} \\ \dfrac{\sinh(av)\cosh(au+b)}{a} + \lambda u \end{pmatrix}$$

$$[1.2ex] = \begin{pmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{A(u,v)}{a^2+1} \\ \dfrac{B(u,v)}{a^2+1} \\ \dfrac{\sinh(av)\cosh(au+b)}{a} + \lambda u \end{pmatrix}, \tag{15}$$

where

$$A(u,v) = \left((a^2+1)\cos(v) + \sinh(au+b)(\cos(v)(a\mu+\lambda)\sinh(av) + \sin(v)(\mu - a\lambda)\cosh(av))\right)$$

$$B(u,v) = \cosh(au+b)(\cos(v)(a\lambda-\mu)\sinh(av) + \sin(v)(a\mu+\lambda)\cosh(av)).$$

2. Case $a = 0$. The case that φ is a constant function gives a helicoidal surface. Indeed, if $\varphi(s) = b$, then (1) yields

$$X(u,v) = \begin{pmatrix} \cos(u)\cos(v) + \sin(v)(-\lambda\cosh(b)\sin(u) + \mu\cos(u)\sinh(b)) \\ \sin(u)\cos(v) + \sin(v)(\lambda\cosh(b)\cos(u) + \mu\sin(u)\sinh(b)) \\ \lambda u + v\cosh(b) \end{pmatrix}. \tag{16}$$

For any $\theta \in \mathbb{R}$, consider the helicoidal motion of axis $(0,0,1)$

$$\Psi^h(\theta) : (x,y,z) \longmapsto \Psi(x,y,z) + \begin{pmatrix} 0 \\ 0 \\ \lambda\theta \end{pmatrix}.$$

Define $\gamma(v) = X(0,v)$. Then it is immediate that $X(u,v) = \Psi^h(u)\cdot\gamma(v)$. Thus, we have $\Psi^h(\theta)\cdot X(u,v) = X(u+\theta,v)$ for all $\theta \in \mathbb{R}$, proving that the surface is invariant by the uniparametric group of helicoidal motions $\{\Psi^h(\theta) : \theta \in \mathbb{R}\}$; that is, the surface is helicoidal.

4.1.2. Timelike helix

The normal vector $\mathbf{n}(s)$ and binormal vector $\mathbf{b}(s)$ of the timelike helix are

$$\mathbf{n}(s) = -(\cos(s), \sin(s), 0) \quad , \quad \mathbf{b}(s) = \frac{1}{\mu}(\lambda\sin(s), -\lambda\cos(s), -1),$$

with $\mu = \sqrt{\lambda^2 - 1}$. A unit spacelike vector field $V(s)$ is expressed as $V(s) = \cos\varphi(s)\mathbf{n}(s) + \sin\varphi(s)\mathbf{b}(s)$. Let $\varphi(s) = as + b$, $a, b \in \mathbb{R}$.

1. Case $a \neq 0$ and $a \neq \pm 1$. Then the parametrization of the minimal surface is

$$X(u,v) = \begin{pmatrix} \cos(u)\cos(v) + \dfrac{(1+a)(\mu-\lambda)A(u,v) + (1-a)(\mu+\lambda)B(u,v)}{2(a^2-1)} \\ \sin(u)\cos(v) - \dfrac{(1+a)(\lambda-\mu)C(u,v) + (1-a)(\mu+\lambda)D(u,v)}{2(a^2-1)} \\ \lambda u + \dfrac{\cos(au+b)\sin(av)}{a} \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} A(u,v) &= \sin((a-1)u+b)\sin((1-a)v), \\ B(u,v) &= \sin((a+1)u+b)\sin((1+a)v), \\ C(u,v) &= \cos((a-1)u+b)\sin((1-a)v), \\ D(u,v) &= \cos((a+1)u+b)\sin((1+a)v). \end{aligned}$$

2. Case $a = \pm 1$. Solution (1) is

$$X(u,v) = \begin{pmatrix} \cos(u)\cos(v) + \dfrac{2av(\lambda - a\mu)\sin(b) - a(\lambda + a\mu)\sin(2au+b)\sin(2v)}{4} \\ \sin(u)\cos(v) + \dfrac{2v(\lambda - a\mu)\cos(b) + (\lambda + a\mu)\cos(2au+b)\sin(2v)}{4} \\ \lambda u + \cos(au+b)\sin(v) \end{pmatrix}. \tag{18}$$

3. Case $a = 0$. The case that $\varphi(s) = b$ is a constant function gives a helicoidal surface, which has the parametrization

$$X(u,v) = \begin{pmatrix} \cos(u)\cos(v) - \sin(v)(\lambda\cos(b)\sin(u) + \mu\cos(u)\sin(b)) \\ \sin(u)\cos(v) + \sin(v)(\lambda\cos(b)\cos(u) - \mu\sin(u)\sin(b)) \\ \lambda u + v\cos(b) \end{pmatrix}. \tag{19}$$

As in the previous case of the spacelike helix, the surface is invariant by the helicoidal group $\{\Psi^h(\theta) : \theta \in \mathbb{R}\}$; that is, $\Psi^h(\theta) \cdot X(u,v) = X(u+\theta, v)$ for all $\theta \in \mathbb{R}$.

4.2. The axis is spacelike

Assume that axis L is $(1,0,0)$. In this case, there are two types of spacelike helices, namely,

$$\begin{aligned} \alpha(s) &= (\lambda s, \cosh(s), \sinh(s)), \quad \lambda > 1, \text{(type I)}, \\ \alpha(s) &= (\lambda s, \sinh(s), \cosh(s)), \quad \lambda > 0, \text{(type II)}, \end{aligned}$$

and also a timelike helix whose parametrization is

$$\alpha(s) = (\lambda s, \cosh(s), \sinh(s)), \quad 0 < \lambda < 1.$$

Because the computations are similar to those in the case of the timelike axis, we will only give the parametrizations of the surfaces. Again we point out that curve α is not parametrized by the arc-length. If the function φ that defines the unit spacelike vector field $V(s)$ is a constant function, the Björling surface is a helicoidal surface. We omit the details.

4.2.1. Spacelike helix

4.2.1.1. Spacelike helix of type I

The normal vector $\mathbf{n}(s)$ and binormal vector $\mathbf{b}(s)$ are

$$\mathbf{n}(s)=(0,-\cosh(s),-\sinh(s)),\qquad \mathbf{b}(s)=-\frac{1}{\mu}(1,\lambda\sinh(s),\lambda\cosh(s)),$$

where $\mu=\sqrt{\lambda^2-1}$. Because $\mathbf{n}(s)$ is spacelike and $\mathbf{b}(s)$ is timelike, the unit spacelike vector field V is written as $V(s)=\cosh\varphi(s)\mathbf{n}(s)+\sinh\varphi(s)\mathbf{b}(s)$. Let us consider $\varphi(s)=as+b,\ a,b\in\mathbb{R}$.

1. Case $a\neq\pm1$ and $a\neq0$. The Björling surface is

$$X(u,v)=\begin{pmatrix} \lambda u-\dfrac{\cosh(au+b)\sinh(av)}{a}\\[2mm] \cosh(u)\cosh(v)+\dfrac{(1+a)(\mu-\lambda)A(u,v)+(1-a)(\mu+\lambda)B(u,v)}{2(a^2-1)}\\[2mm] \cosh(v)\sinh(u)+\dfrac{(1+a)(-\mu+\lambda)C(u,v)-(1-a)(\mu+\lambda)D(u,v)}{2(a^2-1)}\end{pmatrix},\tag{20}$$

where

$$\begin{aligned} A(u,v)&=\sinh((a-1)u+b)\sinh((1-a)v),\\ B(u,v)&=\sinh((a+1)u+b)\sinh((a+1)v),\\ C(u,v)&=\cosh((a-1)u+b)\sinh((1-a)v),\\ D(u,v)&=\cosh((a+1)u+b)\sinh((a+1)v).\end{aligned}$$

2. Case $a=\pm1$. The parametrization of the surface is

$$X(u,v)=\begin{pmatrix} \lambda u-\cosh(b+au)\sinh(v)\\[2mm] \cosh(u)\cosh(v)+\dfrac{2av(\lambda-a\mu)\sinh(b)-a(\lambda+a\mu)\sinh(2au+b)\sinh(2v)}{4}\\[2mm] \sinh(u)\cosh(v)-\dfrac{2v(\lambda-a\mu)\cosh(b)+a(a\lambda+\mu)\cosh(2au+b)\sinh(2v)}{4}\end{pmatrix}.\tag{21}$$

3. Case $a=0$. The parametrization of the surface is

$$X(u,v)=\begin{pmatrix} \lambda u-v\cosh(b)\\ \cosh(u)\cosh(v)-\sinh(v)(\mu\cosh(u)\sinh(b)+\lambda\cosh(b)\sinh(u))\\ \sinh(u)\cosh(v)-\sinh(v)(\mu\sinh(u)\sinh(b)+\lambda\cosh(b)\cosh(u))\end{pmatrix}.\tag{22}$$

If we denote

$$\Theta^h(\theta) : (x, y, z) \longmapsto \Theta(x, y, z) + \begin{pmatrix} \lambda\theta \\ 0 \\ 0 \end{pmatrix},$$

then $X(u, v) = \Theta^h(u) \cdot \gamma(v)$, with $\gamma(v) = X(0, v)$. Then $\Theta^h(\theta) \cdot X(u, v) = X(u + \theta, v)$ for all $\theta \in \mathbb{R}$, proving that the surface is helicoidal.

4.2.1.2. Spacelike helix of type II

Now the normal vector $\mathbf{n}(s)$ and binormal vector $\mathbf{b}(s)$ are given by

$$\mathbf{n}(s) = (0, -\sinh(s), -\cosh(s)), \qquad \mathbf{b}(s) = \frac{1}{\mu}(1, -\lambda\cosh(s), -\lambda\sinh(s)),$$

where $\mu = \sqrt{\lambda^2 + 1}$. A unit spacelike vector field $V(s) = \sinh\varphi(s)\mathbf{n}(s) + \cosh\varphi(s)\mathbf{b}(s)$. Let $\varphi(s) = as + b$, $a, b \in \mathbb{R}$.

1. Case $a \neq \pm 1$. The Björling surface is

$$X(u, v) = \begin{pmatrix} \lambda u + \dfrac{\sinh(au + b)\sinh(av)}{a} \\ \sinh(u)\cosh(v) + \dfrac{(1 + a)(\mu - \lambda)A(u, v) + (1 - a)(\lambda + \mu)B(u, v)}{2(a^2 - 1)} \\ \cosh(u)\cosh(v) + \dfrac{(1 + a)(\mu - \lambda)C(u, v) + (1 - a)(\lambda + \mu)D(u, v)}{2(a^2 - 1)} \end{pmatrix}. \tag{23}$$

2. Case $a = \pm 1$. Then

$$X(u, v) = \begin{pmatrix} \lambda u + \sinh(v)\sinh(au + b) \\ \sinh(u)\cosh(v) - \dfrac{2v(\lambda - a\mu)\sinh(b) + (\lambda + a\mu)\sinh(2au + b)\sinh(2v)}{4} \\ \cosh(u)\cosh(v) - \dfrac{2v(-a\lambda + \mu)\cosh(b) + (a\lambda + \mu)\cosh(2au + b)\sinh(2v)}{4} \end{pmatrix}. \tag{24}$$

3. Case $a = 0$. If $\varphi(s) = b$, then

$$X(u, v) = \begin{pmatrix} \lambda u + v\sinh(b) \\ \sinh(u)\cosh(v) - \sinh(v)(\lambda\cosh(u)\sinh(b) + \mu\cosh(b)\sinh(u)) \\ \cosh(u)\cosh(v) - \sinh(v)(\mu\cosh(b)\cosh(u) + \lambda\sinh(b)\sinh(u)) \end{pmatrix}. \tag{25}$$

Similarly to parametrization (22), we find $\Theta^h(\theta) \cdot X(u, v) = X(u + \theta, v)$ for all $\theta \in \mathbb{R}$, proving that the surface is helicoidal.

4.2.2. Timelike helix

The normal vector $\mathbf{n}(s)$ and binormal vector $\mathbf{b}(s)$ of the timelike helix are

$$\mathbf{n}(s) = (0, \cosh(s), \sinh(s)), \qquad \mathbf{b}(s) = -\frac{1}{\mu}(1, \lambda\sinh(s), \lambda\cosh(s)),$$

where $\mu = \sqrt{1-\lambda^2}$. Because $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are spacelike, the unit spacelike vector field V can be written as $V(s) = \cos\varphi(s)\mathbf{n}(s) + \sin\varphi(s)\mathbf{b}(s)$. Let $\varphi(s) = as + b$, $a, b \in \mathbb{R}$.

1. Case $a \neq 0$. The Björling surface is

$$X(u,v) = \begin{pmatrix} \lambda u + \dfrac{\cos(au+b)\sin(av)}{a} \\ \cosh(u)\cosh(v) + \dfrac{\cosh(u)\sin(au+b)A(v) + \sinh(u)\cos(au+b)B(v)}{a^2+1} \\ \cosh(v)\sinh(u) + \dfrac{\sinh(u)\sin(au+b)A(v) + \cosh(u)\cos(au+b)B(v)}{a^2+1} \end{pmatrix}, \tag{26}$$

   where

$$\begin{aligned} A(v) &= (-\lambda + a\mu)\cosh(v)\sin(av) + (a\lambda + \mu)\cos(av)\sinh(v), \\ B(v) &= (a\lambda + \mu)\cosh(v)\sin(av) + (\lambda - a\mu)\cos(av)\sinh(v). \end{aligned}$$

2. Case $a = 0$. Then

$$X(u,v) = \begin{pmatrix} \lambda u + v\cos(b) \\ \cosh(u)\cosh(v) + \sinh(v)(\mu\cosh(u)\sin(b) + \lambda\cos(b)\sinh(u)) \\ \sinh(u)\cosh(v) + \sinh(v)(\mu\sinh(u)\sin(b) + \lambda\cos(b)\cosh(u)) \end{pmatrix}. \tag{27}$$

   As in the case of spacelike helices of type I and II, this surface is helicoidal.

Remark 4.1 *It is not difficult to see that in this section, the case $a = 0$ (in (16), (19), (22), (25), and (27)) and the case $a = \pm 1$ (in (18), (21), and (24)) coincide with parametrizations (15), (17), (20), (23), and (26) by letting $a \to 0$ or $a \to \pm 1$ depending on the case.*

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