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On the asymptotic behavior of solution of certain systems of Volterra equations

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Abstract: This paper is concerned with the asymptotic property of the solution of a system of the linear Volterra difference equations. The criterion for the existence of a solution of the considered system that is asymptotically equivalent to a given sequence is established. The results presented here improve and generalize the results published by Diblik et al. Unlike in those works, here periodicity of the nonhomogeneous term of the equation is not assumed. Examples illustrate the obtained results.

Key words: Linear Volterra difference equation, asymptotic equivalence

1. Introduction and notation

We consider a Volterra system of difference equations

$$X(n+1) = A(n) + B(n)X(n) + \sum_{i=0}^{n} K(n,i)X(i) \tag{1}$$

where $n \in \mathbb{N} := \{0, 1, 2, \dots\}$. Let $\mathbb{R}$ denote the set of real numbers. Here $A = (a_1, a_2, \dots, a_r)^T$, $B = \operatorname{diag}(b_1, \dots, b_r)$, and $X = (x_1, x_2, \dots, x_r)^T$, where $a_s, x_s \colon \mathbb{N} \to \mathbb{R}$ and $b_s \colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$, $s = 1, \dots, r$. Moreover $K = (K_{sp})_{s,p=1,\dots,r}$, where $K_{sp} \colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ for $s, p \in \{1, \dots, r\}$. By a solution of system (1) we mean a sequence $X = (x_1, x_2, \dots, x_r)^T$ whose terms satisfy (1) for every $n \in \mathbb{N}$.

In the last years, there has been an interest among many authors to study the asymptotic behavior of solutions of Volterra difference equations. The results were published, e.g., by Appleby et al. [1], by Appleby and Patterson [2], by Berezansky et al. [3], by Gajda et al. [7], by Gil and Medina [8], by Gronek and Schmeidel [9], by Györi and Horváth [10], by Györi and Reynolds [11], by Medina [12, 13], by Migda and Migda [14, 17], by Migda et al. [18], and by Song and Baker [19].

In 2018, Migda and Migda [15] presented a new approach to the theory of asymptotic properties of solutions to nonlinear discrete Volterra equations in which a higher order forward difference operator is involved. The method of proof is based on using the iterated remainder operator and asymptotic difference pairs. This approach allows the authors to control the degree of approximation. Migda et al. [16], also in 2018, obtained sufficient conditions for the existence of the solutions with prescribed asymptotic behavior.

The following definition and lemma will be used in the sequel.

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Definition 1 (asymptotically equivalent sequences) *Sequences $x\colon \mathbb{N} \to \mathbb{R}$ and $y\colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ are called asymptotically equivalent if*

$$\lim_{n\to\infty} \frac{x(n)}{y(n)} = 1. \tag{2}$$

If two sequences x and y satisfy Definition 1, notation $x(n) \sim y(n)$ is often used instead of equality (2).

Lemma 1 (Schauder's fixed point theorem, [4]) *Let S be a nonempty, convex, and compact subset of a Banach space and let $T\colon S \to S$ be continuous. Then T has a fixed point in S.*

For $s, k \in \{1, 2, \dots, r\}$, we set

$$\beta_s(n)\colon = \prod_{i=0}^{n-1} b_s(i), \;\; \tilde{a}_s(n)\colon = \sum_{i=0}^{n-1} \frac{a_s(i)}{\beta_s(i+1)}$$

and

$$M_{sk}(n) := \sum_{j=n}^{\infty} \sum_{i=0}^{j} \left| \frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)} \right|. \tag{3}$$

We define matrix $\tilde{M} := (M_{sp}(0))_{s,p=1,\dots,r}$ and vector

$$\tilde{A} := (\sup_{n\in\mathbb{N}} |\tilde{a}_1(n)|, \dots, \sup_{n\in\mathbb{N}} |\tilde{a}_r(n)|)^T.$$

Let us take $c_s \in \mathbb{R}$, $s = 1, \dots, r$ and vector $C = (c_1, c_2, \dots, c_r)^T$. Moreover, $\mathbb{I}$ stands for an $r \times r$ unit matrix. By D_s we denote the matrix formed by replacing the s-th column of $(\mathbb{I} - \tilde{M})$ by the column vector $\tilde{M}(C + \tilde{A})$.

2. Asymptotic formula for solutions of system (1)

In the following theorem we present the sufficient conditions for the existence of the solution of system (1), which is equivalent to a given sequence.

Theorem 1 *Assume that $M_{sk}(0) < \infty$ for all $s, k \in \{1, \dots, r\}$. If for some given constant vector C and $s = 1, \dots, r$, inequalities*

$$\det(\mathbb{I} - \tilde{M}) \neq 0, \quad \frac{\det D_s}{\det(\mathbb{I} - \tilde{M})} > 0, \;\; c_s + \frac{\det D_s}{\det(\mathbb{I} - \tilde{M})} + \sup_{n\in\mathbb{N}} |\tilde{a}_s(n)| > 0 \tag{4}$$

are satisfied, then there exists a solution $X\colon \mathbb{N} \to \mathbb{R}^r$ of (1) such that

$$x_s(n) \sim (c_s + \tilde{a}_s(n))\, \beta_s(n) \tag{5}$$

for any $s = 1, \dots, r$.

Proof We define a vector $(\alpha_1(n), \dots, \alpha_r(n))^T$, $n \in \mathbb{N}$ as

$$\alpha_s(n) = \sum_{k=1}^{r} M_{sk}(n) \left(c_k + \sup_{n\in\mathbb{N}} |\tilde{a}_k(n)| + \alpha_k(0) \right). \tag{6}$$

The terms $\alpha_s(0)$, $s = 1, 2, \ldots, r$ are well defined since (6) for $n = 0$ turns into a system regarding $\alpha_s(0)$:

$$\alpha_s(0) = \sum_{k=1}^{r} M_{sk}(0) \left(c_k + \sup_{n\in\mathbb{N}} |\tilde{a}_k(n)| + \alpha_k(0) \right), \quad s = 1, 2, \ldots, r,$$

which can be rewritten in the form

$$(\mathbb{I} - \tilde{M})\alpha = \tilde{M}(C + \tilde{A}), \tag{7}$$

where $\alpha = (\alpha_1(0), \alpha_2(0), \ldots, \alpha_r(0))^T$. Since $\det(\mathbb{I} - \tilde{M}) \neq 0$, system (7) has a unique solution α, where by (4),

$$\alpha_s(0) = \frac{\det D_s}{\det(\mathbb{I} - \tilde{M})} > 0, \;\; s = 1, 2, \ldots, r.$$

According to the assumption $M_{sk}(0) < \infty$ for $s, k \in \{1, \ldots, r\}$, we have that $\lim_{n\to\infty} M_{sk}(n) = 0$ for $s = 1, \ldots, r$. Hence, by (6), we note that also

$$\lim_{n\to\infty} \alpha_s(n) = 0, \;\; s = 1, 2, \ldots, r. \tag{8}$$

From (3), we get

$$M_{sk}(n) \le M_{sk}(0) \;\; \text{for} \;\; n \in \mathbb{N}.$$

Therefore, for $n \in \mathbb{N}$

$$\sum_{k=1}^{r} M_{sk}(0) \left(c_k + \sup_{n\in\mathbb{N}} |\tilde{a}_k(n)| + \alpha_k(0) \right) \ge \sum_{k=1}^{r} M_{sk}(n) \left(c_k + \sup_{n\in\mathbb{N}} |\tilde{a}_k(n)| + \alpha_k(0) \right).$$

It means that $\alpha_s(0) \ge \alpha_s(n)$, $s = 1, 2, \ldots, r$, for any $n \in \mathbb{N}$.

Let $\mathcal{B}$ be the Banach space of all real bounded sequences

$$z = (z_1, z_2, \ldots, z_r)^T \colon \mathbb{N} \to \mathbb{R}^r$$

equipped with the usual supremum norm. We define a subset $S \subset \mathcal{B}$ as

$$S := \{z(n) \in \mathcal{B} \colon |z_s(n) - (c_s + \tilde{a}_s(n))| \le \alpha_s(0), \;\; s = 1, 2, \ldots, r, \, n \in \mathbb{N}\}.$$

It is not difficult to prove that S is the nonempty, convex, and compact subset of $\mathcal{B}$.

Let us define a mapping $T \colon S \to \mathcal{B}$, $T = (T_1, T_2, \ldots, T_r)$ in the following way:

$$(T_s z)(n) = c_s + \tilde{a}_s(n) - \sum_{k=1}^{r} \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)} z_k(i), \tag{9}$$

for any $s = 1, 2, \ldots, r$ and $n \in \mathbb{N}$.

We will prove that the mapping T has a fixed point in $\mathcal{B}$. We first show that $T(S) \subset S$. For $z \in S$, we get

$$\begin{aligned}|(T_s z)(n) - c_s - \tilde{a}_s(n)| =& \Big|\sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}z_k(i)\Big| \\ \leq& \sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}\right||z_k(i)| \\ \leq& \sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}\right|(c_k + \tilde{a}_k(i) + \alpha_k(0)) \\ \leq& \sum_{k=1}^{r} M_{sk}(n)\big(c_k + \sup_{n\in\mathbb{N}}|\tilde{a}_k(n)| + \alpha_k(0)\big) \\ =& \alpha_s(n) \leq \alpha_s(0), \end{aligned} \tag{10}$$

for any $s = 1, 2, \ldots, r$ and $n \in \mathbb{N}$.

Next, we prove that T is continuous. Let $z^{(q)}$ be a sequence in S such that $z^{(q)} \to z$ as $q \to \infty$. Since S is closed, $z \in S$. By (9), we get

$$\begin{aligned}|(T_s z^{(q)})(n) - (T_s z)(n)| \leq& \sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}\right|\left|z_k^{(q)}(i) - z_k(i)\right| \\ \leq& \sum_{k=1}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}\right|\sup_{i\in\mathbb{N}}|z_k^{(q)}(i) - z_k(i)| \\ \leq& \sum_{k=1}^{r} M_{sk}(0)\sup_{i\in\mathbb{N}}|z_k^{(q)}(i) - z_k(i)| \\ \leq& r \cdot \max_{s,p\in\{1,2,\ldots,r\}}\{M_{sp}(0)\} \cdot \|z^{(q)} - z\|, \end{aligned}$$

and $\lim_{q\to\infty}\|Tz^{(q)} - Tz\| = 0$. This means that T is continuous.

By Schauder's fixed point theorem (see Lemma 1), there exists $z \in S$ such that $z(n) = (Tz)(n)$ for any $n \in \mathbb{N}$. Thus,

$$z_s(n) = c_s + \tilde{a}_s(n) - \sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}z_k(i) \tag{11}$$

for any $s = 1, 2, \ldots, r$ and $n \in \mathbb{N}$.

Due to (8) and (10),

$$\lim_{n\to\infty}|z_s(n) - c_s - \tilde{a}_s(n)| = \lim_{n\to\infty}|T_s z(n) - c_s - \tilde{a}_s(n)| = 0.$$

Hence,

$$z_s(n) \sim c_s + \tilde{a}_s(n), \qquad s = 1, 2, \ldots, n.$$

Finally, we will show that there exists a connection between the fixed point $z \in S$ and the existence of the solution of (1). Considering (11) for $z_s(n+1)$ and $z_s(n)$, $s = 1, 2, \ldots, r$, we get

$$\begin{aligned}\Delta(z_s(n)) =& z_s(n+1) - z_s(n)\\ =& \tilde{a}_s(n+1) - \tilde{a}_s(n) - \sum_{k=1}^{r}\sum_{j=n+1}^{\infty}\sum_{i=0}^{j}\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}z_k(i)\\ &+\sum_{k=1}^{r}\sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K_{sk}(j,i)\beta_k(i)}{\beta_s(j+1)}z_k(i)\\ =& \tilde{a}_s(n+1) - \tilde{a}_s(n) + \sum_{k=1}^{r}\sum_{i=0}^{n}\frac{K_{sk}(n,i)\beta_k(i)}{\beta_s(n+1)}z_k(i), \ n \in \mathbb{N}.\end{aligned}$$

Putting $z_s(n) = \frac{x_s(n)}{\beta_s(n)}$, we obtain

$$\frac{x_s(n+1)}{\beta_s(n+1)} - \frac{x_s(n)}{\beta_s(n)} = \frac{a_s(n)}{\beta_s(n+1)} + \sum_{k=1}^{r}\sum_{i=0}^{n}\frac{K_{sk}(n,i)}{\beta_s(n+1)}x_k(i).$$

It yields

$$x_s(n+1) = a_s(n) + b_s(n)x_s(n) + \sum_{k=1}^{r}\sum_{i=0}^{n}K_{sk}(n,i)x_k(i)$$

for $s = 1, 2, \ldots, r$.

Consequently, X is a solution of (1), and formula (5) is satisfied. $\square$

Putting $r = 1$, and assuming periodicity of B and $\max\limits_{s,p\in\{1,2,\ldots,r\}}\{M_{sp}(0)\} \in (0,1)$, Theorem 1 reduces to Theorem 1 [5].

Theorem 1 improves and generalizes Theorem 1 of [6] in three ways, namely:

1) we do not require ω-periodicity of sequence B sequence;
2) since $\sup\limits_{n\in\mathbb{N}}|a_s(n)|$ is nonnegative, our assumption

$$c_s + \frac{\det D_s}{\det(\mathbb{I}-\tilde{M})} + \sup_{n\in\mathbb{N}}|\tilde{a}_s(n)| > 0$$

   is weaker than assumption $c_s + \frac{\det D_s}{\det(\mathbb{I}-\tilde{M})} > 0$ employed in [6];
3) assumption $c_1c_2\ldots c_r \neq 0$ is omitted.

It means that if just one condition (or even two conditions) chosen from conditions

- sequence B is ω-periodic,
- $c_s + \frac{\det D_s}{\det(\mathbb{I}-\tilde{M})} > 0$,
- $c_1c_2\ldots c_r \neq 0$,

is not satisfied, Theorem 1 still generalizes Theorem 1 of [6]. To obtain Theorem 1 of [6] as a special case of Theorem 1 presented here, all the above conditions should be fulfilled.

The following examples illustrate three cases of system (1) for which Theorem 1 of [6] is not applicable, but our Theorem 1 is. In each example there is different reason why Theorem 1 of [6] cannot be applied.

Example 1 *Let us consider system (1) where $r = 2$ and*

$$a_1(n) = -a_2(n) = \frac{9(-1)^n}{10^{n+1}}, \qquad b_1(n) = b_2(n) = -1,$$

$$K_{11}(n,i) = -K_{22}(n,i) = \frac{3(-1)^i 2^i}{2^{2n+3}}, \qquad K_{12}(n,i) = -K_{21}(n,i) = \frac{9(-1)^i 5^i}{2^{n+2}5^{n+1}},$$

for $i, n \in \mathbb{N}$. It means that (1) takes the following form:

$$\begin{cases} x_1(n+1) = \dfrac{9(-1)^n}{10^{n+1}} - x_1(n) + \displaystyle\sum_{i=0}^{n} \frac{3(-1)^i 2^i}{2^{2n+3}} x_1(i) + \sum_{i=0}^{n} \frac{9(-1)^i 5^i}{2^{n+2}5^{n+1}} x_2(i) \\ x_2(n+1) = \dfrac{9(-1)^{n+1}}{10^{n+1}} - x_2(n) - \displaystyle\sum_{i=0}^{n} \frac{9(-1)^i 5^i}{2^{n+2}5^{n+1}} x_1(i) - \sum_{i=0}^{n} \frac{3(-1)^i 2^i}{2^{2n+3}} x_2(i). \end{cases} \tag{12}$$

Hence, we have

$$\beta_s(n) = \prod_{i=0}^{n-1} b_s(i) = (-1)^n, \quad \sup_{n\in\mathbb{N}} |\tilde{a}_s(n)| = \sum_{i=0}^{\infty} \frac{9}{10^{i+1}} = 1 \text{ for } s = 1, 2,$$

$$M_{11}(0) = M_{22}(0) = 3 \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{2^i}{2^{2j+3}} = 1,$$

$$M_{12}(0) = M_{21}(0) = 9 \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{5^i}{2^{j+2}5^{j+1}} = 1,$$

$\tilde{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, *and* $\tilde{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Taking* $C = \left(-\frac{3}{2}, -\frac{3}{2}\right)^T$, *we get* $\tilde{M}(C + \tilde{A}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

and $\mathbb{I} - \tilde{M} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\det(\mathbb{I} - \tilde{M}) = -1$.

Further, $D_1 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$, $\det D_1 = -1$ *and* $D_2 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$, $\det D_2 = -1$. *We get* $\frac{\det D_s}{\det(\mathbb{I} - \tilde{M})} = 1 > 0$, $c_s + \frac{\det D_s}{\det(\mathbb{I} - \tilde{M})} + \sup_{n\in\mathbb{N}} |\tilde{a}_s(n)| = -\frac{3}{2} + 1 + 1 = \frac{1}{2} > 0$, $s = 1, 2$.

All the assumptions of Theorem 1 are satisfied, so there exists a solution $x \colon \mathbb{N} \to \mathbb{R}^2$ such that

$$x_1(n) \sim \left(-\frac{3}{2} - \left(1 - \frac{1}{10^n}\right)\right)(-1)^n = \left(\frac{5}{2} - \frac{1}{10^n}\right)(-1)^{n+1}$$

and

$$x_2(n) \sim \left(-\frac{3}{2} + \left(1 - \frac{1}{10^n}\right)\right)(-1)^n = \left(\frac{1}{2} + \frac{1}{10^n}\right)(-1)^{n+1}.$$

Notice that in Example 1 we have $c_s + \frac{\det D_s}{\det(\mathbb{I}-\tilde{M})} = -\frac{3}{2}+1 = -\frac{1}{2} < 0$. It means that Theorem 1 of [6] is not applicable here.

In the next two examples periodicity of B is not assumed.

Example 2 *Let us consider a scalar equation of the form* (1)

$$x(n+1) = \frac{n+2}{2^n} + \frac{n+2}{n+1}x(n) + \sum_{i=0}^{n} \frac{n+2}{(i+1)2^{n+i}}x(i). \tag{13}$$

Let us take $c < -2$*. Here* $\beta(n) = n+1$, $\tilde{a}(n) = 2 - \frac{1}{2^{n-1}}$, $\sup_{n\in\mathbb{N}} |\tilde{a}(n)| = 2$, $\tilde{M} = \left[\frac{8}{3}\right]$, $\tilde{M}(C+\tilde{A}) = \left[\frac{8}{3}(c+2)\right]$, $\det(\mathbb{I}-\tilde{M}) = -\frac{5}{3}$, $\det D_1 = \frac{8}{3}(c+2)$. *All assumptions of Theorem 1 are satisfied. Hence, equation* (13) *has a solution* $x(n) \sim \left(c+2-\frac{1}{2^{n-1}}\right)(n+1)$.

Additionally, for system (13), Theorem 1 is applicable whereas Theorem 1 of [5] is not, because of the assumption $\max_{s,p\in\{1,2,\ldots,r\}} \{M_{sp}(0)\} \in (0,1)$.

Example 3 *Put* $r = 2$ *and consider system* (1) *with* $C = (0,0)^T$ *and*

$$\begin{aligned} &a_1(n) = \frac{(n+1)!}{2^n}, \ \ a_2(n) = \frac{4(-1)^n}{3^n}, \quad b_1(n) = n+1, \ \ b_2(n) = 1, \\ &K_{11}(n,i) = (-1)^{n+i}\frac{(n+1)!}{i!\,2^{n+i+3}}, \ \ K_{12}(n,i) = (-1)^i\frac{(n+1)!}{2^{n+i+3}} \\ &K_{21}(n,i) = \frac{(-1)^n}{i!\,2^{n+i+3}}, \ \ K_{22}(n,i) = \frac{1}{2^{n+i+3}}. \end{aligned} \tag{14}$$

Hence, $\beta_1(n) = n!$, $\beta_2(n) = 1$, $\tilde{a}_1(n) = 2-\frac{1}{2^{n-1}}$, $\tilde{a}_2(n) = 3+\left(-\frac{1}{3}\right)^{n-1}$. *Since* $\sup_{n\in\mathbb{N}} |\tilde{a}_1(n)| = 2$, $\sup_{n\in\mathbb{N}} |\tilde{a}_2(n)| = 4$, *and* $M_{11}(0) = M_{12}(0) = M_{21}(0) = M_{22}(0) = \frac{1}{3}$, *we have* $\tilde{M} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, *and* $\mathbb{I}-\tilde{M} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$, $\det(\mathbb{I}-\tilde{M}) = \frac{1}{3} > 0$.

Let us take $c_1 = 0$, $c_2 = 0$*. Then* $\tilde{M}(C+\tilde{A}) = [2,2]^T$ *and* $D_1 = \begin{bmatrix} 2 & -\frac{1}{3} \\ 2 & \frac{2}{3} \end{bmatrix}$, $\det D_1 = 2$, $D_2 = \begin{bmatrix} \frac{2}{3} & 2 \\ -\frac{1}{3} & 2 \end{bmatrix}$, $\det D_2 = 2$. *All the assumptions of Theorem 1 are satisfied, so there exists a solution* $x\colon \mathbb{N} \to \mathbb{R}^2$ *such that*

$$x_1(n) \sim \left(2-\frac{1}{2^{n-1}}\right)n! \ \ \textit{and} \ \ x_2(n) \sim \left(3+\left(-\frac{1}{3}\right)^{n-1}\right).$$

For the system of difference equations given by (14), Theorem 1 of [5] cannot be applied since $c_1c_2 = 0$.

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