Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 3002 – 3009
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doi:10.3906/mat-1807-77

On Hochstadt–Lieberman theorem for impulsive Sturm–Liouville problems with boundary conditions polynomially dependent on the spectral parameter

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Received: 10.07.2018 • **Accepted/Published Online:** 25.09.2018 • **Final Version:** 27.11.2018

Abstract: In the present paper, we consider an inverse problem for the Sturm–Liouville operator with a finite number of discontinuities at interior points and boundary conditions polynomially dependent on the spectral parameter on an arbitrary finite interval, and prove the Hochstadt–Lieberman-type theorem for this problem.

Key words: Sturm–Liouville problem, interior discontinuities, Hochstadt–Lieberman theorem, boundary conditions polynomially dependent on the spectral parameter

1. Introduction

We consider the boundary value problem $\mathcal{S}$ generated by the second-order differential equation of Sturm–Liouville (S-L) type

$$y'' + (\lambda - q(x))y = 0 \tag{1.1}$$

for $x \in [a_0, b_0]$, with the boundary conditions

$$\begin{cases} a(\lambda)y'(a_0,\lambda) - b(\lambda)y(a_0,\lambda) = 0, \\ c(\lambda)y'(b_0,\lambda) - d(\lambda)y(b_0,\lambda) = 0, \end{cases} \tag{1.2}$$

and the transmission (discontinuous) conditions

$$\begin{cases} y(x_p+0) = \alpha_p y(x_p-0), \quad p = 1,2,3,...,\ell, \\ y'(x_p+0) = \alpha_p^{-1} y'(x_p-0), \quad p = 1,2,3,...,\ell, \end{cases} \tag{1.3}$$

where λ is the spectral parameter, q is a real-valued function in $L_2(a_0,b_0)$, $\alpha_p \in \mathbb{R}$ and $\alpha_p \neq 0$ for $p = 1,2,3,...,\ell$, $a_0 < x_1 < x_2 < ... < x_{k-1} < x_k = \frac{a_0+b_0}{2} < x_{k+1} < ... < x_\ell < b_0$, $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ are real polynomials as follows:

$$a(\lambda) = \sum_{j=1}^{m} a_j\lambda^j,\ b(\lambda) = \sum_{j=1}^{m} b_j\lambda^j,\ c(\lambda) = \sum_{j=1}^{r} c_j\lambda^j,\ d(\lambda) = \sum_{j=1}^{r} d_j\lambda^j. \tag{1.4}$$

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2010 *AMS Mathematics Subject Classification:* Primary: 34A55, 34B24; Secondary: 34L05, 47E05

Without loss of generality, we assume that $a_m = c_r = 1$ and $\int_{a_0}^{b_0} q(x)dx = 0$.

S-L problems with interior discontinuities arise from several models such as quantum mechanics models (for example, in the description of delta interactions [1]), physical or geophysical models, and quantum physics (for example, in the oscillation of the Earth [2, 11], or the description of radially symmetric quantum trees [15]).

Inverse problems for the S-L equation (1.1) without discontinuity were studied under various conditions on the potential $q(x)$ by several mathematicians (for example, see [10]). In the case where the problem has one transmission condition, the asymptotic formulas for the eigenvalues and the eigenfunctions were investigated in [13, 14, 16, 17] and the references therein. Moreover, by Weyl–Titchmarsh $\mathfrak{M}$-function, the uniqueness of the solution for the inverse problem $\mathcal{S}$ with Robin boundary conditions (i.e. $a(\lambda) = c(\lambda) = 1$, $b(\lambda) = h$, $d(\lambda) = -H$, where h and H are real numbers) and an arbitrary number of transmission conditions on the interval $(0, 1)$ were studied in [18]. Recently, the asymptotic forms of eigenvalues and eigenfunctions of $\mathcal{S}$ with one discontinuity at $x = 1/2 \in (0, 1)$ were obtained by Keskin and Ozkan [8], and the potential $q(x)$ was reconstructed from nodal points (zeros of eigenfunctions).

In 1987, Hochstadt and Lieberman considered equation (1.1) on (0,1) with Robin boundary conditions, where $q \in L_1(0, 1)$. They proved that the spectrum of the problem and $q|_{(\frac{1}{2},1)}$ determine $q(x)$ uniquely (see [7]). Next, some mathematicians obtained more results and generalized Hochstadt and Lieberman's results under various conditions on S-L operators. In [6], Hald proved a Hochstadt–Lieberman result in the case where the problem has one discontinuous condition. Then, for S-L boundary value problems (BVPs) with a reflection symmetry, Kobayashi proved a similar result [9]. Later, Gesztesy and Simon by partial spectrum and information of $q(x)$ established a generalization of the Hochstadt–Lieberman theorem [5]. Shieh et al. presented some Hochstadt–Lieberman-type theorems for $\mathcal{S}$ on $(0, 1)$ under Robin boundary conditions with arbitrary finite number of discontinuities [19]. Also, in [20], Wang and Koyunbakan studied this for discontinuous BVPs with boundary conditions linearly dependent on the spectral parameter.

The purpose of the presented paper is to discuss some Hochstadt–Lieberman-type theorems for S-L BVPs with an arbitrary finite number of transmission conditions on finite intervals and boundary conditions polynomially dependent on the spectral parameter. First, we present a Hochstadt–Lieberman-type theorem in the case of one transmission condition at $x = 1/2$ inside the interval $(0, 1)$. Then we will generalize our results for the BVP $\mathcal{S}$ with discontinuities at the points $x_1, x_2, ..., x_\ell \in (a_0, b_0)$.

2. Preliminaries

Let $s(x, \lambda)$ and $u(x, \lambda)$ be the solutions of (1.1) on the interval $(0, 1)$ satisfying the initial conditions

$$s(0, \lambda) = a(\lambda), \quad s'(0, \lambda) = b(\lambda), \quad u(1, \lambda) = c(\lambda), \quad u'(1, \lambda) = d(\lambda).$$

Denote the Weyl–Titchmarsh $\mathfrak{M}$-function corresponding to (1.1) as follows:

$$\mathfrak{M}(\lambda) = \frac{s(1, \lambda)}{s'(1, \lambda)},$$

which is a meromorphic function. We know that $\mathfrak{M}$ uniquely determines the potential $q(x)$ (see [3, 18]). Moreover, it is not difficult to show that the solution $s(x, \lambda)$ satisfies the following integral equation (for more details see [12]):

$$s(x, \lambda) = a(\lambda) \cos \sqrt{\lambda}x + b(\lambda) \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin \sqrt{\lambda}(x - t)}{\sqrt{\lambda}} q(t)s(t, \lambda)dt.$$

In order to prove our main results, we need the following lemma.

Lemma 2.1 *([8]) Let $q \in L_2(0,1)$. Then the following asymptotic formula holds:*

$$s(x,\lambda) = \lambda^m\{\cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}(b_m + \frac{1}{2}\int_0^x q(t)dt) + o(\frac{1}{\sqrt{\lambda}}\exp(\zeta x))\}$$

as $|\lambda| \to \infty$, where $\zeta = |Im\sqrt{\lambda}|$. Moreover, in the case where $a_0 = 0$, $b_0 = 1$ and the problem (1.1)–(1.3) has only one transmission condition at $x = 1/2$ (i.e. $p = 1$ and $\alpha_p = \alpha$), the eigenvalues of $ satisfy the following asymptotic representation as $n \to \infty$:

$$\sqrt{\lambda_n(q)} = (n-m-r)\pi + \frac{b_m - d_r - (-1)^{n-m-r}\omega}{(n-m-r)\pi} + o(\frac{1}{n}), \tag{2.1}$$

where

$$\omega = \frac{\alpha^-}{\alpha^+}\{b_m + d_r + \frac{1}{2}\int_0^{\frac{1}{2}} q(t)dt - \frac{1}{2}\int_{\frac{1}{2}}^1 q(t)dt\}, \quad \alpha^{\pm} = \frac{1}{2}(\alpha \pm \frac{1}{\alpha}).$$

For our analysis, we also need the following lemma to establish some Hochstadt–Lieberman-type theorems for $.

Lemma 2.2 *Let $f(z)$ be an entire function that satisfies the following:*

(1) $\sup_{|z|=R_k}|f(z)| \leq C_1\exp(C_2R_k^\rho)$ *for some $0 < \rho < 1$, $C_1, C_2 > 0$, and some sequences $R_k \to \infty$ as $k \to \infty$.*

(2) $\lim_{|x|\to\infty}|f(ix)| = 0$.

Then $f \equiv 0$.

Proof See Proposition B.6 of [5]. □

3. Main results

In this section, first we prove a Hochstadt–Lieberman-type theorem for S-L problems with one interior discontinuity and boundary conditions polynomially dependent on the spectral parameter. Then we will generalize the results of our study to the S-L problems with an arbitrary finite number of transmission conditions on finite intervals.

We denote $N_b^a := \{a_j, b_j\}_{j=1}^m$, $N_d^c := \{c_j, d_j\}_{j=1}^r$, and the characteristic function of the problem $_0 consisting of (1.1)–(1.2) as follows:

$$\begin{aligned}\Delta(N_b^a, N_d^c, q)(\lambda) =& W(u,s)(x,\lambda) = W(u,s)(\frac{1}{2},\lambda)\\ =& A\prod_{n=0}^{\infty}(1 - \frac{\lambda}{\lambda_n^0}),\end{aligned} \tag{3.1}$$

where $W(u,s)(x,\lambda)$ is the Wronskian of u and s, $\{\lambda_n^0\}$ is the spectrum of $\$_0$, and A is a constant. According to Lemma 2.1, we have

$$\Delta(N_b^a, N_d^c, q)(\lambda) = \lambda^m\{-\sqrt{\lambda}\sin\sqrt{\lambda} + (b_m - d_r)\cos\sqrt{\lambda} + O(\frac{1}{\sqrt{\lambda}}\exp(\zeta))\}.$$

Further, $|\sin\sqrt{\lambda}| \geq A_\sigma \exp(\zeta)$ for $\sigma \in (0,\pi)$ and $\sqrt{\lambda} \in \Gamma_\sigma := \{\sqrt{\lambda} \in \mathbb{C} \ : \ |\sqrt{\lambda} - n\pi| > \sigma, \ n \in \mathbb{Z}\}$. Thus, we get for $\sqrt{\lambda} \in \Gamma_\sigma$ and sufficiently large $|\lambda|$

$$|\Delta(N_b^a, N_d^c, q)(\lambda)| \geq A_\sigma |\sqrt{\lambda}| \exp(\zeta).$$

Now, for $x \in [0, \frac{1}{2}]$, let

$$y_1(x) = y(x), \quad y_2(x) = y(1-x), \quad q_1(x) = q(x), \quad q_2(x) = q(1-x). \tag{3.2}$$

Hence, the problem $\$_0$ can be transformed to the BVP $\$_1$ as follows:

$$\begin{cases} \mathcal{Y}'' + (\lambda\mathbf{I} - \mathbf{q}(x))\mathcal{Y} = 0, & x \in (0, \frac{1}{2}), \\ \mathbf{M}_c^a(\lambda)\mathcal{Y}'(0) - \mathbf{M}_d^b(\lambda)\mathcal{Y}(0) = 0, & \\ \mathbf{D}_1\mathcal{Y}'(\frac{1}{2}) + \mathbf{D}_2\mathcal{Y}(\frac{1}{2}) = 0, & \end{cases}$$

where $\mathbf{I}$ is the 2×2 identity matrix and

$$\begin{aligned} \mathcal{Y}(x,\lambda) =& diag(y_1(x,\lambda), y_2(x,\lambda)) = \begin{bmatrix} y_1(x,\lambda) & 0 \\ 0 & y_2(x,\lambda) \end{bmatrix}, \\ \mathbf{q}(x) =& diag(q_1(x), q_2(x)), \\ \mathbf{M}_c^a(\lambda) =& diag(a(\lambda), c(\lambda)), \quad \mathbf{M}_d^b(\lambda) = diag(b(\lambda), d(\lambda)), \\ \mathbf{D}_1 =& \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.1 *Let $\widetilde{\mathbf{q}}(x)$ be a 2×2 matrix-valued function with elements in $L_2(0,\frac{1}{2})$, and let $\widetilde{\mathcal{Y}}(x) := diag(\widetilde{y}_1(x,\lambda), \widetilde{y}_2(x,\lambda))$ be the solution of the matrix-valued equation*

$$\widetilde{\mathcal{Y}}'' + (\lambda\mathbf{I} - \widetilde{\mathbf{q}}(x))\widetilde{\mathcal{Y}} = 0, \qquad x \in (0, \frac{1}{2}),$$

with the initial conditions $\widetilde{\mathcal{Y}}(0,\lambda) = \mathbf{H}_1(\lambda)$, $\widetilde{\mathcal{Y}}'(0,\lambda) = \mathbf{H}_2(\lambda)$, where $\mathbf{H}_1(\lambda) = diag(\widetilde{a}(\lambda), \widetilde{c}(\lambda))$ and $\mathbf{H}_2(\lambda) = diag(\widetilde{b}(\lambda), \widetilde{d}(\lambda))$ are two complex-valued 2×2 matrices and

$$\widetilde{a}(\lambda) = \sum_{j=1}^{m} \widetilde{a}_j\lambda^j, \ \widetilde{b}(\lambda) = \sum_{j=1}^{m} \widetilde{b}_j\lambda^j, \ \widetilde{c}(\lambda) = \sum_{j=1}^{r} \widetilde{c}_j\lambda^j, \ \widetilde{d}(\lambda) = \sum_{j=1}^{r} \widetilde{d}_j\lambda^j,$$

where $\widetilde{a}_m = \widetilde{c}_r = 1$. Then, for sufficiently large $|\lambda|$, the following asymptotic representation holds:

$$\widetilde{\mathcal{Y}}(x,\lambda) = \lambda^m\{\cos\sqrt{\lambda}x\mathbf{I} + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\mathbf{I}(\widetilde{b}_m\mathbf{I} + \frac{1}{2}\int_0^x \mathbf{q}(t)dt) + O(\frac{1}{\sqrt{\lambda}}\exp(\zeta x)).$$

Corollary 3.2 *If* $\widetilde{\mathbf{q}}(x) = \mathbf{q}(x) = diag(q_1(x), q_2(x))$, $\mathbf{H}_1(\lambda) = \mathbf{M}_c^a(\lambda)$, *and* $\mathbf{H}_2(\lambda) = \mathbf{M}_d^b(\lambda)$, *then* $\widetilde{\mathcal{Y}}(x) = diag(y_1(x,\lambda), y_2(x,\lambda))$, *where* $y_i(x,\lambda)$ *is the solution of the initial value problem*

$$\begin{cases} y'' + (\lambda - q_i(x))y = 0, \\ y(0,\lambda) = a(\lambda),\ y'(0,\lambda) = b(\lambda). \end{cases}$$

Moreover, from (3.1), the characteristic function of the problem $\$_1$ *is*

$$\begin{aligned} \Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \widetilde{\mathbf{q}})(\lambda) =& \det(\mathbf{D}_1\widetilde{\mathcal{Y}}'(\frac{1}{2},\lambda) + \mathbf{D}_2\widetilde{\mathcal{Y}}(\frac{1}{2},\lambda)) \\ =& W(y_1,y_2)(\frac{1}{2},\lambda) = W(u,s)(\frac{1}{2},\lambda) \\ =& \Delta(N_b^a(\lambda), N_d^c(\lambda), \mathbf{q})(\lambda). \end{aligned}$$

Now, first we consider the problem $\$_2$ with one discontinuity at $x = \frac{1}{2}$ as follows:

$$\begin{cases} y'' + (\lambda - q(x))y = 0, \quad x \in (0,1), \\ a(\lambda)y'(0,\lambda) - b(\lambda)y(0,\lambda) = 0, \\ c(\lambda)y'(1,\lambda) - d(\lambda)y(1,\lambda) = 0, \\ y(\frac{1}{2}+0) = \alpha_p y(\frac{1}{2}-0), \\ y'(\frac{1}{2}+0) = \alpha_p^{-1} y'(\frac{1}{2}-0), \end{cases}$$

where $q(x) \in L_2(0,1)$, $\int_0^1 q(t)dt = 0$, $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ are defined as (1.4), $\alpha \in \mathbb{R}$, $\alpha \neq 0$. In the following theorem, we prove the first Hochstadt–Lieberman-type theorem of this section. Note, for this purpose, that we use the Weyl–Titchmarsh $\mathfrak{M}$-function techniques from [4, 18, 21].

Theorem 3.3 *Let* $q(x) \in L_2(0,1)$ *and* $\{\lambda_n(q)\}_{n\geq 0}$ *be the set of eigenvalues of* $\$_2$ *defined in (2.1). Then* $\{\lambda_n(q)\}$ *and* $q|_{(\frac{1}{2},1)}$ *determine* $q(x)$ *uniquely.*

Proof By the replacement (3.2), we can transform $\$_2$ to the problem consisting of

$$\mathcal{Y}'' + (\lambda\mathbf{I} - \mathbf{q}(x))\mathcal{Y} = 0, \qquad x \in (0, \frac{1}{2}) \tag{3.3}$$

with the boundary conditions

$$\begin{cases} \mathbf{M}_c^a(\lambda)\mathcal{Y}'(0) - \mathbf{M}_d^b(\lambda)\mathcal{Y}(0) = 0, \\ \mathbf{D}_3\mathcal{Y}'(\frac{1}{2}) + \mathbf{D}_4\mathcal{Y}(\frac{1}{2}) = 0, \end{cases}$$

where $\mathbf{q}(x) = diag(q_1(x), q_2(x))$, $q_1(x) = q(x)$ and $q_2(x) = q(1-x)$ for $x \in (0, \frac{1}{2})$, and

$$\mathbf{D}_3 = \begin{bmatrix} 0 & 0 \\ \alpha^{-1} & 1 \end{bmatrix}, \ \mathbf{D}_4 = \begin{bmatrix} \alpha & -1 \\ 0 & 0 \end{bmatrix}.$$

Let $s_i(x,\lambda)$ denote the solution of the initial value problem

$$\begin{cases} y'' + (\lambda - q_i(x))y = 0, \quad x \in (0, \frac{1}{2}), \\ s_i(0,\lambda) = (a(\lambda))_i,\ y'(0,\lambda) = (b(\lambda))_i,\ i = 1,2, \end{cases}$$

where $(a(\lambda))_i = \sum_{j=1}^{m} a_{ij}\lambda^j$, $(b(\lambda))_i = \sum_{j=1}^{m} b_{ij}\lambda^j$, $a_{im} = 1$, and $y(x,\lambda) = diag(s_1(x,\lambda), s_2(x,\lambda))$. Then, for sufficiently large $|\lambda|$, the characteristic function of (3.3) is

$$\begin{aligned}\Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q})(\lambda) =& \det(\mathbf{D}_3\widetilde{\mathcal{Y}}'(\frac{1}{2},\lambda) + \mathbf{D}_4\widetilde{\mathcal{Y}}(\frac{1}{2},\lambda)) \\ =& \det\begin{bmatrix}\alpha s_1(\frac{1}{2},\lambda) & -s_2(\frac{1}{2},\lambda) \\ \alpha^{-1}s_1'(\frac{1}{2},\lambda) & s_2'(\frac{1}{2},\lambda)\end{bmatrix} \\ =& \alpha s_1(\frac{1}{2},\lambda)s_2'(\frac{1}{2},\lambda) + \alpha^{-1}s_1'(\frac{1}{2},\lambda)s_2(\frac{1}{2},\lambda) \\ =& O(\sqrt{\lambda}\exp(\zeta)).\end{aligned}$$

Suppose now that there are two potentials q and $\widetilde{q}$ such that $\lambda_n(q) = \lambda_n(\widetilde{q})$ and $q(x) = \widetilde{q}(x)$ on the interval $(\frac{1}{2}, 1)$. Then, for the corresponding potential matrices $\mathbf{q}$ and $\widetilde{\mathbf{q}}$, we get

$$\sigma(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q}) = \sigma(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \widetilde{\mathbf{q}}). \tag{3.4}$$

We denote the fundamental matrices of (3.3) corresponding to $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ by $\mathcal{Y}(x,\lambda;\mathbf{q}) = diag(s_1(x,\lambda), s_2(x,\lambda))$ and $\mathcal{Y}(x,\lambda;\widetilde{\mathbf{q}}) = diag(\widetilde{s}_1(x,\lambda), \widetilde{s}_2(x,\lambda)) = diag(\widetilde{s}_1(x,\lambda), s_2(x,\lambda))$, respectively. Hence, it follows from (3.4) that

$$\Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q}) = \Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \widetilde{\mathbf{q}}),$$

and therefore

$$\begin{bmatrix}s_1(\frac{1}{2},\lambda) & s_1'(\frac{1}{2},\lambda) \\ \widetilde{s}_1(\frac{1}{2},\lambda) & \widetilde{s}_1'(\frac{1}{2},\lambda)\end{bmatrix}\begin{bmatrix}\alpha s_2'(\frac{1}{2},\lambda) \\ \alpha^{-1}s_2(\frac{1}{2},\lambda)\end{bmatrix} = \begin{bmatrix}\Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q})(\lambda) \\ \Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q})(\lambda)\end{bmatrix}. \tag{3.5}$$

On the other hand, since $\begin{bmatrix}\alpha s_2'(\frac{1}{2},\lambda) \\ \alpha^{-1}s_2(\frac{1}{2},\lambda)\end{bmatrix}$ never vanishes, using (3.5) we obtain for each $\lambda = \lambda_n(q)$

$$W(s_1, \widetilde{s}_1)(\frac{1}{2},\lambda) = 0.$$

Thus, the function

$$f(\lambda) := \frac{W(s_1, \widetilde{s}_1)(\frac{1}{2},\lambda)}{\Delta(\mathbf{M}_c^a(\lambda), \mathbf{M}_d^b(\lambda), \mathbf{D}_3, \mathbf{D}_4, \mathbf{q})(\lambda)} \tag{3.6}$$

is an entire function. Now, using Lemma 2.2, we obtain $f(\lambda) \equiv 0$. Since f is identically zero, then (3.6) yields

$$\frac{s_1(\frac{1}{2},\lambda)}{s_1'(\frac{1}{2},\lambda)} = \frac{\widetilde{s}_1(\frac{1}{2},\lambda)}{\widetilde{s}_1'(\frac{1}{2},\lambda)}.$$

Therefore, $q_1(x) = \widetilde{q}_1(x)$, and consequently $q(x) = \widetilde{q}(x)$. □

The previous Hochstadt–Lieberman-type theorem can be generalized for the BVP $\$_3$ consisting of (1.1)–(1.3) on the interval $(0,1)$, which has an arbitrary finite number of transmission conditions. In the next theorem, we prove this assertion and show that the number and positions of discontinuities are not important.

Theorem 3.4 *Let* $a_0 = 0$, $b_0 = 1$, $q(x) \in L_2(0,1)$, $0 < x_1 < x_2 < ... < x_{k-1} < x_k = \frac{1}{2} < x_{k+1} < ... < x_\ell < 1$, $\alpha_p \in \mathbb{R}$, *and* $\alpha_p \neq 0$ *for* $p = 1, 2, 3, ..., \ell$. *Assume that* $\sigma := \sigma(\alpha_p, N_b^a, N_d^c, q; p)(\lambda)$ *is the spectrum of the problem* $\$_3$ *consisting of* (1.1)–(1.3) *on the interval* $(0,1)$. *Then* σ *and* $q|_{(\frac{1}{2},1)}$ *uniquely determine the potential* $q(x)$.

Proof Let $s_1(x,\lambda)$ be the solution of (1.1) satisfying $s_1(0,\lambda) = a(\lambda)$, $s_1'(0,\lambda) = b(\lambda)$ and the discontinuity condition (1.3) at $x_1, x_2, ..., x_k = \frac{1}{2}$, and $s_2(x,\lambda)$ be the solution of (1.1) satisfying $s_2(0,\lambda) = c(\lambda)$, $s_2'(0,\lambda) = d(\lambda)$ and the discontinuity condition (1.3) at $x_{k+1}, ..., x_\ell$. By the same arguments as in the proofs of Theorem 3.3, we can uniquely determine the Weyl–Titchmarsh $\mathfrak{M}$-function $s_1(\frac{1}{2},\lambda)/s_1'(\frac{1}{2},\lambda)$ for $\$_3$ on the interval $(0,\frac{1}{2})$. Finally, since the $\mathfrak{M}$-function for the S-L problem $\$_3$, which has arbitrary finite number of interior discontinuities, can uniquely determine the potential $q(x)$ (for more details, see [18]), we arrive at the assertion of Theorem 3.4. □

Finally, in the following theorem, we generalize Theorem 3.4 on an arbitrary finite interval $[a_0, b_0]$ as follows.

Theorem 3.5 *Assume that* $q(x) \in L_2(a_0,b_0)$, $\int_{a_0}^{b_0} q(x)dx = 0$, $a_0 < x_1 < x_2 < ... < x_{k-1} < x_k = \frac{a_0+b_0}{2} < x_{k+1} < ... < x_\ell < b_0$, $\alpha_p \in \mathbb{R}$, *and* $\alpha_p \neq 0$ *for* $p = 1, 2, 3, ..., \ell$. *Let* $\sigma_1 := \sigma(\alpha_p, N_b^a, N_d^c, q; p)(\lambda)$ *be the spectrum of the boundary value problem* $\$$ *generated by* (1.1)–(1.3). *Then* σ_1 *and* $q|_{(\frac{a_0+b_0}{2},b_0)}$ *uniquely determine the potential* $q(x)$.

Proof We denote for $x \in (a_0, \frac{a_0+b_0}{2})$:

$$\begin{cases} y_1(x) = y(x), \\ y_2(x) = y(a_0+b_0-x), \end{cases} \qquad \begin{cases} q_1(x) = q(x), \\ q_2(x) = q(a_0+b_0-x). \end{cases}$$

Then, replacing $(0,1)$ and $\frac{1}{2}$ by (a_0,b_0) and $\frac{a_0+b_0}{2}$, respectively, and applying the same arguments as in the proof of Theorem 3.4, we can conclude the assertion. □

Acknowledgment

This research was partially supported by the University of Kashan under grant number 682482/6.

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