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Radii of uniform convexity of some special functions

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Abstract: In this investigation our main aim is to determine the radii of uniform convexity of selected normalized q-Bessel and Wright functions. Here we consider six different normalized forms of q-Bessel functions and we apply three different kinds of the normalization of the Wright function. We also show that the obtained radii are the smallest positive roots of some functional equations.

Key words: Radius of uniform convexity, Mittag-Leffler expansions, q-Bessel functions, Wright function

1. Introduction and preliminaries

Special and geometric function theories are the most important branches of mathematical analysis. There has been a close relationship between special and geometric function theories since hypergeometric functions were used in the proof of the famous Bieberbach conjecture. Therefore, most mathematicians have considered some of the geometric properties of special functions that can be expressed by the hypergeometric series. Some of the geometric properties of the Bessel, Struve, Lommel, Wright, and q-Bessel functions in particular have been investigated by many authors. The first important results concerning the geometric properties of hypergeometric and related functions can be found in [14, 22, 23, 29]. In fact, there are some relationships between the geometric properties and the zeros of special functions. Due to these relationships, numerous investigations have been done on the zeros of the above mentioned special functions. Comprehensive information about the Bessel function and its q-analogue can be found in [28], and some results on the zeros of some special functions can be found in [10, 18–21, 25, 26]. Recently, some of the geometric properties (like univalence, starlikeness, convexity, and uniform convexity) of the Bessel, Struve, and Lommel functions of the first kind were investigated in [2, 3, 6–9, 11–13, 15, 27, 30]. In addition, the radii of starlikeness and convexity of some normalized q-Bessel functions were studied in [1, 4, 5]. Motivated by the previous works in this field, our aim is to determine the radii of uniform convexity of some normalized q-Bessel and Wright functions.

First we would like to present some basic concepts related to geometric function theory. Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$,

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$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

which satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of functions that belong to $\mathcal{A}$ that are univalent in $\mathbb{D}_r$. On the other hand, the class of convex functions is defined by

$$\mathcal{K} = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r \right\}.$$

The radius of convexity of an analytic locally univalent function $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$r^c(f) = sup\left\{ r > 0 : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r \right\}.$$

Note that $r^c(f)$ is the largest radius for which the image domain $f\left(\mathbb{D}_{r^c(f)}\right)$ is a convex domain in $\mathbb{C}$. For more information about convex functions, we refer to Duren's book [16] and its references.

In [17] the author introduced the concept of uniform convexity for the functions of the form (1). A function $f(z)$ is said to be uniformly convex in $\mathbb{D}$ if $f(z)$ is in the class of usual convex functions and if it has the property that for every circular arc γ contained in $\mathbb{D}$, with the center ζ also in $\mathbb{D}$, the arc $f(\gamma)$ is a convex arc. An analytic description of the uniformly convex functions given by Rønning in [24] reads as follows:

Theorem 1 *Let $f(z)$ be of the form (1). Then f is a uniformly convex function if and only if*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{D}.$$

On the other hand, the concept of the radius of uniform convexity is defined by (see [15])

$$r^{uc}(f) = \sup\left\{ r \in (0, r_f) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{D} \right\}.$$

Thanks to the above theorem, we can determine the radius of uniform convexity for the functions of the form (1). Also, we will need the following lemma in the sequel.

Lemma 1 ([15]) *If $a > b > r \geq |z|$, and $\lambda \in [0, 1]$, then*

$$\left|\frac{z}{b - z} - \lambda\frac{z}{a - z}\right| \leq \frac{r}{b - r} - \lambda\frac{r}{a - r}. \tag{2}$$

The following are very simple consequences of this inequality:

$$\Re\left(\frac{z}{b - z} - \lambda\frac{z}{a - z}\right) \leq \frac{r}{b - r} - \lambda\frac{r}{a - r} \tag{3}$$

and

$$\Re\left(\frac{z}{b - z}\right) \leq \left|\frac{z}{b - z}\right| \leq \frac{r}{b - r}. \tag{4}$$

2. Radius of uniform convexity of some special functions

In this section we focus on some normalized q-Bessel and Wright functions and determine the radii of uniform convexity for these functions.

2.1. Uniform convexity of some normalized q-Bessel functions

Jackson's second and third (or Hahn–Exton) q-Bessel functions are defined as follow:

$$J_{\nu}^{(2)}(z;q)=\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\sum_{n\geq 0}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n}q^{n(n+\nu)}$$

and

$$J_{\nu}^{(3)}(z;q)=\frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}}\sum_{n\geq 0}\frac{(-1)^n z^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n}q^{\frac{1}{2}n(n+1)},$$

where $z\in\mathbb{C},\nu>-1,q\in(0,1)$, and

$$(a;q)_0=1,(a;q)_n=\prod_{k=1}^{n}\left(1-aq^{k-1}\right),(a,q)_{\infty}=\prod_{k\geq 1}\left(1-aq^{k-1}\right).$$

It is known that Jackson's second and third q-Bessel functions are q-extensions of the classical Bessel function of the first kind J_{ν}. Clearly, for fixed z, we have $J_{\nu}^{(2)}\left((1-z)q;q\right)\to J_{\nu}(z)$ and $J_{\nu}^{(3)}\left((1-z)q;q\right)\to J_{\nu}(2z)$ as $q\nearrow 1$.

Because the functions $J_{\nu}^{(2)}(.;q)$ and $J_{\nu}^{(3)}(.;q)$ do not belong to $\mathcal{A}$, we first consider the following six normalized forms as in [5]. For $\nu>-1$,

$$f_{\nu}^{(2)}(z;q)=\left(2^{\nu}c_{\nu}(q)J_{\nu}^{(2)}(z;q)\right)^{\frac{1}{\nu}},\nu\neq 0$$

$$g_{\nu}^{(2)}(z;q)=2^{\nu}c_{\nu}(q)z^{1-\nu}J_{\nu}^{(2)}(z;q),$$

$$h_{\nu}^{(2)}(z;q)=2^{\nu}c_{\nu}(q)z^{1-\frac{\nu}{2}}J_{\nu}^{(2)}(\sqrt{z};q),$$

$$f_{\nu}^{(3)}(z;q)=\left(c_{\nu}(q)J_{\nu}^{(3)}(z;q)\right)^{\frac{1}{\nu}},\nu\neq 0$$

$$g_{\nu}^{(3)}(z;q)=c_{\nu}(q)z^{1-\nu}J_{\nu}^{(3)}(z;q),$$

$$h_{\nu}^{(3)}(z;q)=c_{\nu}(q)z^{1-\frac{\nu}{2}}J_{\nu}^{(3)}(\sqrt{z};q),$$

where $c_{\nu}(q)=(q;q)_{\infty}/(q^{\nu+1};q)_{\infty}$. Consequently, all of the above functions belong to the class $\mathcal{A}$. Of course there exist an infinite number of other normalizations for both the Jackson and Hahn–Exton q-Bessel functions, but the main motivation for considering these six functions is the fact that their limiting cases for Bessel functions appear in the literature. For an example of this see [14] and the references therein.

It is known from [5, Lemma 1., p.972] that, if $\nu > -1$, then the Hadamard factorizations of the functions $z \mapsto J_\nu^{(2)}(z;q)$ and $z \mapsto J_\nu^{(3)}(z;q)$ are of the form

$$J_\nu^{(2)}(z;q) = \frac{z^\nu}{2^\nu c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{j_{\nu,n}^2(q)}\right)$$

and

$$J_\nu^{(3)}(z;q) = \frac{z^\nu}{c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{l_{\nu,n}^2(q)}\right)$$

where $j_{\nu,n}(q)$ and $l_{\nu,n}(q)$ are the nth positive zeros of the functions $J_\nu^{(2)}(z;q)$ and $J_\nu^{(3)}(z;q)$. Also, it is known from [5, Lemma 7., p. 975] that, if $\nu > 0$, then the Hadamard factorizations of the derivatives of the functions $z \mapsto J_\nu^{(2)}(z;q)$ and $z \mapsto J_\nu^{(3)}(z;q)$ are of the form

$$\frac{dJ_\nu^{(2)}(z;q)}{dz} = \frac{\nu\left(\frac{z}{2}\right)^{\nu-1}}{2c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{{j'}_{\nu,n}^2(q)}\right)$$

and

$$\frac{dJ_\nu^{(3)}(z;q)}{dz} = \frac{\nu z^{\nu-1}}{c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{{l'}_{\nu,n}^2(q)}\right)$$

where $j'_{\nu,n}(q)$ and $l'_{\nu,n}(q)$ are the nth positive zeros of the functions $z \mapsto dJ_\nu^{(2)}(z;q)/dz$ and $z \mapsto dJ_\nu^{(3)}(z;q)/dz$.

In addition, for the derivatives of the functions $z \mapsto g_\nu^{(2)}(z;q), z \mapsto h_\nu^{(2)}(z;q), z \mapsto g_\nu^{(3)}(z;q)$, and $z \mapsto h_\nu^{(3)}(z;q)$, the infinite product representations are given, respectively, in [5, Lemma 8, p. 975] as follow:

$$\frac{dg_\nu^{(2)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z^2}{\alpha_{\nu,n}^2(q)}\right), \tag{5}$$

$$\frac{dh_\nu^{(2)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z}{\beta_{\nu,n}^2(q)}\right), \tag{6}$$

$$\frac{dg_\nu^{(3)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z^2}{\gamma_{\nu,n}^2(q)}\right), \tag{7}$$

and

$$\frac{dh_\nu^{(3)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z}{\delta_{\nu,n}^2(q)}\right), \tag{8}$$

where $\alpha_{\nu,n}(q)$ and $\beta_{\nu,n}(q)$ are the nth positive zeros of $z \mapsto z.dJ_\nu^{(2)}(z;q)/dz + (1-\nu)J_\nu^{(2)}(z;q)$ and $z \mapsto z.dJ_\nu^{(2)}(z;q)/dz + (2-\nu)J_\nu^{(2)}(z;q)$, while $\gamma_{\nu,n}(q)$ and $\delta_{\nu,n}(q)$ are the nth positive zeros of $z \mapsto z.dJ_\nu^{(3)}(z;q)/dz + (1-\nu)J_\nu^{(3)}(z;q)$ and $z \mapsto z.dJ_\nu^{(3)}(z;q)/dz + (2-\nu)J_\nu^{(3)}(z;q)$.

Finally, it is known from [5, Lemma 9., p. 975] that, between any two consecutive roots of the function $z \mapsto J_\nu^{(s)}(z;q)$, the function $z \mapsto dJ_\nu^{(s)}(z;q)/dz$ has precisely one zero when $\nu \geq 0$ and $s \in \{2,3\}$.

The following are our first main results concerning the q-Bessel functions.

Theorem 2 *Let $\nu > -1, s \in \{2,3\}$ and $q \in (0,1)$. The following assertions are true.*

a. *Suppose that $\nu > 0$. Then the radius of uniform convexity of the function $z \mapsto f_\nu^{(s)}(z;q)$ is the smallest positive root of the equation*

$$1 + 2r\frac{\left(f_\nu^{(s)}(r;q)\right)''}{\left(f_\nu^{(s)}(r;q)\right)'} = 0.$$

b. *The radius of uniform convexity of the function $z \mapsto g_\nu^{(s)}(z;q)$ is the smallest positive root of the equation*

$$(2\nu-1)(\nu-1)J_\nu^{(s)}(r;q) + (5-4\nu)r\left(J_\nu^{(s)}(r;q)\right)' + 2r^2\left(J_\nu^{(s)}(r;q)\right)'' = 0.$$

c. *The radius of uniform convexity of the function $z \mapsto h_\nu^{(s)}(z;q)$ is the smallest positive root of the equation*

$$(\nu-1)(\nu-2)J_\nu^{(s)}(\sqrt{r};q) + (4-2\nu)\sqrt{r}\left(J_\nu^{(s)}(\sqrt{r};q)\right)' + r\left(J_\nu^{(s)}(\sqrt{r};q)\right)'' = 0.$$

Proof The proofs for the cases $s=2$ and $s=3$ are almost the same. This is why we only present the proof for the case $s=2$.

a. Let $j_{\nu,n}(q)$ and $j'_{\nu,n}(q)$ be the nth positive roots of the functions $z \mapsto J_\nu^{(2)}(z;q)$ and $z \mapsto dJ_\nu^{(2)}(z;q)/dz$, respectively. In [5, p. 979], it was shown that the following equality is valid:

$$1 + z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'} = 1 - \left(\frac{1}{\nu}-1\right)\sum_{n\geq 1}\frac{2z^2}{j_{\nu,n}^2(q)-z^2} - \sum_{n\geq 1}\frac{2z^2}{{j'}_{\nu,n}^2(q)-z^2}.$$

In the first step of our proof we consider the case $\nu \geq 1$. We know that the zeros of Jackson's second and third q-Bessel functions are all real when $\nu > -1$, according to [18, 20]. Also, it is known from [5, Lemma 9, p. 975] that the zeros of the functions $z \mapsto J_\nu^{(s)}(z;q)$ and $z \mapsto dJ_\nu^{(s)}(z;q)/dz$ are interlaced. Here it is important to mention that the nonnegative smallest zero is $z=0$ for Jackson's second and third q-Bessel functions. By taking $\lambda = 1 - \frac{1}{\nu}$ in inequality (3) we have

$$\Re\left(\frac{2z^2}{{j'}_{\nu,n}^2(q)-z^2} - \left(1-\frac{1}{\nu}\right)\frac{2z^2}{j_{\nu,n}^2(q)-z^2}\right) \leq \left(\frac{2r^2}{{j'}_{\nu,n}^2(q)-r^2} - \left(1-\frac{1}{\nu}\right)\frac{2r^2}{j_{\nu,n}^2(q)-r^2}\right),$$

for $|z| \leq r < j'_{\nu,1}(q) < j_{\nu,1}(q)$, and so we get that

$$\Re\left(1 + z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right) \geq 1 + r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}. \quad (9)$$

On the other hand, inequality (2) implies that

$$\left|\frac{2z^2}{j'^2_{\nu,n}(q)-z^2}-\left(1-\frac{1}{\nu}\right)\frac{2z^2}{j^2_{\nu,n}(q)-z^2}\right|\leq\frac{2r^2}{j'^2_{\nu,n}(q)-r^2}-\left(1-\frac{1}{\nu}\right)\frac{2r^2}{j^2_{\nu,n}(q)-r^2},$$

where $|z|\leq r<j'_{\nu,1}(q)<j_{\nu,1}(q)$. Therefore, we find that

$$\left|z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right|\leq -r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}. \tag{10}$$

As a second step, one can easily show that inequalities (9) and (10) hold for $\nu\in(0,1)$. Clearly, by considering inequality (4), we can write that

$$\Re\left(\frac{2z^2}{j'^2_{\nu,n}(q)-z^2}\right)\leq\left|\frac{2z^2}{j'^2_{\nu,n}(q)-z^2}\right|\leq\frac{2r^2}{j'^2_{\nu,n}(q)-r^2}$$

and

$$\Re\left(\frac{2z^2}{j^2_{\nu,n}(q)-z^2}\right)\leq\left|\frac{2z^2}{j^2_{\nu,n}(q)-z^2}\right|\leq\frac{2r^2}{j^2_{\nu,n}(q)-r^2}$$

for $|z|\leq r<j'_{\nu,1}(q)<j_{\nu,1}(q)$. Since $\frac{1}{\nu}-1>0$, the above last two inequalities imply that inequalities (9) and (10) hold true. Consequently, using these two inequalities yields that

$$\Re\left(1+z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right)-\left|z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right|\geq 1+2r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'} \tag{11}$$

for $|z|\leq r<j'_{\nu,1}(q)$. In (11), the equality holds if and only if $z=r$. Thus, it follows that

$$\inf_{|z|<r}\left[\Re\left(1+z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right)-\left|z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right|\right]=1+2r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'},$$

where $r\in(0,j'_{\nu,1}(q))$. The mapping $\Phi_\nu:(0,j'_{\nu,1}(q))\mapsto\mathbb{R}$ defined by

$$\Phi_\nu(r)=1+2r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}=1-2\sum_{n\geq1}\left(\frac{2r^2}{j'^2_{\nu,n}(q)-r^2}-\left(1-\frac{1}{\nu}\right)\frac{2r^2}{j^2_{\nu,n}(q)-r^2}\right)$$

is strictly decreasing since

$$\Phi'_\nu(r)=-2\sum_{n\geq1}\left(\frac{4rj'^2_{\nu,n}(q)}{\left(j'^2_{\nu,n}(q)-r^2\right)^2}-\left(1-\frac{1}{\nu}\right)\frac{4rj^2_{\nu,n}(q)}{\left(j^2_{\nu,n}(q)-r^2\right)^2}\right)<0$$

for $r \in (0, j'_{\nu,1}(q))$. Also, we have the following limits:

$$\lim_{r\searrow 0} \Phi_\nu(r) = 1 \text{ and } \lim_{r\nearrow j'_{\nu,1}(q)} \Phi_\nu(r) = -\infty.$$

As a result of this, we can say that the equation

$$1 + 2r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'} = 0$$

has a unique root r_0 in the interval $(0, j'_{\nu,1}(q))$, which is the radius of uniform convexity $r_0 = r^{uc}\left(f_\nu^{(2)}(z;q)\right)$ of the function $z \mapsto f_\nu^{(2)}(z;q)$.

b. By using the logarithmic derivative of the function $z \mapsto dg_\nu^{(2)}(z;q)/dz$, which is given by (5), we get that

$$z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'} = -\sum_{n\geq 1}\frac{2z^2}{\alpha_{\nu,n}^2(q) - z^2} \tag{12}$$

and

$$1 + z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'} = 1 - \sum_{n\geq 1}\frac{2z^2}{\alpha_{\nu,n}^2(q) - z^2}. \tag{13}$$

Now, for $|z| \leq r < \alpha_{\nu,1}(q)$, using inequality (4) in equalities (13) and (12), respectively, implies that

$$\Re\left(1 + z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right) \geq 1 + r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'} \tag{14}$$

and

$$\left|z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right| \leq -r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'}. \tag{15}$$

From inequalities (14) and (15), we deduce

$$\Re\left(1 + z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right) - \left|z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right| \geq 1 + 2r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'} \tag{16}$$

for $|z| \leq r < \alpha_{\nu,1}(q)$. The equality holds in (16) if and only if $z = r$. As a result, we have

$$\inf_{|z|<r}\left[\Re\left(1 + z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right) - \left|z\frac{\left(g_\nu^{(2)}(z;q)\right)''}{\left(g_\nu^{(2)}(z;q)\right)'}\right|\right] = 1 + 2r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'},$$

where $r \in (0, \alpha_{\nu,1}(q))$. Now consider the function $A_\nu : (0, \alpha_{\nu,1}(q)) \mapsto \mathbb{R}$ defined by

$$A_\nu(r) = 1 + 2r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'} = 1 - \sum_{n\geq 1} \frac{4r^2}{\alpha_{\nu,n}^2(q) - r^2}.$$

The function $A_\nu(r)$ is strictly decreasing since

$$A_\nu'(r) = -\sum_{n\geq 1} \frac{8r\alpha_{\nu,n}^2(q)}{\left(\alpha_{\nu,n}^2(q) - r^2\right)^2} < 0$$

for $r \in (0, \alpha_{\nu,1}(q))$ and in addition

$$\lim_{r\searrow 0} A_\nu(r) = 1 \text{ and } \lim_{r\nearrow \alpha_{\nu,1}(q)} A_\nu(r) = -\infty.$$

Therefore, the equation

$$1 + 2r\frac{\left(g_\nu^{(2)}(r;q)\right)''}{\left(g_\nu^{(2)}(r;q)\right)'} = 0 \tag{17}$$

has a unique root $r_1 \in (0, \alpha_{\nu,1}(q))$ and $r_1 = r^{uc}\left(g_\nu^{(2)}(z;q)\right)$. By using the first and second derivatives of the function $z \mapsto g_\nu^{(2)}(z;q)$, one can easily see that equation (17) is equivalent to

$$(2\nu-1)(\nu-1)J_\nu^{(2)}(r;q) + (5-4\nu)r\left(J_\nu^{(2)}(r;q)\right)' + 2r^2\left(J_\nu^{(2)}(r;q)\right)'' = 0.$$

Thus, the proof is completed.

c. The proof of this part can be done in a similar manner. The logarithmic derivative of the function $z \mapsto dh_\nu^{(2)}(z;q)/dz$, which is given by (6), implies that

$$z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'} = -\sum_{n\geq 1} \frac{z}{\beta_{\nu,n}^2(q) - z} \tag{18}$$

and

$$1 + z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'} = 1 - \sum_{n\geq 1} \frac{z}{\beta_{\nu,n}^2(q) - z}. \tag{19}$$

Now, for $|z| \leq r < \beta_{\nu,1}^2(q)$, by using inequality (4) in equalities (19) and (18), respectively, we get that

$$\Re\left(1 + z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'}\right) \geq 1 + r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'} \tag{20}$$

and

$$\left| z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'} \right| \leq -r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'}. \tag{21}$$

Now summarizing inequalities (20) and (21), we obtain

$$\Re\left(1+z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'}\right) - \left| z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'} \right| \geq 1+2r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'} \tag{22}$$

for $|z| \leq r < \beta_{\nu,1}^2(q)$. The equality holds in (22) if and only if $z=r$. Finally, we have

$$\inf_{|z|<r}\left[\Re\left(1+z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'}\right) - \left| z\frac{\left(h_\nu^{(2)}(z;q)\right)''}{\left(h_\nu^{(2)}(z;q)\right)'} \right|\right] = 1+2r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'},$$

where $r \in \left(0,\beta_{\nu,1}^2(q)\right)$. Now consider the function $B_\nu : \left(0,\beta_{\nu,1}^2(q)\right) \mapsto \mathbb{R}$ defined by

$$B_\nu(r) = 1+2r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'} = 1-\sum_{n\geq 1}\frac{2r}{\beta_{\nu,n}^2(q)-r}.$$

The function $B_\nu(r)$ is strictly decreasing since

$$B_\nu'(r) = -\sum_{n\geq 1}\frac{2\beta_{\nu,n}^2(q)}{\left(\beta_{\nu,n}^2(q)-r\right)^2} < 0$$

for $r \in \left(0,\beta_{\nu,1}^2(q)\right)$ and furthermore

$$\lim_{r\searrow 0} B_\nu(r) = 1 \text{ and } \lim_{r\nearrow \beta_{\nu,1}^2(q)} B_\nu(r) = -\infty.$$

As a result, the equation

$$1+2r\frac{\left(h_\nu^{(2)}(r;q)\right)''}{\left(h_\nu^{(2)}(r;q)\right)'} = 0 \tag{23}$$

has a unique root $r_2 \in \left(0,\beta_{\nu,1}^2(q)\right)$ and $r_2 = r^{uc}\left(h_\nu^{(2)}(z;q)\right)$. By considering the first and second derivatives of the function $z \mapsto h_\nu^{(2)}(z;q)$, we can easily find that equation (23) is equivalent to

$$(\nu-1)(\nu-2)J_\nu^{(2)}(\sqrt{r};q) + (4-2\nu)\sqrt{r}\left(J_\nu^{(2)}(\sqrt{r};q)\right)' + r\left(J_\nu^{(2)}(\sqrt{r};q)\right)'' = 0,$$

which is desired.

□

2.2. Uniform convexity of some normalized Wright functions

In this subsection, we will focus on the function

$$\phi(\rho,\beta,z)=\sum_{n\geq 0}\frac{z^n}{n!\Gamma(n\rho+\beta)} \qquad (\rho>-1 \text{ and } z,\beta\in\mathbb{C})$$

named after the British mathematician E.M. Wright. It is well known that this function was introduced by him for the first time in the case $\rho>0$ in connection with his investigations on the asymptotic theory of partitions [30].

From [12, Lemma 1] we know that under the conditions $\rho>0$ and $\beta>0$, the function $z\mapsto\lambda_{\rho,\beta}(z)=\phi(\rho,\beta,-z^2)$ has an infinite number of zeros, all of which are real. Thus, due to the Hadamard factorization theorem, the expression $\lambda_{\rho,\beta}(z)$ can be written as

$$\Gamma(\beta)\lambda_{\rho,\beta}(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\lambda^2_{\rho,\beta,n}}\right)$$

where $\lambda_{\rho,\beta,n}$ stands for the nth positive zero of the function $\lambda_{\rho,\beta}(z)$ (or the positive real zeros of the function $\Psi_{\rho,\beta}$). Moreover, let $\zeta'_{\rho,\beta,n}$ denote the nth positive zero of $\Psi'_{\rho,\beta}$, where $\Psi_{\rho,\beta}(z)=z^{\beta}\lambda_{\rho,\beta}(z)$, and then the zeros satisfy the chain of inequalities

$$\zeta'_{\rho,\beta,1}<\zeta_{\rho,\beta,1}<\zeta'_{\rho,\beta,2}<\zeta_{\rho,\beta,2}<\ldots.$$

One can easily see that the function $z\mapsto\phi(\rho,\beta,-z^2)$ does not belong to $\mathcal{A}$, and thus first we perform some natural normalization. We define three functions originating $\phi(\rho,\beta,.)$:

$$f_{\rho,\beta}(z)=\left(z^{\beta}\Gamma(\beta)\phi(\rho,\beta,-z^2)\right)^{\frac{1}{\beta}},$$

$$g_{\rho,\beta}(z)=z\Gamma(\beta)\phi(\rho,\beta,-z^2),$$

$$h_{\rho,\beta}(z)=z\Gamma(\beta)\phi(\rho,\beta,-z).$$

Clearly these functions are contained in the class $\mathcal{A}$.

The following are our results regarding the uniform convexity of the functions $f_{\rho,\beta}$, $g_{\rho,\beta}$, and $h_{\rho,\beta}$.

Theorem 3 *Let $\rho>0$ and $\beta>0$.*

a. *The radius of uniform convexity of the function $f_{\rho,\beta}$ is the smallest positive root of the equation*

$$1+2r\frac{\Psi''_{\rho,\beta}(r)}{\Psi'_{\rho,\beta}(r)}+2\left(\frac{1}{\beta}-1\right)\frac{r\Psi'_{\rho,\beta}(r)}{\Psi_{\rho,\beta}(r)}=0,$$

where $\Psi_{\rho,\beta}(z)=z^{\beta}\lambda_{\rho,\beta}(z)$.

b. *The radius of uniform convexity of the function $g_{\rho,\beta}$ is the smallest positive root of the equation*

$$1+2r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}=0.$$

c. *The radius of uniform convexity of the function* $h_{\rho,\beta}$ *is the smallest positive root of the equation*

$$1 + 2r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)} = 0.$$

Proof **a.** Let $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ be the nth positive roots of $\Psi_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$, respectively. In [12, Theorem 5] the following equality was demonstrated:

$$1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1 - \left(\frac{1}{\beta} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\rho,\beta,n} - z^2} - \sum_{n\geq 1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n} - z^2}.$$

In order to prove the theorem, we need to investigate two different cases such as $\beta \in (0,1]$ and $\beta > 1$. First suppose $\beta \in (0,1]$. In this case, with the help of (4) for $\beta \in (0,1]$, we deduce that the inequality

$$\Re\left(1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) \geq 1 - \left(\frac{1}{\beta} - 1\right)\sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\rho,\beta,n} - r^2} - \sum_{n\geq 1}\frac{2r^2}{\zeta'^2_{\rho,\beta,n} - r^2} \tag{24}$$

$$= 1 + \frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}, \quad |z| \leq r < \zeta'_{\rho,\beta,1} < \zeta_{\rho,\beta,1}$$

holds true for $|z| = r$. Moreover, in view of (4), we get

$$\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right| = \left|\sum_{n\geq 1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n} - z^2} + \left(\frac{1}{\beta} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\rho,\beta,n} - z^2}\right| \tag{25}$$

$$\leq \sum_{n\geq 1}\left|\left(\frac{2z^2}{\zeta'^2_{\rho,\beta,n} - z^2} + \left(\frac{1}{\beta} - 1\right)\frac{2z^2}{\zeta^2_{\rho,\beta,n} - z^2}\right)\right|$$

$$\leq \sum_{n\geq 1}\left(\frac{2r^2}{\zeta'^2_{\rho,\beta,n} - r^2} + \left(\frac{1}{\beta} - 1\right)\frac{2r^2}{\zeta^2_{\rho,\beta,n} - r^2}\right)$$

$$= -\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}$$

where $|z| \leq r < \zeta'_{\rho,\beta,1} < \zeta_{\rho,\beta,1}$. On the other hand, in view of inequality (3) we obtain that (24) and (25) are also valid when $\beta \geq 1$ for all $z \in (0, \zeta'_{\rho,\beta,1})$. Here we assume that the zeros of $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ interlace as mentioned before; that is, we have $\zeta'_{\rho,\beta,1} < \zeta_{\rho,\beta,1}$. Eventually, thanks to (24) and (25), we arrive at

$$\Re\left(1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) - \left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right| \geq 1 + 2r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}, \quad |z| \leq r < \zeta'_{\rho,\beta,1}. \tag{26}$$

Due to the minimum principle for harmonic functions, the equality holds if and only if $z = r$. Now, the above deduced inequalities imply for $r \in (0, \zeta'_{\rho,\beta,1})$

$$\inf_{|z|<r}\left\{\Re\left(1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) - \left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right|\right\} = 1 + 2r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.$$

On the other hand, the function $u_{\rho,\beta}:(0,\zeta'_{\rho,\beta,1})\to\mathbb{R}$, defined by

$$u_{\rho,\beta}(r)=1+2\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}=1-2\sum_{n\geq1}\left(\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2}-\left(1-\frac{1}{\beta}\right)\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}\right),$$

is strictly decreasing when $\beta\in(0,1]$. Moreover, it is also strictly decreasing when $\beta>1$ since

$$\begin{aligned}u'_{\rho,\beta}(r)&=-\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{8r\zeta^2_{\rho,\beta,n}}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-\sum_{n\geq1}\frac{8r\zeta'^2_{\rho,\beta,n}}{(\zeta'^2_{\rho,\beta,n}-r^2)^2}\\&<\sum_{n\geq1}\frac{8r\zeta^2_{\rho,\beta,n}}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-\sum_{n\geq1}\frac{8r\zeta'^2_{\rho,\beta,n}}{(\zeta'^2_{\rho,\beta,n}-r^2)^2}<0\end{aligned}$$

for $r\in(0,\zeta'_{\rho,\beta,1})$. Observe also that

$$\lim_{r\searrow0}u_{\rho,\beta}(r)=1\text{ and }\lim_{r\nearrow\zeta'_{\rho,\beta,1}}u_{\rho,\beta}(r)=-\infty.$$

Thus, it follows that the equation

$$1+2r\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}=0$$

has a unique root $r_3\in(0,\zeta'_{\rho,\beta,1})$ and $r_3=r^{uc}(f_{\rho,\beta})$.

b. Let $\vartheta_{\rho,\beta,n}$ be the nth positive zero of the function $g'_{\rho,\beta}(z)$. In [12, Theorem 5] the following equality was proven:

$$1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}=1-\sum_{n\geq1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}. \tag{27}$$

As a result of this equality, the inequality

$$\Re\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)\geq1-\sum_{n\geq1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2},\quad|z|\leq r<\vartheta_{\rho,\beta,1} \tag{28}$$

was shown in [12]. From equality (27) we arrive at

$$\begin{aligned}\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|&=\left|\sum_{n\geq1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}\right|\leq\sum_{n\geq1}\left|\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}\right|\leq\sum_{n\geq1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2}\\&=-\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)},\quad|z|\leq r<\vartheta_{\rho,\beta,1}.\end{aligned} \tag{29}$$

By using inequalities (28) and (29) we obtain

$$\Re\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)-\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|\geq1+2r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)},\quad|z|\leq r<\vartheta_{\rho,\beta,1}.$$

Owing to the minimum principle for harmonic functions, the equality holds if and only if $z = r$. Thus, for $r \in (0, \vartheta_{\rho,\beta,1})$ we get

$$\inf_{|z|<r} \left\{ \Re\left(1 + \frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right) - \left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right| \right\} = 1 + 2r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}.$$

The function $v_{\rho,\beta} : (0, \vartheta_{\rho,\beta,1}) \to \mathbb{R}$, defined by

$$v_{\rho,\beta}(r) = 1 + 2r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)},$$

is strictly decreasing and

$$\lim_{r\searrow 0} v_{\rho,\beta}(r) = 1, \quad \lim_{r\nearrow \vartheta_{\rho,\beta,1}} v_{\rho,\beta}(r) = -\infty.$$

Consequently, the equation

$$1 + 2r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} = 0$$

has a unique root r_4 in $(0, \vartheta_{\rho,\beta,1})$, and $r_4 = r^{uc}(g_{\rho,\beta})$.

c. Let $\tau_{\rho,\beta,n}$ denote the nth positive zero of the function $h_{\rho,\beta}(z)$. In [12, Theorem 5] the following equation was obtained:

$$\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)} = -\sum_{n\geq 1} \frac{z}{\tau_{\rho,\beta,n} - z}, \tag{30}$$

and, in the same paper, with the help of (30), the following inequality was given:

$$\Re\left(1 + \frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right) \geq 1 + \frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}, \quad |z| \leq r < \tau_{\rho,\beta,1} < \lambda_{\rho,\beta,1}. \tag{31}$$

From (30) we get

$$\left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right| = \left|\sum_{n\geq 1} \frac{z}{\tau_{\rho,\beta,n} - z}\right| \leq \sum_{n\geq 1} \left|\frac{z}{\tau_{\rho,\beta,n} - z}\right| \leq \sum_{n\geq 1} \frac{r}{\tau_{\rho,\beta,n} - r} \tag{32}$$
$$= -\frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}, \quad |z| \leq r < \tau_{\rho,\beta,1}.$$

From inequality (31) and (32) we deduce that

$$\Re\left(1 + \frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right) - \left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right| \geq 1 + 2\frac{rh''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}, \quad |z| \leq r < \tau_{\rho,\beta,1}.$$

Due to the minimum principle for harmonic functions, the equality holds if and only if $z = r$. Thus, we find that

$$\inf_{|z|<r} \left\{ \Re\left(1 + \frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right) - \left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right| \right\} = 1 + 2r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)},$$

for every $r \in (0, \tau_{\rho,\beta,1})$. Since the function $w_{\rho,\beta}(r) : (0, \tau_{\rho,\beta,1}) \to \mathbb{R}$ defined by

$$w_{\rho,\beta}(r) = 1 + 2r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)} = 1 - \sum_{n\geq 1} \frac{2r}{\tau_{\rho,\beta,n} - r}$$

is strictly decreasing on $(0, \tau_{\rho,\beta,1})$, and

$$\lim_{r\searrow 0} w_{\rho,\beta}(r) = 1, \quad \lim_{r\nearrow \tau_{\rho,\beta,1}} w_{\rho,\beta}(r) = -\infty,$$

it follows that the equation $w_{\rho,\beta}(r) = 0$ has a unique root $r_5 \in (0, \tau_{\rho,\beta,1})$, and this root is the radius of uniform convexity. □

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