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Almost paracontact structures obtained from $G^*_{2(2)}$ structures

Nülifer ÖZDEMİR[1], **Şirin AKTAY**[1], **Mehmet SOLGUN**[2,*]

[1]Department of Mathematics, Faculty of Science, Eskişehir Technical University, Eskişehir, Turkey

[2]Department of Mathematics, Faculty of Science, Bilecik Şeyh Edebali University, Bilecik, Turkey

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Abstract: In this paper, we construct almost paracontact metric structures by using the fundamental 3-forms of manifolds with $G^*_{2(2)}$ structures. The existence of certain almost paracontact metric structures is investigated due to the properties of the 2-fold vector cross-product. Furthermore, we give some relations between the classes of $G^*_{2(2)}$ structures and almost paracontact metric structures.

Key words: $G^*_{2(2)}$ structure, almost paracontact metric structure

1. Introduction

Almost paracontact structures on manifolds of odd dimension, analogues of the almost contact structures on manifolds, were first introduced by Kaneyuki and Williams in [5]. After the work of Zamkovoy in [10], almost paracontact metric structures have been a widely studied research area. In [11], almost paracontact metric structures were classified into 2^{12} classes taking into consideration the Levi-Civita covariant derivative of the fundamental 2-form of the structure.

Almost contact metric structures induced by G_2 structures were constructed by Matzeu and Munteanu in [7]; see also [1]; and the possible classes that these structures may belong to were considered in [8].

The objective of this manuscript is the investigation of almost paracontact metric structures on manifolds with structure group $G^*_{2(2)}$. First, we construct almost paracontact metric structures induced by $G^*_{2(2)}$ structures. Then we investigate the relation between the classes of almost paracontact metric structures and $G^*_{2(2)}$ structures. In addition, we give an elementary example to support the arguments of the manuscript.

2. Preliminaries

Consider $\mathbb{R}^7$ with the standard basis $\{e_1, ..., e_7\}$. The fundamental 3-form on $\mathbb{R}^7$ is defined as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $\{e^1, ..., e^7\}$ denotes the basis dual to $\{e_1, ..., e_7\}$ and $e^{ijk} = e^i \wedge e^j \wedge e^k$. The Lie group G_2 is defined by

$$G_2 := \{f \in GL(7, \mathbb{R}) \mid f^*\varphi_0 = \varphi_0\};$$

see [3].

*Correspondence: mehmet.solgun@bilecik.edu.tr

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A 7-dimensional oriented manifold M has a G_2 structure if and only if its structure group reduces to G_2. Then there is a 3-form φ on M with the property that $(T_pM, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$, for all $p \in M$, said to be the fundamental 3-form or the G_2 structure on M. Manifolds (M, g) with G_2 structure were classified into 16 classes in [4].

The noncompact dual of G_2 is the group

$$G^*_{2(2)} = \{g \in GL(7, \mathbb{R}) \mid g^*\widetilde{\varphi} = \widetilde{\varphi}\},$$

where

$$\widetilde{\varphi} = -e^{127} - e^{135} + e^{146} + e^{236} + e^{245} - e^{347} + e^{567}$$

and $\{e^1, ..., e^7\}$ denotes the dual to the standard basis of $\mathbb{R}^{4,3} = (\mathbb{R}^7, g_{4,3})$ with the metric $g_{4,3} = (-1, -1, -1, -1, 1, 1, 1)$. A semi-Riemannian manifold M with the metric of signature $(-, -, -, -, +, +, +)$ whose structure group reduces to $G^*_{2(2)}$ is called a manifold with $G^*_{2(2)}$ structure. Similar to the G_2 case, there is the fundamental 3-form (or the $G^*_{2(2)}$ structure) $\widetilde{\varphi}$ on M inducing a metric $g_{4,3}$, a volume form, and a 2-fold vector cross-product $\widetilde{P}$ on M, which can be calculated via

$$\widetilde{\varphi}(X, Y, Z) = g_{4,3}(\widetilde{P}(X, Y), Z); \tag{2.1}$$

see [3]. Similar to the G_2 case, a $G^*_{2(2)}$ structure $\widetilde{\varphi}$ satisfying $\nabla^{g_{4,3}}\widetilde{\varphi} = 0$ is called a parallel $G^*_{2(2)}$ structure and a $G^*_{2(2)}$ structure with $\nabla^{g_{4,3}}_X\widetilde{\varphi}(X, Y, Z) = 0$ is called nearly parallel [6].

For convenience, throughout the paper, a $G^*_{2(2)}$ structure and the induced vector cross-product will be denoted by φ and P, respectively.

A triple (ϕ, ξ, η) on a $2n+1$-dimensional differentiable manifold M^{2n+1} satisfying

$$\phi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.2}$$

where ϕ is a $(1,1)$ tensor field, ξ is a vector field, and η is a 1-form η on M, is called an almost paracontact structure on M and M is called an almost paracontact manifold. As a consequence of (2.2), one can see that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$ on the almost paracontact structure (ϕ, ξ, η).

If an almost paracontact manifold M has a semi-Riemannian metric g of signature $(n, n+1)$ satisfying

$$g(\phi(X), \phi(Y)) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

then M is an almost paracontact metric manifold having the almost paracontact metric structure (ϕ, ξ, η, g) and g is said to be a compatible metric.

The 2-form

$$\Phi(X, Y) := g(\phi(X), Y)$$

is said to be the fundamental 2-form of the almost paracontact metric structure. It is known that on an almost paracontact metric manifold there is an orthonormal basis (called a ϕ-basis) $\{e_1, \phi e_1, \cdots, e_n, \phi e_n, \xi\}$ with

$$g(e_i, e_j) = -g(\phi e_i, \phi e_j) = \delta_{ij}, \quad g(e_i, \phi e_j) = 0, \quad i, j = 1, \cdots, n;$$

see [10]. For the almost contact case, see [2].

Let F be the $(0,3)$ tensor field defined by

$$F(X,Y,Z) = (\nabla_X \Phi)(Y,Z) = g((\nabla_X \phi)Y, Z), \tag{2.4}$$

for $X, Y, Z \in TM$. It can be seen that F has the following properties:

$$\begin{aligned} &F(X,Y,Z) = -F(X,Z,Y), \\ &F(X,\phi(Y),\phi(Z)) = F(X,Y,Z) + \eta(Y)F(X,Z,\xi) - \eta(Z)F(X,Y,\xi). \end{aligned} \tag{2.5}$$

In [11], a classification of almost paracontact metric manifolds was obtained by considering the space $\mathcal{F}$ of tensors F that satisfy (2.5). Initially, this space was decomposed into four subspaces

$$W_1 = \left\{ F \in \mathcal{F} \;\middle|\; \begin{array}{l} F(X,Y,Z) = g(\mathcal{A}_Y^F X, Z), \\ F(\xi,Y,Z) = g(\mathcal{A}_Y^F \xi, Z) = 0, \\ F(X,\xi,Z) = g(\mathcal{A}_\xi^{'F} X, \phi(Z)) = 0 \end{array} \right\}, \tag{2.6}$$

$$W_2 = \left\{ F \in \mathcal{F} \;\middle|\; \begin{array}{ll} F(X,Y,Z) & = \eta(Y)g(\phi(\mathcal{A}_\xi^{'F} X), Z) \\ & \quad +\eta(Z)g(\phi(\mathcal{A}_\xi^{'F} X), Y), \\ \mathcal{A}_\xi^{'F}\xi & = 0 \end{array} \right\}, \tag{2.7}$$

$$W_3 = \mathcal{G}_{11} = \{F \in \mathcal{F} | F(X,Y,Z) = \eta(X)F(\xi, \phi(Y), \phi(Z)\}, \tag{2.8}$$

$$W_4 = \mathcal{G}_{12} = \{F \in \mathcal{F} | F(X,Y,Z) = \eta(X)(\eta(Y)\omega_F(Z) - \eta(Z)\omega_F(Y))\}, \tag{2.9}$$

where $\mathcal{A}_X^F Y = (\nabla_Y \phi)(X), \mathcal{A}_\xi^{'F} X = \nabla_X \xi$ and $\omega_F(X) = F(\xi,\xi,X)$. Then W_1 and W_2 were written as sums of $U(n) \times 1$ irreducible components $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, and $\mathcal{G}_5, \cdots, \mathcal{G}_{10}$ respectively, where $U(n)$ is the paraunitary group, with the following defining relations [11]:

$\mathcal{G}_1$	$F(X,Y,Z) = \frac{1}{2(n-1)}\{g(X,\phi Y)\theta_F(\phi Z) - g(X,\phi Z)\theta_F(\phi Y)$ $+g(\phi X,\phi Z)\theta_F(hY) - g(\phi X,\phi Y)\theta(hZ)\}$
$\mathcal{G}_2$	$F(\phi X,\phi Y,Z) = -F(X,Y,Z), \quad \theta_F = 0$
$\mathcal{G}_3$	$F(\xi,Y,Z) = F(X,\xi,Z) = 0, \quad F(X,Y,Z) = -F(Y,X,Z)$
$\mathcal{G}_4$	$F(\xi,Y,Z) = F(X,\xi,Z) = 0, \quad \mathfrak{S}_{(X,Y,Z)}F(X,Y,Z) = 0$
$\mathcal{G}_5$	$F(X,Y,Z) = \frac{\theta_F(\xi)}{2n}\left[\eta(Y)g(\phi X,\phi Z) - \eta(Z)g(\phi X,\phi Y)\right]$
$\mathcal{G}_6$	$F(X,Y,Z) = -\frac{\theta^*_F(\xi)}{2n}\left[\eta(Y)g(X,\phi Z) - \eta(Z)g(X,\phi Y)\right]$
$\mathcal{G}_7$	$F(X,Y,Z) = -F(Y,Z,X) + F(Z,X,Y) - 2F(\phi X,\phi Y,Z),$ $= -F(\phi X,\phi Y,Z) - F(\phi X,Y,\phi Z)$ $\theta^*_F(\xi) = 0$
$\mathcal{G}_8$	$F(X,Y,Z) = -F(Y,Z,X) - F(Z,X,Y),$ $= -F(\phi X,\phi Y,Z) - F(\phi X,Y,\phi Z)$ $\theta_F(\xi) = 0$
$\mathcal{G}_9$	$F(X,Y,Z) = -F(Y,Z,X) + F(Z,X,Y) + 2F(\phi X,\phi Y,Z),$ $= F(\phi X,\phi Y,Z) + F(\phi X,Y,\phi Z)$
$\mathcal{G}_{10}$	$F(X,Y,Z) = -F(Y,Z,X) - F(Z,X,Y)$ $= F(\phi X,\phi Y,Z) + F(\phi X,Y,\phi Z)$
$\mathcal{G}_{11}$	$F(X,Y,Z) = \eta(X)F(\xi,\phi Y,\phi Z)$
$\mathcal{G}_{12}$	$F(X,Y,Z) = \eta(X)\left[\eta(Y)\omega_F(Z) - \eta(Z)\omega_F(Y)\right]$

where $\theta_F(X) = g^{ij}F(e_i,e_j,X)$, $\theta^*_F(X) = g^{ij}F(e_i,\phi(e_j,X)$, and $h(X) = \phi^2(X)$.

The trivial class denoted by $\mathcal{G}_0$, for which the defining relation is $\nabla\Phi = 0$, is the class of paracosymplectic structures. The classes $\mathcal{G}_5$ and $\mathcal{G}_6$ correspond to α-para-Sasakian and β-para-Kenmotsu structures, respectively. Also, the defining relations of paracontact and almost K-paracontact classes are $d\eta = \Phi$ and $\nabla_\xi\Phi = 0$, respectively.

Let (M,ϕ,ξ,η,g) be an almost paracontact metric manifold. M is called normal if

$$\phi((\nabla_X\phi)(Y)) - (\nabla_{\phi X}\phi)(Y) + (\nabla_X\eta)(Y)\xi = 0; \tag{2.10}$$

see [9].

3. Almost paracontact metric structures and $G^*_{2(2)}$ structures

Consider a 7-dimensional smooth manifold M with a $G^*_{2(2)}$-structure φ inducing the pseudo-Riemannian metric $g_{4,3}$ and the vector cross-product P. Let ξ be a nonzero vector field on M such that $g_{4,3}(\xi,\xi) = -1$. Then the quadruple (ϕ,ξ,η,g), where the endomorphism is

$$\phi(X) = P(\xi,X) \tag{3.1}$$

and $g = -g_{4,3}$, $\eta(X) = g(\xi, X)$, is an almost paracontact metric structure on M. Indeed, we have

$$\begin{aligned}\phi^2 X &= \phi(\phi X) = \phi(P(\xi, X)) = P(\xi, P(\xi, X)) \\ &= -g_{4,3}(\xi,\xi)X + g_{4,3}(\xi, X)\xi = g(\xi,\xi)X - g(\xi, X)\xi \\ &= X - \eta(X)\xi\end{aligned}$$

and

$$\begin{aligned}g(\phi X, \phi Y) &= -g_{4,3}(P(\xi, X), P(\xi, Y) \\ &= -g_{4,3}(\xi,\xi)g_{4,3}(X, Y) + g_{4,3}(\xi, X)g_{4,3}(\xi, Y) \\ &= -g(X, Y) + \eta(X)\eta(Y).\end{aligned}$$

Throughout the paper, unless otherwise stated, (ϕ, ξ, η, g) corresponds to the almost paracontact metric structure (a.p.m.s.) obtained by a $G^*_{2(2)}$ structure φ on M. Note that $\nabla^g = \nabla^{g_{4,3}}$ and we use the notation ∇ for the Levi-Civita covariant derivative ∇^g.

The following proposition gives a relation between the covariant derivatives of the fundamental 2-form of the almost paracontact structure and of the $G^*_{2(2)}$ structure φ.

Proposition 3.1 *For an a.p.m.s. (ϕ, ξ, η, g) on M, the equation*

$$(\nabla_X \Phi)(Y, Z) = -(\nabla_X \varphi)(\xi, Y, Z) - \varphi(\nabla_X \xi, Y, Z) \tag{3.2}$$

holds.

Proof

$$\begin{aligned}(\nabla_X \varphi)(\xi, Y, Z) &= g_{4,3}(\nabla_X P(\xi, Y), Z) - g_{4,3}(P(\nabla_X \xi, Y), Z) - g_{4,3}(P(\xi, \nabla_X Y), Z) \\ &= -g(\nabla_X(\phi Y), Z) - \varphi(\nabla_X \xi, Y, Z) + g(\phi(\nabla_X Y), Z) \\ &= -g((\nabla_X \phi)(Y), Z) - \varphi(\nabla_X \xi, Y, Z) \\ &= -(\nabla_X \Phi)(Y, Z) - \varphi(\nabla_X \xi, Y, Z).\end{aligned}$$

□

The following proposition gives a condition for almost paracontact metric structures induced by $G^*_{2(2)}$ structures to be paracontact.

Proposition 3.2 *An a.p.m.s. (ϕ, ξ, η, g) induced by a $G^*_{2(2)}$ structure is paracontact (i.e. $d\eta = \Phi$) if and only if ξ satisfies*

$$g_{4,3}(P(\xi, X), Y) = \frac{1}{2}(g_{4,3}(\nabla_X \xi, Y) - g_{4,3}(\nabla_Y \xi, X)). \tag{3.3}$$

Proof The exterior derivative of η is:

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X. \tag{3.4}$$

After some calculations, the following is obtained:

$$d\eta(X,Y) = \frac{1}{2}(-g_{4,3}(\nabla_X \xi, Y) + g_{4,3}(\nabla_Y \xi, X)).$$

Besides, for the corresponding almost paracontact metric structure, we have $\Phi(X,Y) = g(\phi(X),Y) = -g_{4,3}(P(\xi,X),Y)$. Thus, $d\eta = \Phi$ if the relation (3.3) holds. □

Theorem 1 *An a.p.m.s. (ϕ, ξ, η, g) induced by a parallel $G^*_{2(2)}$ structure φ on $(M, g_{4,3})$ (i.e $\nabla^{g_{4,3}}\varphi = 0$) is in the class $\mathcal{G}_0(\nabla\Phi = 0)$ (paracosymplectic) if and only if the vector field ξ is parallel.*

Proof Let φ be a parallel structure; that is, $\nabla\varphi = 0$. Then from the equation (3.2), we have

$$(\nabla_X \Phi)(Y,Z) = -\varphi(Y,Z,\nabla_X \xi) = -g_{4,3}(P(Y,Z),\nabla_X \xi),$$

which implies

$$\nabla\Phi = 0 \iff \nabla\xi = 0.$$

□

Theorem 2 *For an a.p.m.s. (ϕ, ξ, η, g), if ξ is not parallel, then the structure is not in W_1.*

Proof Consider the equation

$$g(\mathcal{A}'^F_\xi X, \phi Z) = g(\nabla_X \xi, \phi Z).$$

Letting the vector field ξ not be parallel, then there exists X_0 such that $\nabla_{X_0}\xi \neq 0$ and obviously the third condition of the defining relation (2.6) of W_1 fails. □

Note that, under the assumption of Theorem 2, the structure is not an element of any subclass of $W_1 = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3 \oplus \mathcal{G}_4$.

Theorem 3 *If the $G^*_{2(2)}$ structure φ is nearly parallel and ξ is parallel, then (ϕ, ξ, η, g) is in W_1.*

Proof Let φ be nearly parallel; that is,

$$(\nabla_X \varphi)(X,Y,Z) = 0,$$

and let ξ be parallel, i.e. $\nabla\xi = 0$. Then, from equation (3.2),

$$F(\xi,Y,Z) = -(\nabla_\xi \varphi)(\xi,Y,Z) - \varphi(\nabla_\xi \xi, Y, Z) = 0$$

and

$$F(X,\xi,Z) = -(\nabla_X \varphi)(\xi,\xi,Z) - \varphi(\nabla_X \xi, \xi, Z) = 0.$$

Thus, the definition of W_1 is satisfied. □

Theorem 4 *An a.p.m.s. (ϕ, ξ, η, g) from a nearly parallel structure φ satisfies $\nabla_\xi \Phi = 0$ (almost K-paracontact) if and only if $\nabla_\xi \xi = 0$.*

Proof Let φ be nearly parallel. Then this is an immediate consequence of formula (3.2) and of the definition of the nearly parallel $G^*_{2(2)}$ structure. Indeed,

$$(\nabla_\xi \Phi)(X,Y) = -(\nabla_\xi \varphi)(\xi, X, Y) - \varphi(\nabla_\xi \xi, X, Y) = -\varphi(\nabla_\xi \xi, X, Y).$$

Then

$$\nabla_\xi \Phi = 0 \iff \nabla_\xi \xi = 0.$$

□

Note that an a.p.m.s. (ϕ, ξ, η, g) such that $\nabla_\xi \xi \neq 0$ cannot be in the class W_2 by the definition of W_2. In addition, if ξ is not Killing, the structure is not in the class $\mathcal{G}_5 \oplus \mathcal{G}_8$.

Theorem 5 *If ξ is not parallel, then the structure (ϕ, ξ, η, g) is not an element of $W_3(=\mathcal{G}_{11})$.*

Proof Take $Y = \xi$ in the defining relation (2.8) of the class W_3. Then, as a consequence of the formula (3.2), the left-hand side of (2.8) is

$$\begin{aligned}(\nabla_X \Phi)(\xi, Z) &= X[\Phi(\xi, Z)] - \Phi(\nabla_X \xi, Z) - \Phi(\xi, \nabla_X Z) \\ &= g(\phi Z, \nabla_X \xi),\end{aligned}$$

while the right-hand side vanishes since $\phi(\xi) = 0$. Thus, if $\nabla \xi \neq 0$ (i.e. ξ is not parallel), the structure can not be in the class W_3. □

Theorem 6 *If there exists a vector field $X \in \{\xi\}^\perp$ with the property $\nabla_X \xi \neq 0$, then the structure (ϕ, ξ, η, g) is not in $W_4(=\mathcal{G}_{12})$.*

Proof Let $X \in \{\xi\}^\perp$ with $\nabla_X \xi \neq 0$. Take $Y = \xi$ in the defining relation (2.9) of the class W_4. Then $\eta(X) = 0$ since $X \in \{\xi\}^\perp$, so the right-hand side of the relation (2.9) is zero. On the other hand, from formula (3.2),

$$\begin{aligned}(\nabla_X \Phi)(Y, Z) &= (\nabla_X \Phi)(\xi, Z) \\ &= X[\Phi(\xi, Z)] - \Phi(\nabla_X \xi, Z) - \Phi(\xi, \nabla_X Z) \\ &= g(\phi Z, \nabla_X \xi).\end{aligned}$$

Therefore, $(\nabla_X \Phi)(Y, Z)$ does not have to be zero since $\nabla_X \xi \neq 0$. Hence, the defining relation is not satisfied under the given conditions. □

Example 7 *Consider the seven-dimensional Lie algebra $\mathfrak{L}$ with nonzero brackets*

$$[e_1, e_2] = e_5, \quad [e_1, e_3] = e_6.$$

Then $\mathfrak{L}$ admits the $G^*_{2(2)}$ structure

$$\varphi = e^{567} - e^{512} - e^{534} - e^{613} + e^{624} + e^{714} + e^{723}. \tag{3.5}$$

The metric $g_{4,3}$ induced by φ is

$$g_{4,3}(x, y) = x_5 y_5 + x_6 y_6 + x_7 y_7 - x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4$$

for any vector fields $x=\sum x_i e_i$, $y=\sum y_i e_i$; *see [3]. Note that* $g_{4,3}(e_i,e_i)=-1$ *for* $i=1,2,3,4$ *and* $g_{4,3}(e_i,e_i)=1$ *otherwise. The cross-product of frame elements are obtained via (2.1):*

$$P(e_1,e_2)=-e_5,\quad P(e_1,e_3)=-e_6,\quad P(e_1,e_4)=e_7,\quad P(e_1,e_5)=-e_2,$$

$$P(e_1,e_6)=-e_3,\quad P(e_1,e_7)=e_4,\quad P(e_2,e_3)=e_7,\quad P(e_2,e_4)=e_6,$$

$$P(e_2,e_5)=e_1,\quad P(e_2,e_6)=e_4,\quad P(e_2,e_7)=e_3,\quad P(e_3,e_4)=-e_5,$$

$$P(e_3,e_5)=-e_4,\quad P(e_3,e_6)=e_1,\quad P(e_3,e_7)=-e_2,\quad P(e_4,e_5)=e_3,$$

$$P(e_4,e_6)=-e_2,\quad P(e_4,e_7)=-e_1,\quad P(e_5,e_6)=e_7,\quad P(e_5,e_7)=-e_6, P(e_6,e_7)=e_5.$$

The nonzero Levi-Civita covariant derivatives evaluated by Kozsul's formula are

$$\nabla_{e_1}e_2=\frac{1}{2}e_5,\nabla_{e_1}e_3=\frac{1}{2}e_6,\nabla_{e_1}e_5=\frac{1}{2}e_2,\nabla_{e_1}e_6=\frac{1}{2}e_3,\nabla_{e_2}e_1=\frac{-1}{2}e_5,\nabla_{e_2}e_5=\frac{-1}{2}e_1$$

$$\nabla_{e_3}e_1=\frac{-1}{2}e_6,\nabla_{e_3}e_6=\frac{-1}{2}e_1,\nabla_{e_5}e_1=\frac{1}{2}e_2,\nabla_{e_5}e_2=\frac{-1}{2}e_1,\nabla_{e_6}e_1=\frac{1}{2}e_3,\nabla_{e_6}e_3=\frac{-1}{2}e_1.$$

Now we investigate the existence of certain classes on $\mathfrak{L}$.

Assume that a nonzero vector field $X=a_1e_1+\cdots+a_7e_7$ *is parallel. Then,*

$$\begin{aligned}\nabla_{e_1}X&=a_1\nabla_{e_1}e_1+a_2\nabla_{e_1}e_2+a_3\nabla_{e_1}e_3+a_4\nabla_{e_1}e_4+a_5\nabla_{e_1}e_5+a_6\nabla_{e_1}e_6+a_7\nabla_{e_1}e_7\\&=\frac{a_2}{2}e_5+\frac{a_3}{2}e_6+\frac{a_5}{2}e_2+\frac{a_6}{2}e_3\\&=0\iff a_2=a_3=a_5=a_6=0.\end{aligned}$$

On the other hand,

$$\nabla_{e_2}X=-\frac{a_1}{2}e_5=0\iff a_1=0 \tag{3.6}$$

and there is no other restriction on the coefficients a_i. *Thus,* $X=\sum a_ie_i$ *is parallel iff* $X=a_4e_4+a_7e_7$, *that is, iff* $X\in span\{e_4,e_7\}$.

Note that the $G^*_{2(2)}$ *structure (3.5) is neither parallel (since* $(\nabla_{e_1}\varphi)(e_2,e_3,e_4)=1\neq 0$*) nor nearly parallel (since* $(\nabla_{e_1}\varphi)(e_2,e_3,e_4)+(\nabla_{e_2}\varphi)(e_1,e_3,e_4)=\frac{1}{2}\neq 0$*).*

Now we give an example of an a.p.m.s. such that the characteristic vector field is parallel. Let (ϕ,ξ,η,g) *be the a.p.m.s. induced by the* $G^*_{2(2)}$ *structure (3.5), where* $\xi=e_4$ *and* $g=-g_{4,3}$. *Then from the equation (3.1), we get* $\phi(e_1)=P(e_4,e_1)=-e_7$, $\phi(e_2)=-e_6$, $\phi(e_3)=e_5$, $\phi(e_4)=0$, $\phi(e_5)=e_3$, $\phi(e_6)=-e_2$, $\phi(e_7)=-e_1$. *Since* $(\nabla_{e_1}\phi)(e_2)=-e_3\neq 0$, *this structure is not paracosymplectic. Theorem 1 states that an a.p.m.s. induced by a parallel* $G^*_{2(2)}$ *structure is paracosymplectic if and only if the characteristic vector field is parallel. This example shows that if the* $G^*_{2(2)}$ *structure is not parallel, we can obtain a.p.m. structures that are not paracosymplectic but have parallel characteristic vector fields.*

It is easy to check that this structure is in W_1, *although the* $G^*_{2(2)}$ *structure is not nearly parallel, comparing with Theorem 3.*

Now let (ϕ,ξ,η,g) *be the a.p.m.s. induced by the* $G^*_{2(2)}$ *structure (3.5), where* $\xi=e_2$ *(*ξ *is not parallel in this case) and* $g=-g_{4,3}$. *By Theorem 2, this structure is not in* W_1. *In addition, it is not in* W_3 *by Theorem 5. Also, since* $\nabla_{e_1}e_2=\frac{1}{2}e_5\neq 0$, *this structure is not in* W_4 *by Theorem 6. From the equation (3.1), we have* $\phi(e_1)=e_5$,

$\phi(e_2) = 0$, $\phi(e_3) = e_7$, $\phi(e_4) = e_6$, $\phi(e_5) = e_1$, $\phi(e_6) = e_4$, $\phi(e_7) = e_3$. Since $(\nabla_{e_1}\phi)(e_1) = \frac{1}{2}e_2 \neq 0$, this structure is not paracosymplectic. One can check that $\nabla_\xi \phi = \nabla_{e_2}\phi = 0$; that is, this structure is almost-K-paracontact.

Now we investigate the existence of paracontact structures on $\mathfrak{L}$ induced by the $G^*_{2(2)}$ structure (3.5). Let (ϕ, ξ, η, g) be such a structure with fundamental 2-form Φ; that is, $d\eta = \Phi$. Since $de^5 = e^{12}$, $de^6 = e^{13}$, for $\eta = \sum b_i e^i$, $i = 1, \ldots, 7$, we have $d\eta = b_5 e^{12} + b_6 e^{13} = \Phi$. This implies $\phi(e_5) = 0$. From the equation

$$g(\phi(e_5), \phi(e_5)) = -g(e_5, e_5) + \eta^2(e_5),$$

we obtain $\eta^2(e_5) = -1$, which is a contradiction. Therefore, there is no paracontact structure on $\mathfrak{L}$ induced by the given $G^*_{2(2)}$ structure.

Finally, we study the existence of α-para-Sasakian structures on $\mathfrak{L}$ induced by the $G^*_{2(2)}$ structure (3.5). Let (ϕ, ξ, η, g) be an α-para-Sasakian structure induced by (3.5). Note that $g = -g_{4,3}$. The characteristic vector field ξ is Killing. From the equation

$$g(\nabla_{e_i}\xi, e_j) + g(\nabla_{e_j}\xi, e_i) = 0, \tag{3.7}$$

we obtain that ξ is Killing if and only if $a_1 = a_2 = a_3 = 0$. Thus, $\xi = a_4 e_4 + \ldots + a_7 e_7$. From the definition of an α-para-Sasakian structure, we have $\phi(X) = \frac{1}{\alpha}\nabla_x \xi$ for all vector fields X. Then $\phi(e_2) = -\frac{a_5}{2\alpha}e_1$ and $\phi(e_3) = -\frac{a_6}{2\alpha}e_1$. The equation

$$g(\phi(e_2), \phi(e_3)) = -g(e_2, e_3) + \eta(e_2)\eta(e_3)$$

implies $a_5 a_6 = 0$. Thus, $\phi(e_1) = 0$ or $\phi(e_2) = 0$. Assume without loss of generality that $\phi(e_1) = 0$. Since

$$0 = g(\phi(e_1), \phi(e_1)) \neq -g(e_1, e_1) + \eta^2(e_1) = -1,$$

there is no α-para-Sasakian structure induced by (3.5).

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