Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 3034 – 3049
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doi:10.3906/mat-1801-96

Linearized four-step implicit scheme for nonlinear parabolic interface problems

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Received: 29.01.2018 • **Accepted/Published Online:** 03.10.2018 • **Final Version:** 27.11.2018

Abstract: We present the solution of a second-order nonlinear parabolic interface problem on a quasiuniform triangular finite element with a linearized four-step implicit scheme used for the time discretization. The convergence of the scheme in L^2-norm is established under certain regularity assumptions using interpolation and elliptic projection operators. A numerical experiment is presented to support the theoretical result. It is assumed that the interface cannot be fitted exactly.

Key words: Four-step implicit, interface, almost-optimal, nonlinear parabolic equation

1. Introduction

Parabolic interface problems are frequently encountered in scientific computing and industrial applications. A typical example is provided in the modeling of heat diffusion, which involves two or more materials with different properties [8]. The most well-known linear parabolic partial differential equation (PDE) is the heat equation. However, the linear heat equation has some limitations that could be addressed by nonlinear generalizations [5]. It is therefore necessary to investigate the solution of nonlinear PDEs on bounded domains. The problem becomes an interface problem when more than one material medium with different properties is involved.

Many contributions have been made towards the development of the finite element method (FEM) for linear parabolic interface problems, e.g., [2, 4, 13–16, 21]. Semilinear parabolic interface problems were considered in [6, 17]. The finite element solution of nonlinear parabolic interface problems with time discretization based on the θ method was discussed in [7]. With necessary assumptions that guarantee the uniqueness of solutions, it was shown that the scheme preserves the discrete maximum principle. The results were based on the algebraic discrete maximum principle for suitable ODE systems.

Yang [20] proposed and analyzed a linearized 2-step backward difference-finite element method for the solution of the nonlinear parabolic interface problem with linear source term. With the assumption that the coefficient $\sigma(u)$ is positive and smooth with respect to $u \in \mathbb{R}$ but not continuous across the interface, the author proved a convergence rate of almost optimal order in the L^2-norm. Solution of the quasilinear parabolic interface problem using the antisymmetric interior penalty discontinuous Galerkin method was proposed in [18]. Again the time discretization was based on a second-order linearized backward difference scheme. Use was made of the over-penalized method to improve the L^2-norm error to optimal order with the assumption that

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2010 *AMS Mathematics Subject Classification:* 65M60, 65M12, 65M06

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the diffusion coefficient was only continuous on each subdomain and the interface could be fitted exactly (using triangles with curved edges).

It is known that spatial and time discretizations are sources of errors in the FEM; however, research has largely focused on the use of the FEM for parabolic interface problems with emphasis on the improvement of the spatial discretization. In this work, we consider a nonlinear parabolic interface problem with nonlinear source term. The unknown function is approximated by piecewise linear functions on quasiuniform triangular elements with a four-step implicit scheme for time discretization. We consider the case where the triangulation cannot perfectly fit the interface and obtain a convergence rate of almost optimal order for a fully discrete scheme in $L^2(\Omega)$-norm. In this study, the linear theories of interface and noninterface problems and the Sobolev embedding inequality are used. Other technical tools used in this paper are approximation properties of the linear interpolation operator and projection operator.

In this work, we use the standard notations for Sobolev spaces and norms. For a given Banach space B, we define

$$W^{m,p}(0,T;B) = \begin{cases} u(t) \in B \text{ for a.e. } t \in (0,T) \quad \text{and} \ \displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt < 0 & \text{for } 1 \leq p < \infty \\ u(t) \in B \text{ for a.e. } t \in (0,T) \quad \text{and} \ \displaystyle\sum_{i=0}^{m} \operatorname*{ess\ sup}_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B < 0 & \text{for } p = \infty \end{cases}$$

equipped with the norms

$$\|u\|_{W^{m,p}(0,T;B)} = \begin{cases} \left[\displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt \right]^{1/p} & 1 \leq p < \infty \\ \displaystyle\sum_{i=0}^{m} \operatorname*{ess\ sup}_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B & p = \infty. \end{cases}$$

We write $L^2(0,T;B) = W^{0,2}(0,T;B)$ and $H^m(0,T;B) = W^{m,2}(0,T;B)$. We use the definition and notation in [1] when m is negative or fractional. We shall need the following space:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \quad \forall\, v \in X.$$

1.1. Problem specification

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ be another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$; see the Figure. We consider the parabolic interface problem

$$u_t - \nabla \cdot (a(x,u)\nabla u) = f(x,u) \quad \text{in} \quad \Omega \times (0,T] \tag{1.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \end{cases} \tag{1.2}$$

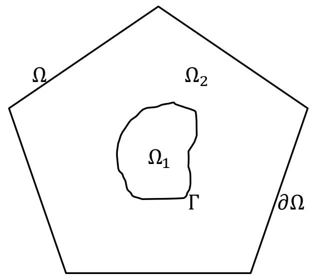

Figure 1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ.

and interface conditions

$$\left\{ \begin{array}{rcl} [u]_\Gamma & = & 0 \\ \left[a(x,u) \dfrac{\partial u}{\partial n} \right]_\Gamma & = & g(x,t) \end{array} \right. \tag{1.3}$$

where $0 < T < \infty$, the symbol $[u]$ is a jump of a quantity u across the interface Γ, and n is the unit outward normal to the boundary $\partial\Omega_1$. The interface conditions are defined as the difference of the limiting values from each side of the interface. The input functions $a(x,u)$ and $f(x,u)$ are assumed continuous on each domain but discontinuous across the interface for $t \in [0,T]$. We impose the following:

Assumption 1.1 A_1 *Ω is a bounded convex polygonal domain in $\mathbb{R}^2$; the interface $\Gamma \subset \Omega$ and the boundary $\partial\Omega$ are piecewise smooth, Lipschitz continuous, and 1-dimensional.*

A_2 *Functions $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable and bounded with respect to their first variable $x \in \Omega$ and continuously differentiable with respect to their second variable $\eta \in \mathbb{R}$, $g(x,t) \in L^2(0,T; H^2(\Gamma)) \cap H^1(0,T; H^{1/2}(\Gamma))$.*

A_3 *Functions a and f satisfy*

$$0 < \mu_1 \le a(x,\xi) \le \mu_2, \qquad \left| \frac{\partial a}{\partial \xi}(x,\xi) \right| + \left| \frac{\partial f}{\partial \xi}(x,\xi) \right| \le \mu_3 \,,$$

for $\xi \in \mathbb{R}$, $x \in \Omega$ with positive constants μ_1, μ_2, and μ_3 independent of (x,ξ).

In [12], we investigated the nonlinear interface problem (1.1) − (1.3). Under certain assumptions on the input data, we obtained regularity estimates that were used to establish convergence rates of almost optimal order in $H^1(\Omega)$-norm for both semi and full discretizations of the problem. The time discretization was based on an implicit Euler scheme and the implementation was based on predictor-corrector method due to the presence of the nonlinear term. This is computationally time-consuming as a time step will be computed twice. In this present work, a linearized four-step implicit scheme is proposed and analyzed to ease the computational stress and improve the accuracy.

The weak form of (1.1) − (1.3) is:
Find $u(t) \in H_0^1(\Omega)$, $t \in (0,T]$ such that

$$(u_t, v) + A(u : u, v) = (f, v) + \langle g, v \rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega),\ t \in (0,T] \tag{1.4}$$

where

$$(\phi, \psi) = \int_\Omega \phi\psi \, dx \qquad A(\xi : \phi, \psi) = \int_\Omega a(x, \xi)\nabla\phi \cdot \nabla\psi \, dx \qquad \langle \phi, \psi \rangle_\Gamma = \int_\Gamma \phi\psi \, d\Gamma.$$

We define

$$|f| \;=\; \sup_{x\in\Omega, \xi\in\mathbb{R}} |f(x, \xi)| \,.$$

For (1.1) − (1.3), we have the following regularity estimates (cf [12]):

Theorem 1.2 *Supposing* $u_0(x) \in H_0^1(\Omega)$ *and that the conditions of Assumption 1.1 are satisfied for every* $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, *and* $g \in L^2(0, T; H^{1/2}(\Gamma))$, *there exists a constant* C *depending on* μ_1, μ_2, μ_3, T, *and* Ω *such that*

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} + \|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le C \left(\|g\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|u_0\|_{L^2(\Omega)} + |f|\right)$$

and

$$\|u\|_{L^2(0,T;X)} \le C \left(\|g\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|u_0\|_{H^1(\Omega)} + |f|\right) \quad \textit{if } u_0(x) \in X \cap H^1(\Omega).$$

The paper is organized as follows. In Section 2, we describe a finite element discretization of the problem and state some auxiliary results. The linearized 4-step implicit scheme is presented in Section 3 and the almost optimal convergence rate is established. Numerical examples are presented in Section 4.

Throughout this paper, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences. The boundary value of $u \in H^1(\Omega)$ is defined in the sense of trace. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{L^2(\partial\Omega)} \le \|z\|_{H^{1/2}(\partial\Omega)} \le c_0\|z\|_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega). \tag{1.5}$$

See [1, 3] for more information on trace operators.

2. Finite element discretization

We adopt the discretization used in [2, 4]. $\mathcal{T}_h$ denotes a partition of Ω into disjoint triangles K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle. The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries, respectively.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ:

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}.$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$.

The triangulation $\mathcal{T}_h$ of the domain Ω satisfies the following conditions:

(i) $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} \bar{K}$.

(ii) If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$, and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

(iii) Each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.

(iv) For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles, respectively. It is assumed that, for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \leq h \leq C_1 \bar{r}_K \quad \forall\, h \in (0, h_0).$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$. The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x) \ ,$$

where each basis function ϕ_j, $(j = 1, 2, \ldots, N_h)$ is a pyramid function with unit height. For the approximation $\hat{g}(t)$, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h.

Let $\pi_h : C(\bar{\Omega}) \to S_h$ be the Lagrange interpolation operator corresponding to the space S_h. The standard interpolation theory cannot be applied because the solutions of interface problems are nonsmooth or even discontinuous across the interface. We have:

Lemma 2.1 *For the linear interpolation operator $\pi_h : C(\bar{\Omega}) \to S_h$, we have, for $m = 0, 1$ and $0 < h < 1$,*

$$\|u - \pi_h u\|_{H^m(\Omega)} \leq C h^{2-m} \left(1 + \frac{1}{|\log h|}\right)^{1/2} \|u\|_X \qquad \forall\, u \in X. \tag{2.1}$$

Proof See [2]. □

Remark 2.2 *In Lemma 2.1, it is assumed that the mesh cannot perfectly fit the interface. However, with the assumption that the interface can be fitted exactly using interface elements with curved edges, the optimal convergence rate is possible (see [14] for an example). In practice, the use of curved interface elements that perfectly fit the interface may be computationally difficult or impossible, particularly when the interface is of irregular shape. The convergence rate of optimal order is also obtainable when the approximation to the interface and the finite element spaces meet certain conditions [11]. Such conditions include $\Omega_h^\star \in S_\delta$ where S_δ is a δ-neighborhood of the interface, with $\delta = O(h^2)$. With this condition, interface elements need to divide more rapidly than noninterface elements to guarantee the optimal convergence rate.*

We recall some existing results that will be used in our analysis. See [4, 12, 16] for proofs.

Lemma 2.3 *Letting $\Omega_h^\star$ be the union of all interface elements, $\pi_h : C(\Omega) \to S_h$ be the interpolation operator, and $g \in H^2(\Gamma)$, we have*

$$\begin{aligned} \|v\|_{H^1(\Omega_h^\star)} &\leq Ch^{1/2}\|v\|_X \quad \forall\, v \in X && (2.2)\\ |\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| &\leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^\star)} \qquad \forall\, v_h \in S_h && (2.3)\\ |A(\xi : \nu_h, \omega_h) - A_h(\psi : \nu_h, \omega_h)| &\leq \mu_3\|\nabla\nu_h\|_{L^\infty(\Omega)}\|\xi - \psi\|_{L^2(\Omega)}\|\omega_h\|_{H^1(\Omega)} \\ &\quad + Ch\|\nu_h\|_{H^1(\Omega_h^\star)}\|\omega_h\|_{H^1(\Omega_h^\star)}. && (2.4)\end{aligned}$$

Let $P_h : X \cap H_0^1(\Omega) \to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(u : P_h\nu, \phi) = A(u : \nu, \phi) \quad \forall\, \phi \in S_h,\ t \in [0, T]. \tag{2.5}$$

It follows from (2.5) that there exists $C > 0$,

$$\|P_h\nu\|_{H^1(\Omega)} \leq C\|\nu\|_{H^1(\Omega)} \qquad \forall\, \nu \in H^1(\Omega). \tag{2.6}$$

For this projection, we have:

Lemma 2.4 *Let $a(x, u)$ satisfy the conditions of Assumption 1.1 and $u \in X \cap H_0^1(\Omega)$ for $t \in (0, T]$. Letting $P_h u$ be defined as in (2.5), then*

$$\begin{aligned} \|P_h u - u\|_{H^1(\Omega)} &\leq Ch\left(1 + \frac{1}{|\log h|}\right)^{1/2}\|u\|_X, && (2.7)\\ \|P_h u - u\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \frac{1}{|\log h|}\right)\|u\|_X. && (2.8)\end{aligned}$$

Proof For $\rho > 0$, we have:

$$\begin{aligned} \rho\|P_h u - u\|^2_{H^1(\Omega)} &\leq A_h(u : P_h u - u, P_h u - u)\\ &\leq |A(u : u, P_h u - \phi) - A_h(u : u, P_h u - \phi)|\\ &\quad + |A_h(u : P_h u - u, \phi - u)| \quad \phi \in S_h\\ &\leq Ch\|u\|_{H^1(\Omega)}\|P_h u - \phi\|_{H^1(\Omega)} + \|P_h u - u\|_{H^1(\Omega)}\|\phi - u\|_{H^1(\Omega)}\\ &\leq \varepsilon Ch^2\|u\|^2_{H^1(\Omega)} + \frac{3}{4\varepsilon}\|P_h u - u\|^2_{H^1(\Omega)} + \varepsilon\|\phi - u\|^2_{H^1(\Omega)}.\end{aligned}$$

(2.7) follows, using (2.1) with $\varepsilon = 2/\rho$ and $\phi = \pi_h u$.
Now consider the dual problem

$$-\nabla \cdot (a(x, u)\nabla\psi) = P_h u - u \qquad \text{in } \Omega,\ \psi = 0 \text{ on } \partial\Omega,$$

whose weak form is

$$A(u : \psi, \phi) = (P_h u - u, \phi) \quad \forall\, \phi \in H_0^1(\Omega). \tag{2.9}$$

By Assumption 1.1, it follows from a similar argument of Thomee [19, pg. 233] that

$$\|\psi\|_X \leq C\|P_h u - u\|_{L^2(\Omega)}. \tag{2.10}$$

Now, from (2.9), we obtain

$$\begin{aligned}\|P_h u - u\|^2_{L^2(\Omega)} &= A(u : P_h u - u, \psi)\\ &= A(u : P_h u - u, \psi - \phi) + A(u : P_h u - u, \phi) \quad \phi \in S_h\\ &\leq C\|P_h u - u\|_{H^1(\Omega)}\|\psi - \phi\|_{H^1(\Omega)} + |A(u : P_h u, \phi) - A_h(u : P_h u, \phi)|.\end{aligned}$$

It follows from (2.1), (2.4), (2.7), and (2.2) with $\phi = \pi_h \psi$ that

$$\|P_h u - u\|^2_{L^2(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\log h|}\right)\|u\|_X\|\psi\|_X + Ch^2\|P_h u\|_{H^1(\Omega)}\|\pi_h\psi\|_{H^1(\Omega)}.$$

(2.8) follows using (2.10), (2.6), and the fact that $\|\pi_h\psi\| \leq C\|\psi\|$ in the last inequality. □

Lemma 2.5 *Let $a(x,u)$ satisfy the conditions of Assumption 1.1, $u_{tt} \in L^2(\Omega)$, $u \in X \cap H^1_0(\Omega)$ for $t \in (0,T]$, and assume a_t is uniformly bounded. Letting $P_h u$ be defined as in (2.5), then*

$$\|(P_h u - u)_t\|_{H^1(\Omega)} \leq Ch\left(1 + \frac{1}{|\log h|}\right)^{1/2}(\|u\|_X + \|u_t\|_X), \tag{2.11}$$

$$\|(P_h u - u)_t\|_{L^2(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\log h|}\right)(\|u\|_X + \|u_t\|_X). \tag{2.12}$$

Proof Let $\xi = P_h u - u$, and assume that a_t is uniformly bounded. Following the argument of Thomee [19], we have

$$\begin{aligned}\rho\|\xi_t\|^2_{H^1(\Omega)} &\leq A(u : \xi_t, \xi_t)\\ &= A(u : \xi_t, \phi - u_t) + A(u : \xi_t, (P_h u)_t - \phi)\\ &= A(u : \xi_t, \phi - u_t) + \int_\Omega \left[\frac{\partial}{\partial t}(a\nabla\xi) - \frac{\partial a}{\partial t}\nabla\xi\right]\cdot\nabla((P_h u)_t - \phi)\, dx\\ &\leq \|\xi_t\|_{H^1(\Omega)}\|\phi - u_t\|_{H^1(\Omega)} + \|\xi\|_{H^1(\Omega)}\|(P_h u)_t - \phi\|_{H^1(\Omega)}.\end{aligned}$$

Take $\phi = \pi_h u_t$. Using (2.1), (2.7), and Young's inequality, we obtain

$$\|(P_h u - u)_t\|^2_{H^1(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\log h|}\right)(\|u\|^2_X + \|u_t\|^2_X). \tag{2.13}$$

Following the duality argument above, it is easy to see that

$$\|(P_h u - u)_t\|^2_{L^2(\Omega)} \leq Ch^4\left(1 + \frac{1}{|\log h|}\right)^2(\|u\|^2_X + \|u_t\|^2_X).$$

□

Remark 2.6 *Usually for the parabolic interface problem, the solution $u \in L^2(0,T;X) \cap H^1(0,T;Y)$ where $Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$ [4, 14]; however, the assumptions of Lemma 2.5 guarantee that $u_t \in X$ for $t \in (0,T]$. To see this, differentiate (1.1) with respect to t:*

$$\nabla\cdot(a\nabla u_t) = u_{tt} - \nabla\cdot(a_t\nabla u) - f_t \in L^2(\Omega_i) \quad \textit{for} \quad i = 1, 2, \quad t \in (0,T]. \tag{2.14}$$

Differentiate (1.4) with respect to t:

$$(u_{tt}, v) + A(u : u_t, v) + \int_\Omega a_t \nabla u \cdot \nabla v = (f_t, v) + \langle g_t, v\rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega),\ t \in (0,T].$$

Using (1.5) with $v = u_t$ and the fact that a_t is bounded, we have

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} + \frac{\mu_1}{2}\|u_t\|^2_{H^1(\Omega)} \le (\mu_1+\mu_3)\|u_t\|^2_{L^2(\Omega)} + \frac{c_1^2}{\mu_1}\|u\|^2_{H^1(\Omega)} + \frac{c_0^2}{\mu_1}\|g_t\|^2_{H^{1/2}(\Omega)}. \tag{2.15}$$

It follows from (2.15) and Theorem 1.2 that

$$\max_{0\le t\le T}\|u_t\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(0,T;H^1(\Omega))} \le C\left(\|g\|_{H^1(0,T;H^{1/2}(\Gamma))} + \|u_0\|_{L^2(\Omega)} + |f|\right). \tag{2.16}$$

We conclude from (2.14) and (2.16) that $u_t \in L^2(0,T;X)$.

3. Error estimate

In this section we propose a fully discrete scheme based on a four-step backward difference approximation. The almost optimal order error estimate is analyzed in $L^2(\Omega)$-norm. The finite element analysis of nonlinear noninterface problems is given in [19] and the references therein.

The interval $[0,T]$ is divided into M equally spaced (for simplicity) subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T,$$

with $t_n = nk$, $k = T/M$ being the time step. Let $I_n = (t_{n-1}, t_n]$ be the nth subinterval and let

$$u^n = u(x,t_n) \quad \text{and} \quad g^n = g(x,t_n)\ .$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotients defined by

$$\begin{aligned}
\partial^1 w^n &= \frac{w^n - w^{n-1}}{\tau_1} && n = 1,2,\ldots,M\\
\partial^2 w^n &= \frac{3w^n - 4w^{n-1} + w^{n-2}}{2\tau_2} && n = 2,3,\ldots,M\\
\partial^3 w^n &= \frac{11w^n - 18w^{n-1} + 9w^{n-2} - 2w^{n-3}}{6\tau_3} && n = 3,4,\ldots,M\\
\partial^4 w^n &= \frac{25w^n - 48w^{n-1} + 36w^{n-2} - 16w^{n-3} + 3w^{n-4}}{12k} && n = 4,6,\ldots,M.
\end{aligned}$$

The fully discrete finite element approximation to (1.4) is defined as follows:
Let $U_h^0 = \pi_h u_0$, and find $U_h^n \in S_h$, such that

$$(\partial^1 U_h^1, v_h)_h + A_h(U_h^0 : U_h^1, v_h) = (f(U_h^0, x), v_h)_h + \langle g_h^1, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \tag{3.1}$$

$$(\partial^2 U_h^2, v_h)_h + A_h(2U_h^1 - U_h^0 : U_h^2, v_h) = (f(2U_h^1 - U_h^0, x), v_h)_h + \langle g_h^2, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \tag{3.2}$$

$$(\partial^3 U_h^3, v_h)_h + A_h(3U_h^2 - 3U_h^1 + U_h^0 : U_h^3, v_h) = (f(3U_h^2 - 3U_h^1 + U_h^0, x), v_h)_h + \langle g_h^3, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \tag{3.3}$$

$$
\begin{aligned}
&(\partial^4 U_h^n, v_h)_h + A_h(4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4} : U_h^n, v_h) \\
&\qquad = (f(4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4}, x), v_h)_h + \langle g_h^n, v_h \rangle_{\Gamma_h} \\
&\qquad\qquad \forall\, v_h \in S_h \quad n = 4, 5, \ldots, M. \qquad (3.4)
\end{aligned}
$$

The scheme (3.1) − (3.4) is zero-stable. To see this, we obtain the first characteristic polynomials:

$$
\begin{aligned}
\rho_1(y) &= y - 1 \\
\rho_2(y) &= \frac{3}{2}y^2 - 2y + \frac{1}{2} \\
\rho_3(y) &= \frac{11}{6}y^3 - 3y^2 + \frac{3}{2}y - \frac{1}{3} \\
\rho_4(y) &= \frac{25}{12}y^4 - 4y^3 + 3y^2 - \frac{4}{3}y + \frac{1}{4}.
\end{aligned}
$$

The roots of these polynomials have moduli less than one and the roots with modulus one are simple. See [10] for more information on zero-stability.

The analysis of this work is done with the assumption that $\|\frac{\partial^i u}{\partial t^i}\|_{L^2(\Omega)}$ exist (for $i = 1, \ldots, 5$). It can be shown using Taylor expansion that

$$
\left\{
\begin{aligned}
\|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)} &\leq (\Delta t)^2 \lambda_0 \\
\|U_h^n - 3U_h^{n-1} + 3U_h^{n-2} - U_h^{n-3}\|_{L^2(\Omega)} &\leq (\Delta t)^3 \lambda_1 \\
\|U_h^n - 4U_h^{n-1} + 6U_h^{n-2} - 4U_h^{n-3} + U_h^{n-4}\|_{L^2(\Omega)} &\leq (\Delta t)^4 \lambda_2.
\end{aligned}
\right. \tag{3.5}
$$

The following is our main result.

Theorem 3.1 *Let u^n and U_h^n be the solutions of (1.4) and (3.1) − (3.4) at t_n, respectively. Suppose $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g(x,t)$ satisfy the conditions of Assumption 1.1, and $\frac{\partial^5 u}{\partial t^5}$ is defined for $\Omega \times [0, T]$. There exists a positive constant C independent of h and k such that*

$$
\|u^n - U_h^n\|_{L^2(\Omega)} \leq \left[k^4 + h^2 \left(1 + \frac{1}{|\log h|}\right)\right] B(u_0, u, g)
$$

where

$$
\begin{aligned}
B(u_0, u, g) &= sqrt\left[\max\left\{ C\left[1 + \int_0^{t_n} \left(\sum_{j=2}^{5} \|\frac{\partial^j u}{\partial t^j}\|^2_{L^2(\Omega)}\right) dt\right],\right.\right. \\
&\qquad \left.\left. C\left[\|u_0\|_X^2 + \int_0^{t_n} \left[\|u\|_X^2 + \|u_t\|_X^2 + \|g\|^2_{H^2(\Gamma)} + |f|^2\right] dt\right]\right\}\right].
\end{aligned}
$$

Proof Letting $z^n = P_h u^n - U_h^n$, from (1.4) and (3.4), we have

$$
(\partial^4 z^n, v_h)_h + A_h(z^n, v_h) = B_1 + B_2 + B_3 \tag{3.6}
$$

where

$$\begin{aligned} B_1 &= (\partial^4(P_hu^n - u^n), v_h)_h + (\partial^4 u^n - u_t^n, v_h) + (\partial^4 u^n, v_h)_h - (\partial^4 u^n, v_h) \\ B_2 &= (f(x,u^n), v_h) - (f^n, v_h)_h + \langle g^n, v_h\rangle_\Gamma - \langle g_h^n, v_h\rangle_{\Gamma_h} \\ B_3 &= A_h(4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4} : P_hu^n, v_h) - A_h(u^n : P_hu^n, v_h) \end{aligned}$$

and $f^n = f(x, 4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4})$.
With $v_h = z^n$, we have

$$\begin{aligned} B_1 &\leq \|\partial^4(P_hu^n - u^n)\|^2_{L^2(\Omega)} + \frac{1}{2}\|z^n\|^2_{L^2(\Omega)} + \|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)} \\ &\quad + \gamma C h^4 \|\partial^4 u^n\|_X^2 + \frac{1}{4\gamma}\|z^n\|^2_{H^1(\Omega).} \end{aligned} \tag{3.7}$$

Using Lemma 2.3 and (3.5) with the fact that $D^\alpha z^n = 0$ for $|\alpha| = 2$, we have

$$\begin{aligned} B_2 &\leq Ch^{3/2}|f|\|z^n\|_{H^1(\Omega_h^\star)} + \mu_3\|u^n - U_h^n\|_{L^2(\Omega)}\|z^n\|_{L^2(\Omega)} \\ &\quad + Ch^{3/2}\|g^n\|_{H^2(\Gamma)}\|z^n\|_{H^1(\Omega_h^\star)} \\ &\quad + \mu_3\|U_h^n - (4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4})\|_{L^2(\Omega)}\|z^n\|_{L^2(\Omega)} \\ &\leq Ch^2|f|\|z^n\|_{H^1(\Omega)} + \left(\mu_3 + \frac{1}{2}\right)\|z^n\|^2_{L^2(\Omega)} + C\|P_hu^n - u^n\|^2_{L^2(\Omega)} \\ &\quad + Ch^2\|g^n\|_{H^2(\Gamma)}\|z^n\|_{H^1(\Omega)} + \mu_3\lambda_2 k^4\|z^n\|_{L^2(\Omega)} \\ &\leq C(\gamma)h^4\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^n\|_X^2 + \|g^n\|^2_{H^2(\Gamma)} + |f|^2\right) + \frac{1}{2\gamma}\|z^n\|^2_{H^1(\Omega)} \\ &\quad + C\|z^n\|^2_{L^2(\Omega)} + Ck^8. \end{aligned} \tag{3.8}$$

For B_3, let $\|\nabla P_hu^n\|_{L^2(\Omega)} = \beta$ and use Assumption 1.1:

$$\begin{aligned} B_3 &\leq |a(x, 4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4}) - a(x,u^n)| \sum_{K\in\mathcal{T}_h}\int_K |P_hu^n \cdot \partial^4 z^n|\,dx \\ &\leq \beta\mu_3\|(4U_h^{n-1} - 6U_h^{n-2} + 4U_h^{n-3} - U_h^{n-4}) - u^n\|_{L^2(\Omega)}\|z^n\|_{H^1(\Omega)} \\ &\leq \mu_3\lambda_2 k^4\|z^n\|_{H^1(\Omega)} + \beta\mu_3\|P_hu^n - u^n\|_{L^2(\Omega)}\|z^n\|_{H^1(\Omega)} \\ &\quad + \beta\mu_3\|z^n\|_{L^2(\Omega)}\|z^n\|_{H^1(\Omega)} \\ &\leq \gamma\beta^2\mu_3^2\|z^n\|^2_{L^2(\Omega)} + \frac{3}{4\gamma}\|z^n\|^2_{H^1(\Omega)} + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\|u^n\|_X^2 + Ck^8. \end{aligned} \tag{3.9}$$

Substituting (3.7) – (3.9) into (3.6), we have, for $c_1 > 0$,

$$\begin{aligned}\frac{1}{k}\|z^n\|^2_{L^2(\Omega)} + \mu_1\|z^n\|^2_{H^1(\Omega)} \quad \leq \quad & \frac{C}{k}\left(\|z^n\|_{L^2(\Omega)}\|z^{n-1}\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)}\|z^{n-2}\|_{L^2(\Omega)}\right.\\ & \left. + \|z^n\|_{L^2(\Omega)}\|z^{n-3}\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)}\|z^{n-4}\|_{L^2(\Omega)}\right)\\ & + \|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + C_0\|z^n\|^2_{L^2(\Omega)}\\ & + \|\partial_k u^n - u_t^n\|^2_{L^2(\Omega)} + Ch^4\|\partial^4 u^n\|^2_X\\ & + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^n\|^2_X + \|g^n\|^2_{H^2(\Gamma)} + |f|^2\right)\\ & + \frac{1}{\gamma}\|z^n\|^2_{H^1(\Omega)} + Ck^8,\end{aligned}$$

where $C_0 = \dfrac{1}{2} + \mu_3 + \gamma\beta^2\mu_3^2$. With $\gamma = \dfrac{1}{\mu_1}$, we obtain

$$\begin{aligned}&(1 - C_0 k)\,\|z^n\|^2_{L^2(\Omega)}\\ &\qquad \leq C\left(\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)} + \|z^{n-3}\|^2_{L^2(\Omega)} + \|z^{n-4}\|^2_{L^2(\Omega)}\right)\\ &\qquad + C\left[k\|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)} + kh^4\|\partial^4 u^n\|^2_X\right]\\ &\qquad + Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^n\|^2_X + \|g^n\|^2_{H^2(\Gamma)} + |f|^2\right) + Ck^9.\end{aligned}$$

For $0 < k < \min\left\{1, \dfrac{1}{C_0}\right\}$, there is a $C > 0$ such that $(1 - C_0 k)^{-1} \leq C$, and therefore

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \quad \leq \quad & C\left[\|z^{n-1}\|^2_{L^2(\Omega)} + \|z^{n-2}\|^2_{L^2(\Omega)} + \|z^{n-3}\|^2_{L^2(\Omega)} + \|z^{n-4}\|^2_{L^2(\Omega)}\right.\\ & + k\|\partial^4(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)} + kh^4\|\partial^4 u^n\|^2_X\\ & + Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^n\|^2_X + \|g^n\|^2_{H^2(\Gamma)} + |f|^2\right) + Ck^9\end{aligned}$$

for $n = 4, \ldots, M$.
By iteration on n, we have

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \quad \leq \quad & C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)} + \|z^3\|^2_{L^2(\Omega)}\right]\\ & + Ck\sum_{j=4}^{n}\|\partial^4(u^j - P_h u^j)\|^2_{L^2(\Omega)} + Ck^9\\ & + Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2\sum_{j=4}^{n}(\|u^j\|^2_X + \|g^j\|^2_{H^2(\Gamma)} + |f|^2)\\ & + Ck\sum_{j=4}^{n}\|\partial^4 u^j - u_t^j\|^2_{L^2(\Omega)} + Ch^4 k\sum_{j=4}^{n}\|\partial^4 u^j\|^2_X.\end{aligned}$$

After a simple calculation, we have

$$
\begin{aligned}
\|z^n\|_{L^2(\Omega)}^2 \leq\ & C\left[\|z^0\|_{L^2(\Omega)}^2+\|z^1\|_{L^2(\Omega)}^2+\|z^2\|_{L^2(\Omega)}^2+\|z^3\|_{L^2(\Omega)}^2\right] \\
& +C\int_0^{t_n}\|(u-P_hu)_t\|_{L^2(\Omega)}^2\,dt+Ck^8\int_0^{t_n}\|\frac{\partial^5 u}{\partial t^5}\|_{L^2(\Omega)}^2\,dt \\
& +Ch^4\left(1+\frac{1}{|\log h|}\right)^2\int_0^{t_n}\left[\|u\|_X^2+\|u_t\|_X^2+\|g\|_{H^2(\Gamma)}^2+|f|^2\right]\,dt \\
& +Ck^9 \\
\leq\ & C\left[\|z^0\|_{L^2(\Omega)}^2+\|z^1\|_{L^2(\Omega)}^2+\|z^2\|_{L^2(\Omega)}^2+\|z^3\|_{L^2(\Omega)}^2\right] \\
& +Ch^4\left(1+\frac{1}{|\log h|}\right)^2\int_0^{t_n}\left[\|u\|_X^2+\|u_t\|_X^2+\|g\|_{H^2(\Gamma)}^2+|f|^2\right]\,dt \\
& +Ck^8\int_0^{t_n}\|\frac{\partial^5 u}{\partial t^5}\|_{L^2(\Omega)}^2\,dt+Ck^9.
\end{aligned}
\tag{3.10}
$$

Letting $z^1=P_hu^1-U_h^1$, from (1.4) and (3.1), we have

$$
\begin{aligned}
(\partial^1 z^1,v_h)_h+A_h(z^1,v_h) =\ & (\partial^1(P_hu^1-u^1),v_h)_h+(\partial^1u^1-u_t^1,v_h) \\
& +(\partial^1u^1,v_h)_h-(\partial^1u^1,v_h) \\
& +A_h(U_h^0:P_hu^1,v_h)-A_h(u^1:P_hu^1,v_h) \\
& +(f(x,u^1),v_h)-(f(x,U_h^0),v_h)_h+\langle g^1,v_h\rangle_\Gamma-\langle g_h^1,v_h\rangle_{\Gamma_h}.
\end{aligned}
$$

With $v_h=z^1$, we have

$$
\begin{aligned}
\frac{1}{\tau_1}\|z^1\|_{L^2(\Omega)}^2+\mu_1\|z^1\|_{H^1(\Omega)}^2 \leq\ & \frac{1}{\tau_1}\|z^0\|_{L^2(\Omega)}\|z^1\|_{L^2(\Omega)}+\|\partial^1(P_hu^1-u^1)\|_{L^2(\Omega)}^2+Ch^4|f|^2 \\
& +\left(\frac{1}{2}+\mu_1\right)\|z^1\|_{L^2(\Omega)}^2+\|\partial^1u^1-u_t^1\|_{L^2(\Omega)}^2+Ch^4\|\partial^1u^1\|_X^2 \\
& +Ch^4\|g^1\|_{H^2(\Gamma)}^2+\frac{1}{\gamma}\|z^1\|_{H^1(\Omega)}^2+\gamma\beta^2\mu_3^2\|U_h^0-u^1\|_{L^2(\Omega)}^2 \\
\leq\ & \frac{1}{\tau_1}\|z^0\|_{L^2(\Omega)}\|z^1\|_{L^2(\Omega)}+\|\partial^1(P_hu^1-u^1)\|_{L^2(\Omega)}^2 \\
& +\left(\frac{1}{2}+\mu_1+\gamma\beta^2\mu_3^2\right)\|z^1\|_{L^2(\Omega)}^2+\|\partial^1u^1-u_t^1\|_{L^2(\Omega)}^2 \\
& +Ch^4\|\partial^1u^1\|_X^2+Ch^4\|g^1\|_{H^2(\Gamma)}^2+\frac{1}{\gamma}\|z^1\|_{H^1(\Omega)}^2 \\
& +C\tau_1+Ch^4\left(1+\frac{1}{|\log h|}\right)^2\left(\|u^1\|_X^2+|f|^2\right).
\end{aligned}
$$

With $\gamma = \dfrac{1}{\mu_1}$, we obtain

$$
\begin{aligned}
(1 - C_0\tau_1)\,\|z^1\|^2_{L^2(\Omega)} \quad \leq \quad & \|z^0\|^2_{L^2(\Omega)} + \tau_1\|\partial^1(P_h u^1 - u^1)\|^2_{L^2(\Omega)} + \tau_1\|\partial^1 u^1 - u^1_t\|^2_{L^2(\Omega)} \\
& + C\tau_1 h^4\|\partial^1 u^1\|^2_X + C\tau_1^2 \\
& + \tau_1 C h^4\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^1\|^2_X + \|g^1\|^2_{H^2(\Gamma)} + |f|^2\right).
\end{aligned}
$$

For $0 < \tau_1 < \min\left\{1, \dfrac{1}{C_0}\right\}$, there is a $C > 0$ such that $(1 - C_0\tau_1)^{-1} \leq C$; therefore,

$$
\begin{aligned}
\|z^1\|^2_{L^2(\Omega)} \quad \leq \quad & C\Big[\|z^0\|^2_{L^2(\Omega)} + \tau_1\|\partial^1(P_h u^1 - u^1)\|^2_{L^2(\Omega)} + \tau_1\|\partial^1 u^1 - u^1_t\|^2_{L^2(\Omega)} \\
& + \tau_1 h^4\|\partial^1 u^1\|^2_X + \tau_1 h^4\left(1 + \frac{1}{|\log h|}\right)^2\left(\|u^1\|^2_X + \|g^1\|^2_{H^2(\Gamma)} + |f|^2\right) + \tau_1^2\Big] \\
\leq \quad & C\|z^0\|^2_{L^2(\Omega)} + C\int_0^{t_1}\|(u - P_h u)_t\|^2_{L^2(\Omega)}\,dt + C\tau_1^2\int_0^{t_1}\|u_{tt}\|^2_{L^2(\Omega)}\,dt \\
& + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\int_0^{t_1}\left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2\right]\,dt + C\tau_1^2 \\
\leq \quad & C\|z^0\|^2_{L^2(\Omega)} + C\tau_1^2\int_0^{t_1}\|u_{tt}\|^2_{L^2(\Omega)}\,dt + C\tau_1^2 \\
& + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\int_0^{t_1}\left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2\right]\,dt. \qquad (3.11)
\end{aligned}
$$

By similar arguments to the one that led to (3.11), we have

$$
\begin{aligned}
\|z^2\|^2_{L^2(\Omega)} \quad \leq \quad & C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right] + C\tau_2^4\int_0^{t_2}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt \\
& + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\int_0^{t_2}\left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2\right]\,dt \\
& + C\tau_2^4 \qquad (3.12) \\
\|z^3\|^2_{L^2(\Omega)} \quad \leq \quad & C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + \|z^2\|^2_{L^2(\Omega)}\right] + C\tau_3^6\int_0^{t_3}\|u_{tttt}\|^2_{L^2(\Omega)}\,dt \\
& + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\int_0^{t_3}\left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2\right]\,dt \\
& + C\tau_3^6. \qquad (3.13)
\end{aligned}
$$

From (3.10) − (3.13) with $\tau_1 \le k^4$, $\tau_2 \le k^2$, and $\tau_3 \le k^{4/3}$, we have

$$\begin{aligned}\|z^n\|^2_{L^2(\Omega)} \quad &\le \quad C\|z^0\|^2_{L^2(\Omega)} + Ck^8 \int_0^{t_n} \left\{ \sum_{j=2}^{5} \|\frac{\partial^j u}{\partial t^j}\|^2_{L^2(\Omega)} \right\} dt + Ck^9 \\ &\quad + Ch^4 \left(1 + \frac{1}{|\log h|}\right)^2 \int_0^{t_n} \left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2 \right] dt.\end{aligned}$$

Now, from Lemma 2.3 and the last inequality,

$$\begin{aligned}\|u^n - U_h^n\|^2_{L^2(\Omega)} \quad &\le \quad 2\|u^n - P_h u^n\|^2_{L^2(\Omega)} + 2\|z^n\|^2_{L^2(\Omega)} \\ &\le \quad Ck^8 \left[1 + \int_0^{t_n} \left(\sum_{j=2}^{5} \|\frac{\partial^j u}{\partial t^j}\|^2_{L^2(\Omega)} \right) dt \right] \\ &\quad + Ch^4 \left(1 + \frac{1}{|\log h|}\right)^2 \left\{ \|u_0\|^2_X + \int_0^{t_n} \left[\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + |f|^2 \right] dt \right\}.\end{aligned}$$

The result follows immediately. □

Remark 3.2 *The initial three steps of the scheme are constructed using a low-order time discretization scheme; however, this does not affect the convergence rate since they are used once. Moreover, in the error analysis, the step sizes of these low-order discretizations are chosen to be sufficiently small to guarantee the convergence rate.*

4. Numerical experiment

For the numerical experiment, globally continuous piecewise linear finite element functions based on the quasiuniform triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM++ [9].

Example 4.1 *We discuss the result of a two-dimensional nonlinear parabolic interface problem in the domain* $\Omega = (-1,1) \times (-1,1)$, *where* Ω_1 *is a circle centered at* $(0,0)$ *with radius* $r = \sqrt{x^2+y^2} = 0.5$, $\Omega_2 = \Omega \setminus \Omega_1$ *and the interface* Γ *is a circle of radius* 0.5 *and therefore* $\Gamma \ne \Gamma_h$.

On $\Omega \times (0,50]$, *we consider the nonlinear problem* (1.1) − (1.3) *whose exact solution is*

$$u = \begin{cases} \frac{1}{8}(1-4r^2)\sin(t) & \text{in} \quad \Omega_1 \times (0,50] \\ \frac{1}{4}(1-x^2)(1-y^2)(1-4r^2)\sin(2t) & \text{in} \quad \Omega_2 \times (0,50]. \end{cases}$$

The source function f, *interface function* g, *and initial data* u_0 *are determined from the choice of* u *with*

$$a = \begin{cases} \dfrac{u^2}{1+u^2} & \text{in} \quad \Omega_1 \times (0,50] \\ \dfrac{1}{1+u^2} & \text{in} \quad \Omega_2 \times (0,50]. \end{cases}$$

Table 1. Error estimates in L^2-norm for Example 4.1.

h	Error ($k = 0.04$)
0.181985	9.47053×10^{-3}
0.0950432	2.33584×10^{-3}
0.0475216	5.62814×10^{-4}

k	Error ($h = 0.0950432$)
0.016	4.62950×10^{-3}
0.008	2.33580×10^{-3}
0.004	2.335798×10^{-4}

Table 2. Error estimates in L^2-norm for Example 4.2.

h	Error ($k = 0.03125$)
0.108621	2.55671×10^{-2}
0.055215	6.26186×10^{-3}
0.0273063	1.58762×10^{-3}

k	Error ($h = 0.055215$)
0.125	6.75108×10^{-3}
0.0625	6.27771×10^{-3}
0.03125	6.26186×10^{-3}

Errors in L^2-norm at $t = 12$ for various step sizes h and time steps k are presented in Table 1. The data indicate that

$$\|Error\|_{L^2(\Omega)} = O\left(k^{3.954} + h^{1.991}\left(1 + \frac{1}{|\log h|}\right)\right).$$

These numerical results match the convergence rate as given in Theorem 3.1.

Example 4.2 *We consider (1.1)–(1.3) on the domain $\Omega = (-1,1) \times (-1,1)$, where Ω_1 is the ellipse $4x^2 + 16y^2 < 1$, $\Omega_2 = \Omega \setminus \Omega_1$, and the interface Γ is the ellipse $4x^2 + 16y^2 = 1$ and therefore $\Gamma \neq \Gamma_h$.*
For the exact solution, we choose

$$u = \begin{cases} \frac{1}{8}(1 - 4x^2 - 16y^2)t\exp(\sin t) & in \quad \Omega_1 \times (0,10] \\ \frac{1}{2}(1 - x^2)(1 - y^2)(4x^2 + 16y^2 - 1)\sin t & in \quad \Omega_2 \times (0,10]. \end{cases}$$

The source function f, interface function g, and initial data u_0 are determined from the choice of u with

$$a = \begin{cases} 5 & in \quad \Omega_1 \times (0,10] \\ \dfrac{1}{1+u^2} & in \quad \Omega_2 \times (0,10]. \end{cases}$$

Errors in L^2-norm at $t = 5$ for various step sizes h and time steps k are presented in Table 2. The data indicate that

$$\|Error\|_{L^2(\Omega)} = O\left(k^{4.900} + h^{1.982}\left(1 + \frac{1}{|\log h|}\right)\right).$$

These numerical results match the convergence rates as given in Theorem 3.1.

Acknowledgment

The authors would like to thank the anonymous referees, whose comments improved the original version of this manuscript.

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