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# Several Hardy-type inequalities with weights related to Baouendi–Grushin operators

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**Abstract:** In this paper we shall prove several weighted $L^p$ Hardy-type inequalities associated to the Baouendi–Grushin-type operators $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\,\Delta_y$, where $\Delta_x$ and $\Delta_y$ are the classical Laplace operators in the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, respectively, and $\gamma$ is a positive real number.

**Key words:** Baouendi–Grushin operator, weighted Hardy inequality

## 1. Introduction

The well-known Hardy inequality in $\mathbb{R}^n$, $n \geq 3$, asserts that for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla u|^2\, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx. \tag{1.1}$$

Though the constant $\left(\frac{n-2}{2}\right)^2$ is sharp, equality in (1.1) is never achieved by any function $u \in H_0^1(\mathbb{R}^n)$. Hardy [13] originally discovered this inequality in 1920 for the one-dimensional case. Since then it has attracted the attention of many mathematicians and has been comprehensively analyzed in several directions; see, for example, [2, 3, 5, 10, 11, 14, 15, 21] and the references therein.

The sharp Hardy inequality (1.1) arises very naturally in the study of degenerate elliptic differential operators and it was first extended in [9] by Garofalo to the Baouendi–Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ with $n, k \geq 1$, $\gamma > 0$. To be explicit, the author in [9] proved the following Hardy inequality

$$\int_{\mathbb{R}^{n+k}} |\nabla_\gamma u|^2 dxdy \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^{n+k}} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} \frac{u^2}{\rho^2} dxdy \tag{1.2}$$

for every $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^k \setminus \{(0,0)\})$. Here, $Q = n + (1+\gamma)k$ is the homogeneous dimension, $\nabla_\gamma = (X_1, \dots, X_n, Y_1, \dots, Y_k)$ is the subelliptic gradient, and $\rho = \left(|x|^{2(1+\gamma)} + (1+\gamma)^2 |y|^2\right)^{\frac{1}{2(1+\gamma)}}$ is the *gauge* function

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2010 *AMS Mathematics Subject Classification:* 26D10, 35H10, 46E35 associated to the fundamental solution for the subelliptic operator $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y$, where $\Delta_x$ and $\Delta_y$ are the Laplacians on $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively.

After the seminal work of Garofalo [9], much effort has been devoted to Hardy-type inequalities for the Baouendi–Grushin vector fields; see [6–8, 16–20] . For instance, in [6], D'Ambrosio obtained a weighted $L^p$ analogue of (1.2). Later, Niu et al. [19] considered various types of Hardy inequalities on some special domains in $\mathbb{R}^n \times \mathbb{R}^k$ via a Picone-type identity for the Baouendi–Grushin vector fields $\{X_i, Y_j\}$. In [16], Kombe proved weighted Hardy-type inequalities with remainder terms. On the other hand, Laptev et al. [18] recently established weighted Hardy inequalities for the quadratic form of the magnetic Baouendi–Grushin operator with Aharonov–Bohm-type magnetic field. More recently, Kombe and Yener [17] introduced a unified approach to the weighted Hardy inequalities related to Baouendi–Grushin operators.

In the literature, Hardy-type inequalities mostly involve the weights of the form $\rho^\alpha |x|^\beta$ for some real numbers $\alpha$ and $\beta$. Our goal in this paper is to prove several $L^p$ Hardy-type inequalities with more general and nonstandard weights associated to the Baouendi–Grushin-type operators $\Delta_\gamma$. For this goal we shall mainly use a technique developed in [17].

## 2. Preliminaries and notations

In this section we shall introduce some notations and preliminary facts. Let $\mathbb{R}^N$ be split in $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^k$ with $n, k \geq 1$ and $N = n + k$. We denote by $|x|$ and $|y|$ the usual Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. The corresponding vector field is

$$\nabla_\gamma = (X_1, \ldots, X_n, Y_1, \ldots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y),$$

where

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, k \tag{2.1}$$

and $\gamma$ is a positive real number. The Baouendi–Grushin operator is of the form

$$\Delta_\gamma = \sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \nabla_\gamma \cdot \nabla_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y.$$

Here, $\Delta_x$ and $\Delta_y$ stand for the usual Laplace operators in the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, respectively. The operator $\Delta_\gamma$ was first studied by Baouendi [1] and Grushin [12] when $\gamma$ is a positive integer. We note that if $\gamma$ is not an even positive integer, then $\Delta_\gamma$ is not a sum of squares of $C^\infty$ vector fields satisfying Hörmander's condition:

$$\text{rank Lie}\,[X_1, \ldots, X_n, Y_1, \ldots, Y_k] = N.$$

The nonlinear $p$-degenerate subelliptic operator associated with the vector fields (2.1) is defined by

$$\Delta_{\gamma,p} u = \nabla_\gamma \cdot \left( |\nabla_\gamma u|^{p-2} \nabla_\gamma u \right), \quad p > 1.$$

The operator $\Delta_\gamma$ possesses a natural family of anisotropic dilations, namely

$$\delta_\lambda(z) = (\lambda x, \lambda^{\gamma+1} y), \quad \lambda > 0, \quad z = (x, y) \in \mathbb{R}^N.$$

It is easy to verify that

$$d\delta_\lambda(x,y) = \lambda^Q dxdy = \lambda^Q dz,$$

where

$$Q = n + (1+\gamma)k$$

is the homogeneous dimension with respect to the dilation $\delta_\lambda$ and $dz = dxdy$ denotes the Lebesgue measure on $\mathbb{R}^N$. For $z = (x,y) \in \mathbb{R}^N$, define the distance between $z$ and the origin of $\mathbb{R}^N$ as follows:

$$\rho = \rho(z) = \left(|x|^{2(1+\gamma)} + (1+\gamma)^2|y|^2\right)^{\frac{1}{2(1+\gamma)}}.$$

We remark that $\rho$ is positive, smooth away from the origin, and symmetric.

A simple calculation shows that

$$\nabla_\gamma \rho = \frac{|x|^\gamma}{\rho^{2\gamma+1}} \left(|x|^\gamma x_1, \ldots, |x|^\gamma x_n, (1+\gamma)y_1, \ldots, (1+\gamma)y_k\right)$$

and hence

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}.$$

A function $u$ on $\mathbb{R}^N$ is said to be radial when $u$ has the form $u = u(\rho)$. If $u$ is radial, then by a straightforward computation we have

$$|\nabla_\gamma u(\rho)| = |\nabla_\gamma \rho| \, |u'(\rho)|$$

and

$$\Delta_\gamma u(\rho) = |\nabla_\gamma \rho|^2 \left[u''(\rho) + (Q-1)\frac{u'(\rho)}{\rho}\right].$$

In particular, when $u(\rho) = \rho^\alpha$ one gets

$$|\nabla_\gamma \rho^\alpha| = |\alpha| \, |\nabla_\gamma \rho| \, \rho^{\alpha-1} \tag{2.2}$$

and

$$\Delta_\gamma \rho^\alpha = \alpha (Q+\alpha-2) |\nabla_\gamma \rho|^2 \rho^{\alpha-2} \tag{2.3}$$

with $\alpha \in \mathbb{R}$. Moreover, the *gauge* $\rho$ is infinite harmonic in $\mathbb{R}^N - \{0\}$ (see [4]); that is, $\rho$ is the solution of the following equation:

$$\nabla_\gamma(|\nabla_\gamma \rho|^2) \cdot \nabla_\gamma \rho = 0. \tag{2.4}$$

## 3. Weighted Hardy-type inequalities

For the convenience of the reader, we begin by quoting a known result from [17].

**Theorem 3.1** *Let $\vartheta \in C^1(\mathbb{R}^N)$ and $w \in L^1_{loc}(\mathbb{R}^N)$ be nonnegative functions and $f \in C^\infty(\mathbb{R}^N)$ be a positive function such that*

$$-\nabla_\gamma \cdot \left(\vartheta \, |\nabla_\gamma f|^{p-2} \, \nabla_\gamma f\right) \geq w f^{p-1} \tag{3.1}$$

*almost everywhere in* $\mathbb{R}^N$. *Then for all* $u \in C_0^\infty\left(\mathbb{R}^N\right)$ *and* $p>1$, *one has*

$$\int_{\mathbb{R}^N} \vartheta \left|\nabla_\gamma u\right|^p dz \geq \int_{\mathbb{R}^N} w\left|u\right|^p dz. \tag{3.2}$$

**Remark 3.2** *Let* $\epsilon>0$ *be given. Define*

$$\rho_\epsilon := \left(|x|_\epsilon^{2(1+\gamma)} + (1+\gamma)^2|y|^2\right)^{\frac{1}{2(1+\gamma)}},$$

*where* $|x|_\epsilon := \left(\epsilon^2+\sum_{i=1}^n x_i^2\right)^{1/2}$. *In the proof of the following theorems, if the functions* $\vartheta$ *and* $f$ *are not smooth enough, by standard argument one can consider* $\rho_\epsilon$ *a regularization of* $\rho$ *and after the computation takes the limit as* $\epsilon \longrightarrow 0$. *However, for the sake of simplicity we will proceed formally.*

Recall that the following weighted Hardy-type inequalities in the Euclidean setting were proved by Ghoussoub and Moradifam [10]: Let $a,b>0$ and $\alpha,\beta,m$ be real numbers.

- If $\alpha\beta>0$ and $m \leq \frac{n-2}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{\left(a+b\left|x\right|^\alpha\right)^\beta}{\left|x\right|^{2m}} \left|\nabla u\right|^2 dx \geq \left(\frac{n-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{\left(a+b\left|x\right|^\alpha\right)^\beta}{\left|x\right|^{2m+2}} u^2 dx. \tag{3.3}$$

- If $\alpha\beta<0$ and $2m-\alpha\beta \leq n-2$, then for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{\left(a+b\left|x\right|^\alpha\right)^\beta}{\left|x\right|^{2m}} \left|\nabla u\right|^2 dx \geq \left(\frac{n+\alpha\beta-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{\left(a+b\left|x\right|^\alpha\right)^\beta}{\left|x\right|^{2m+2}} u^2 dx. \tag{3.4}$$

We now generalize and improve the above inequalities (3.3) and (3.4) to the $L^p$ case for the Baouendi–Grushin operator $\Delta_\gamma$ by specializing the choice of functions $\vartheta$ and $f$ in the differential inequality (3.1). Here is the first result in this direction.

**Theorem 3.3** *Let* $a,b>0$ *and* $\alpha,\beta,m \in \mathbb{R}$. *If* $\alpha\beta>0$ *and* $1<p\leq Q-pm$, *then for all* $u \in C_0^\infty(\mathbb{R}^N)$ *one has*

$$\int_{\mathbb{R}^N} \frac{\left(a+b\rho^\alpha\right)^\beta}{\rho^{pm}} \left|\nabla_\gamma u\right|^p dz \geq C_{Q,p,m}^p \int_{\mathbb{R}^N} \frac{\left(a+b\rho^\alpha\right)^\beta}{\rho^{pm+p}} \left|\nabla_\gamma \rho\right|^p |u|^p dz$$
$$+ C_{Q,p,m}^{p-1}\alpha\beta b \int_{\mathbb{R}^N} \frac{\left(a+b\rho^\alpha\right)^{\beta-1}}{\rho^{pm+p-\alpha}} \left|\nabla_\gamma \rho\right|^p |u|^p dz,$$

*where* $C_{Q,p,m} = \left(\frac{Q-pm-p}{p}\right)$.

**Proof** Set $\vartheta = \dfrac{\left(a+b\rho^\alpha\right)^\beta}{\rho^{pm}}$ and $f=\rho^{-\left(\frac{Q-pm-p}{p}\right)}$ in Theorem 3.1. Clearly,

$$\nabla_\gamma f = -\left(\frac{Q-pm-p}{p}\right)\rho^{-\left(\frac{Q-pm}{p}\right)}\nabla_\gamma \rho.$$

Noting that $1 < p \leq Q - pm,$ one gets

$$|\nabla_\gamma f|^{p-2} = \left(\frac{Q-pm-p}{p}\right)^{p-2} \rho^{-\left(\frac{Q-pm}{p}\right)(p-2)} |\nabla_\gamma \rho|^{p-2}.$$

Hence,

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f :=& - \nabla_\gamma \cdot \left(\vartheta \, |\nabla_\gamma f|^{p-2} \nabla_\gamma f\right) \\ =& \, C_{Q,p,m}^{p-1} \nabla_\gamma \cdot \left( \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}} \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right), \end{aligned} \tag{3.5}$$

where $C_{Q,p,m} := \left(\frac{Q-pm-p}{p}\right)$. Using the identities (2.3) and (2.4), a direct calculation yields

$$\begin{aligned} \nabla_\gamma \left( \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}} \right) &= \frac{\alpha\beta b \rho^{\alpha+pm-1} (a+b\rho^\alpha)^{\beta-1} \nabla_\gamma \rho - pm\rho^{pm-1} (a+b\rho^\alpha)^\beta \nabla_\gamma \rho}{\rho^{2pm}} \\ &= \alpha\beta b \,(a+b\rho^\alpha)^{\beta-1} \rho^{\alpha-pm-1} \nabla_\gamma \rho - pm \,(a+b\rho^\alpha)^\beta \rho^{-pm-1} \nabla_\gamma \rho \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \nabla_\gamma \cdot \left( \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right) =& - \left(\frac{Q-pm}{p}\right)(p-1)\, \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)-1} |\nabla_\gamma \rho|^p \\ &+ \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} \left( \nabla_\gamma \left( |\nabla_\gamma \rho|^{p-2} \right) \cdot \nabla_\gamma \rho + |\nabla_\gamma \rho|^{p-2} \Delta_\gamma \rho \right) \\ =& \left( \frac{Q-pm-p}{p} + pm \right) \rho^{\frac{Q-pm-p}{p} - Q + pm} |\nabla_\gamma \rho|^p. \end{aligned} \tag{3.7}$$

It therefore follows from (3.5), (3.6), and (3.7) that

$$\begin{aligned} -\Delta_{\gamma,p,\vartheta} f =& \, C_{Q,p,m}^{p-1} \nabla_\gamma \left( \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}} \right) \cdot \left( \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right) \\ &+ C_{Q,p,m}^{p-1} \frac{(a+b\rho^\alpha)^\beta}{\rho^{pm}} \nabla_\gamma \cdot \left( \rho^{-\left(\frac{Q-pm}{p}\right)(p-1)} |\nabla_\gamma \rho|^{p-2} \nabla_\gamma \rho \right) \\ =& \, C_{Q,p,m}^{p-1} \alpha\beta b \,(a+b\rho^\alpha)^{\beta-1} \rho^{\frac{Q-pm-p}{p} - Q + \alpha} |\nabla_\gamma \rho|^p \\ &- C_{Q,p,m}^{p-1} pm \,(a+b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ &+ C_{Q,p,m}^{p-1} \left( \frac{Q-pm-p}{p} + pm \right) (a+b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ =& \, C_{Q,p,m}^{p} (a+b\rho^\alpha)^\beta \rho^{\frac{Q-pm-p}{p} - Q} |\nabla_\gamma \rho|^p \\ &+ C_{Q,p,m}^{p-1} \alpha\beta b \,(a+b\rho^\alpha)^{\beta-1} \rho^{\frac{Q-pm-p}{p} - Q + \alpha} |\nabla_\gamma \rho|^p. \end{aligned}$$

Since

$$f^{p-1} = \rho^{\frac{Q-pm-p}{p} - Q + pm + p},$$

one can put

$$w = C^{p}_{Q,p,m} \frac{(a + b\rho^{\alpha})^{\beta}}{\rho^{pm-p}} |\nabla_{\gamma}\rho|^{p} + C^{p-1}_{Q,p,m} \alpha\beta b \frac{(a + b\rho^{\alpha})^{\beta-1}}{\rho^{pm+p-\alpha}} |\nabla_{\gamma}\rho|^{p}.$$

Hence, by Theorem 3.1 we complete the proof. □

**Theorem 3.4** *Let $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta < 0$ and $1 < p \leq Q + \alpha\beta - pm$, then for all $u \in C^{\infty}_{0}(\mathbb{R}^{N})$ one has*

$$\int_{\mathbb{R}^N} \frac{(a + b\rho^{\alpha})^{\beta}}{\rho^{pm}} |\nabla_{\gamma}u|^{p} \, dz \geq C^{p}_{Q,p,m,\alpha,\beta} \int_{\mathbb{R}^N} \frac{(a + b\rho^{\alpha})^{\beta}}{\rho^{pm+p}} |\nabla_{\gamma}\rho|^{p} |u|^{p} dz$$
$$- C^{p-1}_{Q,p,m,\alpha,\beta} \alpha\beta a \int_{\mathbb{R}^N} \frac{(a + b\rho^{\alpha})^{\beta-1}}{\rho^{pm+p}} |\nabla_{\gamma}\rho|^{p} |u|^{p} dz,$$

*where $C_{Q,p,m,\alpha,\beta} = \left(\frac{Q+\alpha\beta-pm-p}{p}\right)$.*

**Proof** Choose $\vartheta = \dfrac{(a + b\rho^{\alpha})^{\beta}}{\rho^{pm}}$ and $f = \rho^{-\left(\frac{Q+\alpha\beta-pm-p}{p}\right)}$, where $a, b > 0$, $\alpha\beta < 0$ and $1 < p \leq Q + \alpha\beta - pm$. One easily sees

$$\nabla_{\gamma} f = -\left(\frac{Q + \alpha\beta - pm - p}{p}\right) \rho^{-\left(\frac{Q+\alpha\beta-pm}{p}\right)} \nabla_{\gamma}\rho$$

and

$$-\Delta_{\gamma,p,\vartheta} f = -\nabla_{\gamma} \cdot \left(\vartheta |\nabla_{\gamma} f|^{p-2} \nabla_{\gamma} f\right)$$
$$= C^{p-1}_{Q,p,m,\alpha,\beta} \nabla_{\gamma} \cdot \left(\frac{(a + b\rho^{\alpha})^{\beta}}{\rho^{pm}} \rho^{-\left(\frac{Q+\alpha\beta-pm}{p}\right)(p-1)} |\nabla_{\gamma}\rho|^{p-2} \nabla_{\gamma}\rho\right), \tag{3.8}$$

where $C_{Q,p,m,\alpha,\beta} = \left(\frac{Q+\alpha\beta-pm-p}{p}\right)$. By a similar computation as in the proof of Theorem 3.3, we derive

$$\nabla_{\gamma} \cdot \left(\rho^{-\left(\frac{Q+\alpha\beta-pm}{p}\right)(p-1)} |\nabla_{\gamma}\rho|^{p-2} \nabla_{\gamma}\rho\right) = \left(\frac{Q + \alpha\beta - pm - p}{p} + pm - \alpha\beta\right) \rho^{\frac{Q+\alpha\beta-pm-p}{p} - Q - \alpha\beta + pm} |\nabla_{\gamma}\rho|^{p}. \tag{3.9}$$

Combining (3.6) and (3.9) with (3.8), we deduce that

$$
\begin{aligned}
-\Delta_{\gamma,p,\vartheta} f &= C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta b \left(a+b\rho^{\alpha}\right)^{\beta-1} \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta+\alpha} \left|\nabla_{\gamma}\rho\right|^{p} \\
&\quad - C_{Q,p,m,\alpha,\beta}^{p-1} pm \left(a+b\rho^{\alpha}\right)^{\beta} \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta} \left|\nabla_{\gamma}\rho\right|^{p} \\
&\quad + C_{Q,p,m,\alpha,\beta}^{p-1} \left(\tfrac{Q+\alpha\beta-pm-p}{p}+pm-\alpha\beta\right) \frac{\left(a+b\rho^{\alpha}\right)^{\beta}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} \left|\nabla_{\gamma}\rho\right|^{p} \\
&= C_{Q,p,m,\alpha,\beta}^{p} \left(a+b\rho^{\alpha}\right)^{\beta} \rho^{\frac{Q+\alpha\beta-pm-p}{p}-Q-\alpha\beta} \left|\nabla_{\gamma}\rho\right|^{p} \\
&\quad + C_{Q,p,m,\alpha,\beta}^{p-1} \left(\alpha\beta b\rho^{\alpha}-\alpha\beta\left(a+b\rho^{\alpha}\right)\right) \frac{\left(a+b\rho^{\alpha}\right)^{\beta-1}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} \left|\nabla_{\gamma}\rho\right|^{p} \\
&= C_{Q,p,m,\alpha,\beta}^{p} \frac{\left(a+b\rho^{\alpha}\right)^{\beta}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} \left|\nabla_{\gamma}\rho\right|^{p} \\
&\quad - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta a \frac{\left(a+b\rho^{\alpha}\right)^{\beta-1}}{\rho^{Q+\alpha\beta}} \rho^{\frac{Q+\alpha\beta-pm-p}{p}} \left|\nabla_{\gamma}\rho\right|^{p}.
\end{aligned}
$$

This shows that one can put

$$
w = C_{Q,p,m,\alpha,\beta}^{p} \frac{\left(a+b\rho^{\alpha}\right)^{\beta}}{\rho^{pm+p}} \left|\nabla_{\gamma}\rho\right|^{p} - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta a \frac{\left(a+b\rho^{\alpha}\right)^{\beta-1}}{\rho^{pm+p}} \left|\nabla_{\gamma}\rho\right|^{p}
$$

and hence finishes the proof. □

**Remark 3.5** *Note that if* $\alpha = 0$ *or* $\beta = 0$ *in the above two inequalities, then they reduce to Hardy-type inequalities with classical weights. Therefore, we are interested in the case where* $\alpha\beta \neq 0$.

Even though the literature has mostly focused on power radial weights $\rho^{\alpha}|x|^{\beta}$ for some $\alpha, \beta \in \mathbb{R}$, we now establish $L^p$ Hardy-type inequalities with nonradial weights related to Baouendi–Grushin operator $\Delta_{\gamma}$. Here is the first result in this direction.

**Theorem 3.6** *For any* $u \in C_0^{\infty}(\Omega)$ *and* $p > 1$, *one has*

$$
\int_{\Omega} \left(\frac{y_1}{|x|^{\gamma}}\right)^{p-2} \log x_1 \left|\nabla_{\gamma} u\right|^{p} dz \geq \int_{\Omega} \frac{|x|^{2\gamma}\log x_1}{y_1^2 \log^{p-1} y_1} |u|^{p} dz, \tag{3.10}
$$

*where* $\Omega = \left\{z = (x,y) \in \mathbb{R}^N : x_1 > 1, y_1 > 1\right\}$.

**Proof** Let us first set $\vartheta = \left(\frac{y_1}{|x|^{\gamma}}\right)^{p-2} \log x_1$ and $f = \log y_1$ in Theorem 3.1 with $x_1, y_1 > 1$. Observing that

$$
X_i\left(\log y_1\right) = \frac{\partial}{\partial x_i} \log y_1 = 0 \quad \forall i = 1, \ldots, n
$$

and

$$
Y_j\left(\log y_1\right) = \left(|x|^{\gamma}\frac{\partial}{\partial y_j}\right)\left(\log y_1\right) = \begin{cases} \frac{|x|^{\gamma}}{y_1}, & \text{if } j = 1, \\ 0, & \text{if } j \neq 1, \end{cases}
$$

we immediately have

$$\nabla_\gamma \log y_1 = \left(0, \ldots, 0, \frac{|x|^\gamma}{y_1}, 0, \ldots, 0\right).$$

It follows that

$$|\nabla_\gamma \log y_1|^{p-2} = \left(\frac{|x|^\gamma}{y_1}\right)^{p-2}$$

and therefore

$$\begin{aligned} -\nabla_\gamma \cdot \left(\vartheta \left|\nabla_\gamma f\right|^{p-2} \nabla_\gamma f\right) &= -\nabla_\gamma \cdot \left(0, \ldots, 0, \frac{|x|^\gamma \log x_1}{y_1}, 0, \ldots, 0\right) \\ &= -Y_1 \left(\frac{|x|^\gamma \log x_1}{y_1}\right) \\ &= -|x|^{2\gamma} \log x_1 \frac{\partial}{\partial y_1} \left(\frac{1}{y_1}\right) \\ &= \frac{|x|^{2\gamma} \log x_1}{y_1^2}. \end{aligned}$$

This means that one can put

$$w = \frac{|x|^{2\gamma} \log x_1}{y_1^2 \log^{p-1} y_1}.$$

We have thus proved the inequality (3.10). □

**Theorem 3.7** *For any $u \in C_0^\infty(\Omega)$ and $p > 1$, one has*

$$\int_\Omega \cosh^\alpha y_1 \left|\nabla_\gamma u\right|^p dz \geq (p-1) \int_\Omega \frac{\cosh^\alpha y_1}{x_1^p \log^{p-1} x_1} |u|^p \, dz, \tag{3.11}$$

*where $\Omega = \left\{z = (x, y) \in \mathbb{R}^N : x_1 > 1\right\}$ and $\alpha \in \mathbb{R}$.*

**Proof** Let us now choose $\vartheta = \cosh^\alpha y_1$ and $f = \log x_1$ with $x_1 > 1$ and $\alpha \in \mathbb{R}$. Then we directly compute

$$X_i (\log x_1) = \frac{\partial}{\partial x_i} \log x_1 = \begin{cases} \frac{1}{x_1}, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1 \end{cases} \tag{3.12}$$

and

$$Y_j (\log x_1) = \left(|x|^\gamma \frac{\partial}{\partial y_j}\right) (\log x_1) = 0 \quad \forall j = 1, \ldots, k. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\nabla_\gamma \log x_1 = \left(\frac{1}{x_1}, 0, \ldots, 0\right), \quad |\nabla_\gamma \log x_1|^{p-2} = x_1^{2-p}.$$

Therefore,

$$\begin{aligned}-\nabla_\gamma \cdot \left(\vartheta \left|\nabla_\gamma f\right|^{p-2} \nabla_\gamma f\right) &= -\nabla_\gamma \cdot \left(\frac{\cosh^\alpha y_1}{x_1^{p-1}}, 0, \ldots, 0\right) \\ &= -X_1\left(\frac{\cosh^\alpha y_1}{x_1^{p-1}}\right) \\ &= -\cosh^\alpha y_1 \frac{\partial}{\partial x_1}\left(x_1^{1-p}\right) \\ &= (p-1)\frac{\cosh^\alpha y_1}{x_1^p \log^{p-1} x_1} f^{p-1}.\end{aligned}$$

By Theorem 3.1, the inequality (3.11) is deduced. □

Finally, we now prove the following $L^p$ Hardy-type inequality including the power of the hyperbolic sine function of $\rho$ as a weight.

**Theorem 3.8** *Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $Q+\alpha+\beta > p > 1$. Then the following inequality holds:*

$$\int_{\mathbb{R}^N} \rho^\alpha \sinh^\beta \rho \left|\nabla_\gamma u\right|^p dz \geq \left(\frac{Q+\alpha+\beta-p}{p}\right)^p \int_{\mathbb{R}^N} \rho^\alpha \sinh^\beta \rho \left|\nabla_\gamma \rho\right|^p \frac{|u|^p}{\rho^p} dz$$

*for all $u \in C_0^\infty(\mathbb{R}^N)$.*

**Proof** Letting $\vartheta = \rho^\alpha \sinh^\beta \rho$ and $f = \rho^{-\left(\frac{Q+\alpha+\beta-p}{p}\right)}$ in Theorem 3.1 with $\beta \geq 0$ and $Q+\alpha+\beta > p > 1$, we have

$$\nabla_\gamma f = -\left(\frac{Q+\alpha+\beta-p}{p}\right)\rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)} \nabla_\gamma \rho$$

and

$$\left|\nabla_\gamma f\right|^{p-2} = \left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-2} \rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-2)} \left|\nabla_\gamma \rho\right|^{p-2}.$$

It therefore follows that

$$\begin{aligned}-\Delta_{\gamma,p,\vartheta} f &= -\nabla_\gamma \cdot \left(\vartheta \left|\nabla_\gamma f\right|^{p-2} \nabla_\gamma f\right) \\ &= \left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-1} \nabla_\gamma \cdot \left(\rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-1)+\alpha} \sinh^\beta \rho \left|\nabla_\gamma \rho\right|^{p-2} \nabla_\gamma \rho\right). \end{aligned} \tag{3.14}$$

From the identities (2.3) and (2.4), it is not hard to see that

$$\nabla_\gamma \cdot \left(\left|\nabla_\gamma \rho\right|^{p-2} \nabla_\gamma \rho\right) = (Q-1)\frac{\left|\nabla_\gamma \rho\right|^p}{\rho}. \tag{3.15}$$

On the other hand, using the formula (2.2), one can readily obtain

$$\begin{aligned}\nabla_\gamma\left(\rho^{-\left(\frac{Q+\alpha+\beta}{p}\right)(p-1)+\alpha} \sinh^\beta \rho\right) &= \left(\frac{Q+\alpha+\beta}{p} - Q - \beta\right)\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1} \sinh^\beta \rho \nabla_\gamma \rho \\ &\quad + \beta \rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta} \sinh^{\beta-1} \rho \cosh \rho \nabla_\gamma \rho. \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) with (3.14) leads to

$$
\begin{aligned}
-\Delta_{\gamma,p,\vartheta} f = & \left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-1}\left(\frac{Q+\alpha+\beta}{p}-Q-\beta\right)\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p} \\
& +\left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-1}\beta\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta}\sinh^{\beta-1}\rho\cosh\rho\,|\nabla_{\gamma}\rho|^{p} \\
& +\left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-1}(Q-1)\,\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p} \\
= & \left(\frac{Q+\alpha+\beta-p}{p}\right)^{p}\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p} \\
& +\left(\frac{Q+\alpha+\beta-p}{p}\right)^{p-1}\left(\rho\frac{\cosh\rho}{\sinh\rho}-1\right)\beta\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p} \\
\geq & \left(\frac{Q+\alpha+\beta-p}{p}\right)^{p}\rho^{\frac{Q+\alpha+\beta}{p}-Q-\beta-1}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p},
\end{aligned}
$$

where we have used the fact that $\rho\coth\rho\geq 1$. Hence,

$$
-\nabla_{\gamma}\cdot\left(\vartheta\,|\nabla_{\gamma}f|^{p-2}\,\nabla_{\gamma}f\right)\geq\left(\frac{Q+\alpha+\beta-p}{p}\right)^{p}\rho^{\alpha-p}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p}\,f^{p-1};
$$

that is, according to the assumption in Theorem 3.1, we can put

$$
w=\left(\frac{Q+\alpha+\beta-p}{p}\right)^{p}\rho^{\alpha-p}\sinh^{\beta}\rho\,|\nabla_{\gamma}\rho|^{p}.
$$

This ends the proof. □

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