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On the solution of an inverse Sturm–Liouville problem with a delay and eigenparameter-dependent boundary conditions

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Abstract: In this paper, a boundary value problem consisting of a delay differential equation of the Sturm–Liouville type with eigenparameter-dependent boundary conditions is investigated. The asymptotic behavior of eigenvalues is studied and the parameter of delay is determined by eigenvalues. Then we obtain the connection between the potential function and the canonical form of the characteristic function.

Key words: Delay differential equation, characteristic function, eigenvalues, inverse problem

1. Introduction

Delay differential equations have multiple applications in science and engineering and are used as models for a variety of phenomena in physics, chemistry, technology, life sciences, etc. Therefore, this field of differential equations may be of interest for applied mathematics, multidisciplinary audiences, computational scientists, and engineers [1,3,6,13]. Moreover, boundary value problems with eigenparameters in boundary conditions appear in such problems of mathematical physics or mathematical chemistry [12,14].

In this paper, we consider the boundary value problem $L := L(q(x), \alpha)$ consisting of the following second-order differential equation of Sturm–Liouville type,

$$y''(x) + q(x)y(\alpha(x-a)) + \lambda^2 y(x) = 0, \tag{1}$$

on the finite interval $[a, b]$, together with the following boundary conditions, which depend on the spectral parameter $\lambda > 0$:

$$y'(a) + \lambda^{r_1} y(a) = 0, \tag{2}$$

$$y'(b) - \lambda^{r_2} y(b) = 0, \tag{3}$$

$$y(h(x,\alpha)) = y(a)\psi(h(x,\alpha)), \quad h(x,\alpha) < a, \tag{4}$$

where $\alpha \in (0,1]$ is the delay coefficient, $0 < r_1 < r_2 \leq 2r_1$, $h(x,\alpha) = (1-\alpha)a + \alpha x$, $q(x)$ is a real continuous function on $[a, b]$, and $\psi(x)$ is an initial function that is continuous and satisfies $\psi(a) = 1$.

Boundary value problems consisting of (1) with $\alpha = 1$ without eigenparameters in boundary conditions have been studied since the 1930s (for example, see [2,9–11,17,18], and for more details see also [4,7]). In [8,15],

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inverse Sturm–Liouville problems with boundary conditions depending on an eigenparameter without delay parameter were investigated. Furthermore, in the special case $r_1 = r_2 = 0$, problem (1)–(3) was studied in [5,16].

In this paper, the asymptotic behavior of positive eigenvalues of delay boundary value problem L are investigated by the characteristic function associated with L. Then we will answer the following question: "Which spectral characteristics of L uniquely determine the parameter of delay α?" Furthermore, we obtain the connection between the potential q and infinite product representation of the characteristic function.

2. Preliminary results

We consider the boundary value problem $L = L(q(x), \alpha)$ of the form (1)–(4). Let $\zeta(x, \lambda)$ be the unique solution of (1) on $[a, b]$, satisfying (4) and the following initial conditions:

$$\zeta(a, \lambda) = 1, \qquad \zeta'(a, \lambda) = -\lambda^{r_1}. \tag{5}$$

Equation (1) with the conditions (5) is equivalent to the following integral equation:

$$\zeta(x, \lambda) = \cos \lambda^{r_1}(x - a) - \sin \lambda^{r_1}(x - a) - \frac{1}{\lambda^{r_1}} \int_a^x q(t) \sin \lambda^{r_1}(x - t)\zeta(h(t, \alpha), \lambda)dt. \tag{6}$$

Now we consider $\zeta(h(x, \alpha), \lambda) \equiv \psi(h(x, \alpha))$ while $h(x, \alpha) < a$. Therefore, we have the following theorem.

Theorem 1 *The eigenvalues of the boundary value problem L are simple.*

Proof Let λ_0 be an eigenvalue of L and let $y(x, \lambda_0)$ be an eigenfunction corresponding to λ_0. It follows from (2) and (5) that the Wronskian of $y(x, \lambda_0)$ and $\zeta(x, \lambda_0)$ is zero, so $y(x, \lambda_0)$ and $\zeta(x, \lambda_0)$ are linearly dependent on $[a, b]$, i.e. $\zeta(x, \lambda_0)$ is an eigenfunction corresponding to λ_0. This completes the proof. □

According to Theorem 1, the nontrivial solution $\zeta(x, \lambda)$ satisfies condition (4) at the left point. Moreover, from (3), the characteristic function of L is obtained as follows:

$$C(\lambda) \equiv \zeta'(b, \lambda) - \lambda^{r_2}\zeta(b, \lambda). \tag{7}$$

Thus, the eigenvalue set of L coincides with the set of real zeros of $C(\lambda)$. Put $c_0 = (1 - \alpha)(b - a)$ and $q_b = \int_a^b |q(t)|dt$. Assume that $\psi(x)$ is extended to the interval $[a - c_0, a]$ continuously.

Lemma 1 *Let $\lambda^{r_1} \geq 2q_b$. Then,*

$$|\zeta(x, \lambda)| \leq \max\{\psi_0, 2\sqrt{2}\}, \qquad x \in [a - c_0, a],$$

where

$$\psi_0 := \max_{a - c_0 \leq x \leq a} |\psi(x)|.$$

Proof Put $\zeta_\lambda = \max |\zeta(x, \lambda)|$. Therefore, according to (4) and (6), we have one of the following inequalities:

$$\zeta_\lambda \leq \sqrt{2} + \frac{1}{\lambda^{r_1}}\zeta_\lambda q_b, \qquad \zeta_\lambda \leq \sqrt{2} + \frac{1}{\lambda^{r_1}}\psi_0 q_b,$$

for every $\lambda > 0$. Under the hypothesis of Lemma 1 we have $\zeta_\lambda \leq \max\{\psi_0, 2\sqrt{2}\}$. This together with (4) completes the proof. □

Theorem 2 *The boundary value problem L has an infinite number of positive eigenvalues.*

Proof According to (6), we have

$$\zeta'(x,\lambda) = \frac{\partial \zeta}{\partial x} = -\lambda^{r_1}\sin\lambda^{r_1}(x-a) - \lambda^{r_1}\cos\lambda^{r_1}(x-a) - \int_a^x q(t)\cos\lambda^{r_1}(x-t)\zeta(h(t,\alpha),\lambda)dt. \tag{8}$$

Substituting (6) and (8) into (7), and dividing both of sides of equation by λ^{r_1}, we obtain

$$-\lambda^{r_2-r_1}\{\cos\lambda^{r_1}(b-a) - \sin\lambda^{r_1}(b-a) - \frac{1}{\lambda^{r_1}}\int_a^b q(t)\sin\lambda^{r_1}(b-t)\zeta(h(t,\alpha),\lambda)dt\} \tag{9}$$

$$-\sin\lambda^{r_1}(b-a) - \cos\lambda^{r_1}(b-a) - \frac{1}{\lambda^{r_1}}\int_a^b q(t)\cos\lambda^{r_1}(b-t)\zeta(h(t,\alpha),\lambda)dt = 0.$$

Hence,

$$-\lambda^{r_2-r_1}\{\cos\lambda^{r_1}(b-a) - \sin\lambda^{r_1}(b-a)\} + O(1) = 0,$$

and consequently,

$$-\lambda^{r_2-r_1}\sin(\frac{\pi}{4} - \lambda^{r_1}(b-a)) + O(1) = 0. \tag{10}$$

Put $\lambda^{r_1} = \eta + \frac{\pi}{4(b-a)}$. Then (10) can be written as follows:

$$\eta\sin(\eta(b-a)) + O(1) = 0. \tag{11}$$

Since equation (11) has an infinite number of zeros at large values of η, Theorem 2 is proved. □

3. Main results

In this section, we examine the asymptotic behavior of the eigenvalues of the boundary value problem L for sufficiently large λ. Also, the coefficient of delay α is uniquely determined by the given sequence of eigenvalues of L, and we obtain the connection between the potential q and canonical form of the characteristic function $C(\lambda)$.

According to Lemma 1 and Theorem 2, we have

$$\zeta(x,\lambda) = O(1), \qquad a - c_0 \leq x \leq b, \tag{12}$$

and moreover, while $\lambda < \infty$, $\frac{\partial}{\partial\lambda}\zeta(x,\lambda)$ is continuous in $a \leq x \leq b$. Also, for $a - c_0 \leq x \leq a$ and arbitrary λ, $\zeta(x,\lambda) \equiv \psi(x)$ and $\frac{\partial}{\partial\lambda}(\zeta(x,\lambda)) \equiv 0$.

Lemma 2 *For* $0 < r_1 \le 1$ *and* $a - c_0 \le x \le b$, *the following equality is valid:*

$$\frac{\partial}{\partial\lambda}\zeta(x,\lambda) = O(1). \tag{13}$$

Proof Let us derive (6) with respect to λ, and then

$$\frac{\partial\zeta}{\partial\lambda}(x,\lambda) = -\frac{1}{\lambda^{r_1}}\int_a^x q(t)\sin\lambda^{r_1}(x-t)\frac{\partial\zeta}{\partial\lambda}(h(t,\alpha),\lambda)dt + F(x,\lambda), \tag{14}$$

where $|F(x,\lambda)| \le F_0$, and F_0 is a constant. Take

$$\tau_\lambda := \max_{a-c_0\le x\le b}|\frac{\partial\zeta}{\partial\lambda}(x,\lambda)|.$$

Since $\frac{\partial\zeta}{\partial\lambda}(x,\lambda)$ is continuous in $[a-c_0,b]$, τ_λ exists. Relation (14) gives us

$$\tau_\lambda \le \frac{1}{\lambda^{r_1}}q_b\tau_\lambda + F_0.$$

Now assume $\lambda^{r_1} > 2q_b$. Therefore, $\tau_\lambda \le 2F_0$, and we arrive at (13). □

Notation 1 *Let* n *be a large enough natural number. If* $|n^2 - \lambda^{r_1}| < \frac{\pi}{4(b-a)}$, *then we say* η *is in proximity to* n^2.

Theorem 3 *For sufficiently large values of natural number* n, *the boundary value problem* L *has only a unique eigenvalue in proximity of* n^2.

Proof By (9) and (10), the expression

$$-\lambda^{-2r_1}\int_a^b q(t)\sin\lambda^{r_1}(b-t)(h(t,\alpha),\lambda)dt - \lambda^{-r_1}\int_a^b q(t)\cos\lambda^{r_1}(b-t)(h(t,\alpha),\lambda)dt \tag{15}$$

is indicated by $O(1)$. Hence, relations (12)–(13) imply that the derivative of (15) is bounded at sufficiently large λ. On the other hand, for sufficiently large η, the zeros of (10) are in the proximity of integers. Now we consider the function $C(\eta)$ as

$$C(\eta) = \eta\sin(\eta(b-a)) + O(1).$$

Then, for sufficiently large values of n, $C'(\eta) \ne 0$ and η is in the proximity of n^2. Hence, the proof is completed by Rolle's theorem and (11). □

Relation (10) has a central role to approximate the eigenvalues of L. Thus, for sufficiently large n, we denote the eigenvalues in proximity of n^{2/r_1} with $\lambda_n = (\eta_n + \frac{\pi}{4(b-a)})^{1/r_1}$ by Theorem 3, where $\eta_n = n^2 + \varepsilon_n$ for sufficiently small ε_n. Consequently,

$$\lambda_n^{r_1} = n^2 + \varepsilon_n + \frac{\pi}{4(b-a)}. \tag{16}$$

Let $b-a=m\pi$, $m\in\mathbb{N}$. Thus, substituting η_n into (11) we obtain

$$(n^2+\delta_n)|\sin(m\delta_n\pi)|=O(1).$$

Hence, for sufficiently large n, $\sin(m\delta_n\pi)=O(\frac{1}{n^2})$, and so $\delta_n=O(\frac{1}{n^2})$. Thus,

$$\eta_n=n^2+O(\frac{1}{n^2}). \tag{17}$$

For obtaining more certain asymptotic expressions that depend on the coefficient of delay α, the following lemma is useful.

Lemma 3 *Assume that $q'(x)$ exists and is bounded on $[a,b]$. Then, for $a\le x\le b$,*

$$\int_a^x q(t)\sin\lambda^{r_1}(t+h(t,\alpha))dt=O(\lambda^{-r_1}),$$
$$\int_a^x q(t)\cos\lambda^{r_1}(t+h(t,\alpha))dt=O(\lambda^{-r_1}).$$

Theorem 4 *If $b-a=m\pi$, $m\in\mathbb{N}$, $q'(x)$ exists and is bounded on $[a,b]$. Then, for sufficiently large n, the asymptotic form of the eigenvalues of the boundary value problem L is*

$$\lambda_n=\{n^2+\frac{1}{4m}-\frac{1}{2m\pi n^2}(1+\int_a^{a+m\pi}q(t)\cos\{(n^2+\frac{1}{4m})(1-\alpha)(t-a)\}dt)+O(\frac{1}{n^4})\}^{1/r_1}. \tag{18}$$

Proof According to (6) and (12) we have

$$\zeta(x,\lambda)=\sqrt{2}\sin(\frac{\pi}{4}-\lambda^{r_1}(x-a))+O(\lambda^{-2r_1}).$$

Hence,

$$\zeta(h(t,\alpha),\lambda)=\sqrt{2}\sin(\frac{\pi}{4}-\lambda^{r_1}(h(t,\alpha)-a))+O(\lambda^{-2r_1}). \tag{19}$$

Substituting (19) into (9) we obtain

$$\begin{aligned}
\lambda^{r_2-r_1}\{\cos(m\pi\lambda^{r_1})-\sin(m\pi\lambda^{r_1})\}&-\lambda^{r_2-2r_1}\sin(\lambda^{r_1}b)\int_a^b q(t)\cos(\lambda^{r_1}t)\{\cos(\lambda^{r_1}h(t,\alpha))-\sin(\lambda^{r_1}h(t,\alpha))\}dt\\
&-\lambda^{r_2-2r_1}\cos(\lambda^{r_1}b)\int_a^b q(t)\sin(\lambda^{r_1}t)\{\cos(\lambda^{r_1}h(t,\alpha))-\sin(\lambda^{r_1}h(t,\alpha))\}dt\\
&-\frac{1}{\lambda^{r_1}}\cos(\lambda^{r_1}b)\int_a^b q(t)\cos(\lambda^{r_1}t)\{\cos(\lambda^{r_1}h(t,\alpha))-\sin(\lambda^{r_1}h(t,\alpha))\}dt\\
&-\frac{1}{\lambda^{r_1}}\sin(\lambda^{r_1}b)\int_a^b q(t)\sin(\lambda^{r_1}t)\{\cos(\lambda^{r_1}h(t,\alpha))-\sin(\lambda^{r_1}h(t,\alpha))\}dt\\
&-\sin(m\pi\lambda^{r_1})-\cos(m\pi\lambda^{r_1})+O(\lambda^{r_2-4r_1})=0.
\end{aligned}$$

This together with the following identities,

$$\sin(\lambda^{r_1}t)\sin(\lambda^{r_1}h(t,\alpha)) = \frac{1}{2}\{\cos\lambda^{r_1}(t-h(t,\alpha)) - \cos\lambda^{r_1}(t+h(t,\alpha))\},$$

$$\cos(\lambda^{r_1}t)\cos(\lambda^{r_1}h(t,\alpha)) = \frac{1}{2}\{\cos\lambda^{r_1}(t-h(t,\alpha)) + \cos\lambda^{r_1}(t+h(t,\alpha))\},$$

$$\sin(\lambda^{r_1}t)\cos(\lambda^{r_1}h(t,\alpha)) = \frac{1}{2}\{\sin\lambda^{r_1}(t-h(t,\alpha)) + \sin\lambda^{r_1}(t+h(t,\alpha))\},$$

$$\cos(\lambda^{r_1}t)\sin(\lambda^{r_1}h(t,\alpha)) = \frac{1}{2}\{\sin\lambda^{r_1}(t+h(t,\alpha)) - \sin\lambda^{r_1}(t-h(t,\alpha))\},$$

and with Lemma 3, yields

$$\begin{aligned}&\sin(\frac{\pi}{4} - m\pi\lambda^{r_1})\{\lambda^{r_2-r_1} + \frac{1}{2}\int_a^{a+m\pi} q(t)\sin\lambda^{r_1}(t-h(t,\alpha))dt\} \\ &-\cos(\frac{\pi}{4} - m\pi\lambda^{r_1})\{1 + \frac{1}{2}\int_a^{a+m\pi} q(t)\cos\lambda^{r_1}(t-h(t,\alpha))dt\} = O(\lambda^{r_1-r_2}).\end{aligned} \tag{20}$$

Take

$$\eta = \lambda^{r_1} - \frac{1}{4m}. \tag{21}$$

Then equality (20) gives us

$$\tan(m\pi\eta) = -\frac{4m}{1+4m\eta}\{1 + \frac{1}{2}\int_a^{a+m\pi} q(t)\cos\lambda^{r_1}(t-h(t,\alpha))dt\} + O(\frac{1}{\eta^2}).$$

Hence, it follows from (17) and assuming

$$\eta_n = n^2 + \varepsilon_n \tag{22}$$

that

$$\tan((n^2+\varepsilon_n)m\pi) = \tan(m\pi\varepsilon_n) = -\frac{1}{n^2}\{1 + f(n,\alpha)\} + O(\frac{1}{n^4}),$$

where

$$f(n,\alpha) = \frac{1}{2}\int_a^{a+m\pi} q(t)\cos\{(n^2 + \frac{1}{4m})(t-h(t,\alpha))\}dt. \tag{23}$$

Therefore, for sufficiently large n we obtain

$$\varepsilon_n = -\frac{1}{m\pi n^2}\{1 + f(n,\alpha)\} + O(\frac{1}{n^4}).$$

Substituting ε_n into (22) together with (21), we arrive at (18). □

Now we ask: *Does the given sequence of eigenvalues λ_n, $n \geq 1$, uniquely determine the parameter of delay α?* Under some additional conditions, we will show that the answer of this question is positive.

Theorem 5 *Let $r_2 = 2r_1$, $b - a = m\pi$, $m \in \mathbb{N}$. If the eigenvalues of the boundary value problem L of the form (18) are given, then the parameter of delay $\alpha \in (0,1)$ is unambiguously determined.*

Proof First, we denote the sequences $C^{\pm}(\lambda_n)$ as follows:

$$C^{\pm}(\lambda_n) = \{\lambda_{n\pm 1}^{r_1} - (n \pm 1)^2 - \frac{1}{4m}\}(n \pm 1)^2.$$

Therefore, according to (18), we get

$$C^{\pm}(\lambda_n) = -\frac{1}{m\pi}\{1 + f(n \pm 1, \alpha)\} + O(\frac{1}{n^2}),$$

where $f(n, \alpha)$ is defined in (23). Hence, we obtain the following system:

$$\begin{aligned} 2m\pi(C^{\pm}(\lambda_n) - 1) = \sin\Theta \int_a^{a+m\pi} & q(t) \sin\Theta^{\pm}(t,n)dt \\ & - \cos\Theta \int_a^{a+m\pi} q(t)\cos\Theta^{\pm}(t,n)dt + O(\frac{1}{n^2}), \end{aligned} \tag{24}$$

where

$$\Theta = (1 + \frac{1}{4m})\alpha a, \quad \Theta^{\pm}(t,n) = \{(n \pm 1)^2 + \frac{1}{4m}\}(1-\alpha)t + (n^2 \pm 2n)(\alpha - 1)a - (1 + \frac{1}{4m})a.$$

Let us choose the subsequence n_k of the sequence n, $n \in \mathbb{N}$, such that

$$\begin{aligned} C_{n_k} := & \int_a^{a+m\pi} q(t)\sin\Theta^{+}(t,n_k)dt \ . \int_a^{a+m\pi} q(t)\cos\Theta^{-}(t,n_k)dt \\ & - \int_a^{a+m\pi} q(t)\sin\Theta^{-}(t,n_k)dt \ . \int_a^{a+m\pi} q(t)\cos\Theta^{+}(t,n_k)dt + O(\frac{1}{n_k^2}) \neq 0. \end{aligned}$$

Then the sequence

$$\begin{aligned} \widetilde{C}(\lambda_{n_k}) := \frac{2m\pi}{C_{n_k}}\{ & (C^{+}(\lambda_{n_k}) - 1)\int_a^{a+m\pi} q(t)\sin\Theta^{-}(t,n_k)dt \\ & -(C^{-}(\lambda_{n_k}) - 1)\int_a^{a+m\pi} q(t)\sin\Theta^{+}(t,n_k)dt\} + O(\frac{1}{n_k^2}) \end{aligned}$$

is well defined. Moreover, from (24), we obtain

$$\cos\Theta = \lim_{k\to\infty} \widetilde{C}(\lambda_{n_k}) =: \gamma.$$

Consequently,

$$\alpha = \frac{4m}{a(1+4m)} \arccos\gamma,$$

and the determination of α is proved. □

In the next theorem, we obtain a relation between the potential function q and the infinite product form of the characteristic function $C(\lambda)$.

Theorem 6 *Let $r_1 = 2$, $b - a = m\pi$, $m \in \mathbb{N}$; then the following relation is valid:*

$$\{m\lambda(\prod_{n=1}^{\infty}\frac{\lambda_n^2}{n^2})\prod_{n=1}^{\infty}(1-\frac{m^2\lambda^2}{\lambda_n^2}) - \sin(m\pi\lambda)\} = \frac{\sin(m\pi\lambda)}{2m\pi}\int_a^{a+m\pi} q(t)B_1(\lambda,t,\alpha)dt$$
$$+\frac{\pi\cos(m\pi\lambda)}{8m^2\lambda} + \frac{\sin(m\pi\lambda)}{m^2\lambda^2}\{m^2\lambda^3\ell(\lambda,\alpha) + \frac{1}{8m} - \frac{\pi^2}{6} - 1\}$$
$$-\frac{\pi\cos(m\pi\lambda)}{2m^3\lambda^3} + \frac{1}{m^4\lambda^4}\{\frac{\sin(m\pi\lambda)}{2} - \frac{\pi^2}{6}\} + O(\frac{\cos(m\pi\lambda)}{\lambda^5}), \tag{25}$$

as $\lambda \to +\infty$.

Proof Since for $r_1 = 2$ the characteristic function $C(\lambda)$ is entire in λ (see (7)), using Hahamard's theorem, $C(\lambda)$ can be represented by its zeros λ_n, $n \in \mathbb{N}$, in the form of an infinite product as follows:

$$C(\lambda) = C(\lambda,\alpha) = H\prod_{n=1}^{\infty}(1-\frac{m^2\lambda^2}{\lambda_n^2})$$
$$= \frac{H}{m\pi}(\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n^2})\ .\ m\pi\prod_{n=1}^{\infty}\{(1-\frac{m^2\lambda^2}{n^2}) + \frac{\lambda_n^2 - n^2}{n^2}\}. \tag{26}$$

Put $H_1 = \frac{H}{m\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n^2}$. Then we have

$$C(\lambda,\alpha) = m\pi H\{\prod_{n=1}^{\infty}(1-\frac{m^2\lambda^2}{n^2}) + \sum_{k=1}^{\infty}\prod_{n\neq k}(1-\frac{m^2\lambda^2}{n^2})\frac{\lambda_k^2-k^2}{k^2}$$
$$+\sum_{i=2}^{\infty}\sum_{k_1<k_2<\ldots<k_i}\prod_{n\neq k_1,k_2,\ldots,k_i}(1-\frac{m^2\lambda^2}{n^2})\prod_{s=1}^{i}\frac{\lambda_{k_s}^2-k_s^2}{k_s^2}\}$$
$$= H_1\{\frac{\sin(m\pi\lambda)}{\lambda} + \frac{\sin(m\pi\lambda)}{\lambda}\sum_{n=1}^{\infty}\frac{\lambda_n^2-n^2}{n^2-m^2\lambda^2} + \ell(\lambda,\alpha)\}, \quad \lambda \in C\setminus\mathbb{Z}. \tag{27}$$

It follows from (11) and (27) that $H_1 = \lambda^2$. Hence, $H = m\pi\lambda^2\prod_{n=1}^{\infty}\frac{\lambda_n^2}{n^2}$. Substituting this into (26), and using (27), equation (26) becomes

$$\sum_{n=1}^{\infty}\frac{\lambda_n^2-n^2}{n^2-m^2\lambda^2} = \frac{(m\pi\prod_{n=1}^{\infty}\frac{\lambda_n^2}{n^2})\prod_{n=1}^{\infty}(1-\frac{m^2\lambda^2}{n^2})}{\sin(m\pi\lambda)} - 1 - \frac{\lambda\ell(\lambda,\alpha)}{\sin(m\pi\lambda)}, \quad \lambda \in C\setminus\mathbb{Z}, \tag{28}$$

where

$$\ell(\lambda,\alpha) = \lambda\sin(m\pi\lambda)\sum_{i=2}^{\infty}\sum_{1\leq k_1<k_2<\ldots<k_i}\prod_{s=1}^{i}\frac{\lambda_{k_s}^2-k_s^2}{k_s^2-m^2\lambda^2}.$$

On the other hand, we can write

$$\sum_{n=1}^{\infty} \frac{\lambda_n^2 - n^2 - \frac{1}{4m}}{n^2 - m^2\lambda^2} = \sum_{n=1}^{\infty} \frac{\lambda_n^2 - n^2 - \frac{1}{4m} - \frac{\widetilde{f}(n,\alpha)}{n^2}}{n^2 - m^2\lambda^2} + \sum_{n=1}^{\infty} \frac{\widetilde{f}(n,\alpha)}{n^2(n^2 - m^2\lambda^2)}$$

$$= \sum_{n=1}^{\infty} \frac{\widetilde{f}(n,\alpha)}{n^2(n^2 - m^2\lambda^2)} + \frac{1}{m^2\lambda^2} \sum_{n=1}^{\infty} (\lambda_n^2 - n^2 - \frac{1}{4m} - \frac{\widetilde{f}(n,\alpha)}{n^2}) \frac{n^2}{n^2 - m^2\lambda^2} - \frac{1}{m^2\lambda^2} Q(\alpha), \tag{29}$$

where

$$\widetilde{f}(n,\alpha) = \frac{1}{m\pi}(1 + f(n,\alpha)), \tag{30}$$

$$Q(\alpha) = \sum_{n=1}^{\infty} (\lambda_n^2 - n^2 - \frac{1}{4m} - \frac{\widetilde{f}(n,\alpha)}{n^2}).$$

Moreover, from (18) with $r_1 = 2$,

$$\lambda_n^2 - n^2 - \frac{1}{4m} - \frac{\widetilde{f}(n,\alpha)}{n^2} = O(\frac{1}{n^4}).$$

Thus,

$$\frac{1}{m^2\lambda^2} \sum_{n=1}^{\infty} (\lambda_n^2 - n^2 - \frac{1}{4m} - \frac{\widetilde{f}(n,\alpha)}{n^2}) \frac{n^2}{n^2 - m^2\lambda^2} \approx \frac{1}{m^2\lambda^2} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - m^2\lambda^2)}$$

$$= -\frac{\pi^2}{6m^4\lambda^4} - \frac{\pi \cot(m\pi\lambda)}{2m^5\lambda^5} + \frac{1}{2m^6\lambda^6} = -\frac{\pi^2}{6m^4\lambda^4} + O(\frac{\cot(m\pi\lambda)}{m^5\lambda^5}), \quad \lambda \to \infty. \tag{31}$$

Based on (29) and (31) we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n^2 - n^2 - \frac{1}{4m}}{n^2 - m^2\lambda^2} = \sum_{n=1}^{\infty} \frac{\widetilde{f}(n,\alpha)}{n^2(n^2 - m^2\lambda^2)} - \frac{1}{m^2\lambda^2} Q(\alpha)$$

$$-\frac{\pi^2}{6m^4\lambda^4} + O(\frac{\cot(m\pi\lambda)}{m^5\lambda^5}), \quad \lambda \to \infty. \tag{32}$$

Further, using (23) and (30), we calculate

$$\sum_{n=1}^{\infty} \frac{\widetilde{f}(n,\alpha)}{n^2(n^2 - m^2\lambda^2)} = \frac{1}{2m\pi} \int_a^{a+m\pi} q(t) B_1(\lambda, t, \alpha) dt + B_0(\lambda), \tag{33}$$

where

$$B_0(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - m^2\lambda^2)} = -\frac{\pi^2}{6m^2\lambda^2} - \frac{\pi}{2m^3\lambda^3} \cot(m\pi\lambda) + \frac{1}{2m^4\lambda^4},$$

$$B_1(\lambda, t, \alpha) = \sum_{n=1}^{\infty} \frac{\cos\{(n^2 + \frac{1}{4m})(t - h(t,\alpha))\}}{n^2(n^2 - m^2\lambda^2)}.$$

On the other hand,

$$\sum_{n=1}^{\infty}\frac{\lambda_n^2-n^2-\frac{1}{4m}}{n^2-m^2\lambda^2}=\sum_{n=1}^{\infty}\frac{\lambda_n^2-n^2}{n^2-m^2\lambda^2}-\frac{1}{4m}\Big(\frac{1}{2m^2\lambda^2}-\frac{\pi}{2m\lambda}\cot(m\pi\lambda)\Big).$$

This together with (28) yields

$$\sum_{n=1}^{\infty}\frac{\lambda_n^2-n^2-\frac{1}{4m}}{n^2-m^2\lambda^2}=\frac{m\pi}{\sin(m\pi\lambda)}\Big(\prod_{n=1}^{\infty}\frac{\lambda_n^2}{n^2}\Big)\prod_{n=1}^{\infty}\Big(1-\frac{m^2\lambda^2}{\lambda_n^2}\Big)-1-\frac{\lambda\ell(\lambda,\alpha)}{\sin(m\pi\lambda)}$$
$$-\frac{1}{8m^3\lambda^2}+\frac{\pi}{8m^2\lambda}\cot(m\pi\lambda). \tag{34}$$

Finally, substituting (33) into (32), and in view of the right side of (34), we arrive at (25). □

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