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Closed range properties of Li–Stević integral-type operators between Bloch-type spaces and their essential norms

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Abstract: We investigate closed range properties of certain integral-type operators introduced by Li and Stević. The operators are considered between Bloch-type spaces. We also give the essential norm of such operators. Our results are given in a general setting and we also give the essential norm of Li–Stević integral-type operators between other well-known spaces of analytic functions.

Key words: Closed range operators, essential norms, integral-type operators, weighted composition operators, Bloch-type spaces

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Let X be a Banach space of analytic functions on $\mathbb{D}$. For an analytic self-map φ of $\mathbb{D}$ and $u \in H(\mathbb{D})$, the *weighted composition operator* uC_φ on X is defined by

$$uC_\varphi f = u \cdot f \circ \varphi, \quad \text{for all } f \in X.$$

When $u = 1$ we get the *composition operator* C_φ on X given by

$$C_\varphi f = f \circ \varphi, \quad \text{for all } f \in X.$$

Weighted composition operators appear in the study of dynamical systems. Moreover, it is known that isometries on many analytic function spaces are of the canonical forms of weighted composition operators. In general, certain operator theoretic properties of weighted composition operators can be studied by considering various conditions on their inducing functions. For more information about these types of operators acting on various spaces of analytic functions, see [2–5, 8, 9, 11, 18–21, 23, 24] and the references therein.

For $g \in H(\mathbb{D})$, the *integral operator* I_g on X is defined by

$$(I_g f)(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D},$$

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for all $f \in X$. As generalizations of integral operator I_g, the products of composition operator C_φ and integral operator I_g, called *Li–Stević integral-type operators*, are given by

$$(I_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D},$$

$$(C_\varphi I_g f)(z) = \int_0^{\varphi(z)} f'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D},$$

for all $f \in X$. These operators were essentially defined by Li and Stević in [12] and later studied along with their n-dimensional relatives in many papers; see, for example, [11, 13–17, 22, 25, 30] and the references therein. By letting $\varphi(z) = z$, we have $I_g C_\varphi = I_g = C_\varphi I_g$. Also, note that

$$I_{g\circ\varphi} C_\varphi f = C_\varphi I_g f - \int_0^{\varphi(0)} f'(\zeta) g(\zeta) d\zeta \tag{1.1}$$

for all $f \in H(\mathbb{D})$, and therefore if $\varphi(0) = 0$ then $I_{g\circ\varphi} C_\varphi = C_\varphi I_g$.

Integral operators of the type $I_g C_\varphi$ and $C_\varphi I_g$ and other similar types of operators have been studied by several authors between different spaces of analytic functions on the unit disk of complex plane $\mathbb{C}$ or the unit ball of the complex vector space $\mathbb{C}^n$. See, for example, [12, 14, 16, 22, 25–29] and the references therein. In this paper, weighted composition operators and Li–Stević integral-type operators are mainly considered between *weighted-type spaces of analytic functions* and *Bloch-type spaces*, defined as follows.

By a *weight* v we mean a strictly positive continuous function on $\mathbb{D}$ that is *radial* (that is, $v(z) = v(|z|)$ for all $z \in \mathbb{D}$) and decreasing with respect to $|z|$. We also assume that the weight v tends to zero at the boundary of $\mathbb{D}$; that is, $\lim_{|z|\to 1} v(z) = 0$. A weight v is said to be *analytic* if $v(z) = \frac{1}{f(|z|)}$ for some $f \in H(\mathbb{D})$ that takes real values on $[0, 1)$ and is such that $|f(z)| \leq f(|z|)$ for all $z \in \mathbb{D}$.

For a weight v, the *associated weight* $\tilde{v}$ is defined by

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_{H_v^\infty} \leq 1\}} = \frac{1}{\|\delta_z\|}, \quad z \in \mathbb{D}, \tag{1.2}$$

where δ_z denotes the point evaluation functional (at z) on H_v^∞. It is known that $\tilde{v}$ is also a weight such that $v(z) \leq \tilde{v}(z)$ for all $z \in \mathbb{D}$. The weight v is called *essential* if there exists a constant $C \geq 1$ such that

$$v(z) \leq \tilde{v}(z) \leq C v(z), \quad z \in \mathbb{D}. \tag{1.3}$$

It is known that if v is an analytic weight, then $v = \tilde{v}$ and therefore every analytic weight is essential. Following Domanski and Lindström [6, Lemma 1], a weight v is called *normal* if it satisfies properties (L1) and (L2) as follows:

(L1) $$\inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0,$$

(L2) $$\limsup_n \frac{v(1 - 2^{-n-k})}{v(1 - 2^{-n})} < 1, \quad \text{for some } k \in \mathbb{N}.$$

It is worth mentioning that for each $0 < \alpha < \infty$, the *standard weights* $v_\alpha(z) = (1-|z|^2)^\alpha$ are normal and essential. For more information and examples of weights satisfying the above conditions, see [1, 2, 6] and the references therein.

For a weight v, the *weighted-type space of analytic functions* H_v^∞ is defined by

$$H_v^\infty = \{f \in H(\mathbb{D}) : \|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Also, the *Bloch-type space* $\mathcal{B}_v$ is defined as follows:

$$\mathcal{B}_v = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_v} = |f(0)| + \|f\|_v < \infty\},$$

where $\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f'(z)|$. The spaces H_v^∞ and $\mathcal{B}_v$ are Banach spaces endowed with the norm $\|\cdot\|_{H_v^\infty}$ and $\|\cdot\|_{\mathcal{B}_v}$, respectively. The closed subspace of $\mathcal{B}_v$ consisting of those f for which $f(0) = 0$ is denoted by $\widetilde{\mathcal{B}}_v$.

In this paper we investigate Li–Stević integral-type operators between Bloch-type spaces. In Section 2, we describe closed range properties of the operators I_gC_φ and $C_\varphi I_g$ between Bloch-type spaces $\mathcal{B}_v$. Our descriptions are given in terms of the existence of specific properties for the inducing functions. In Section 3, using the concept of weighted composition operators, we give the essential norm of the operators I_gC_φ and $C_\varphi I_g$ between Bloch-type spaces $\mathcal{B}_v$. We give the results of Section 3 in a general setting that consequently, besides Bloch-type spaces, also gives the essential norm of I_gC_φ and $C_\varphi I_g$ between some other well-known spaces of analytic functions.

2. Closed range properties of Li–Stević integral type operators

It is well known that a bounded one-to-one operator T on a Banach space $(X, \|\cdot\|_X)$ has closed range if and only if it is bounded below; that is, there exists a positive constant $\delta > 0$ such that $\|Tx\|_X \geq \delta\|x\|_X$ for every $x \in X$. When X is a Banach space of analytic functions on $\mathbb{D}$ and φ is a nonconstant analytic self-map of $\mathbb{D}$, every well-defined composition operator C_φ on X is one-to-one. Therefore, study of closed range bounded composition operators on a large class of Banach function spaces reduces to the study of bounded below composition operators. Bounded below composition operators on the Bloch spaces were first studied by Ghatage et al. in [8]. Closed range operators and connections with their operator theoretic properties have been studied by many authors on different spaces of analytic functions [3, 9, 21, 31, 32]. In this section we study conditions under which the operators I_gC_φ and $C_\varphi I_g$ between Bloch-type spaces $\mathcal{B}_v$ have closed range.

Note that the operator $P : \mathcal{B}_v \to \mathcal{B}_w$ given in (1.1) by $Pf = \left(\int_0^{\varphi(0)} f'(\zeta)g(\zeta)d\zeta\right)1$ is bounded. To see this, consider a compact set $K_\varphi \subset \mathbb{D}$ containing the line segment joining 0 and $\varphi(0)$. Therefore, by letting $R_\varphi(g) := \sup_{z\in K_\varphi} |g(z)| < \infty$ and $r_\varphi(v) := \inf_{z\in K_\varphi} v(z) > 0$ we have

$$\|Pf\|_{\mathcal{B}_w} = |(Pf)(0)| = \left|\int_0^{\varphi(0)} f'(\zeta)g(\zeta)d\zeta\right| \leq \frac{\|f\|_v}{r_\varphi(v)} R_\varphi(g) \leq \frac{R_\varphi(g)}{r_\varphi(v)}\|f\|_{\mathcal{B}_v},$$

which implies boundedness of the operator $P : \mathcal{B}_v \to \mathcal{B}_w$. Consequently, (1.1) implies that $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ is bounded if and only if $I_{g\circ\varphi}C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ is bounded.

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. It is known that for the weights v and w, the weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))} < \infty; \tag{2.1}$$

see [4, Proposition 3.1]. For each weight v, consider the *differentiation operator* $D_v : \mathcal{B}_v \to H_v^\infty$ and the *integration operator* $S_v : H_v^\infty \to \mathcal{B}_v$, given by $(D_v f)(z) = f'(z)$ and $(S_v f)(z) = \int_0^z f(\zeta)d\zeta$, for all $z \in \mathbb{D}$. Clearly, $D_v : \mathcal{B}_v \to H_v^\infty$ is a bounded operator with $\|D_v\| \leq 1$ and $S_v : H_v^\infty \to \mathcal{B}_v$ is an isometry. Using these operators and applying the previous discussion along with (2.1) yields the following theorem, characterizing boundedness of the operators $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ and $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$. In order to simplify the notations, for the weights v and w, define

$$\tau_{\varphi,g}^{v,w}(z) = \frac{w(z)|\varphi'(z)||g(z)|}{\tilde{v}(\varphi(z))}, \quad z \in \mathbb{D};$$

see, for example, [3, 31, 32].

Theorem 2.1 *Suppose that v and w are weights, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then:*

(i) $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ is bounded if and only if $\sup_{z\in\mathbb{D}} \tau_{\varphi,g}^{v,w}(z) < \infty$.

(ii) $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ is bounded if and only if $\sup_{z\in\mathbb{D}} \tau_{\varphi,g\circ\varphi}^{v,w}(z) < \infty$.

Considering the operator $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$, let $I_g \tilde{C}_\varphi$ denote the restriction of $I_g C_\varphi$ to $\widetilde{\mathcal{B}}_v$; that is, $I_g \tilde{C}_\varphi = I_g C_\varphi|_{\widetilde{\mathcal{B}}_v} : \widetilde{\mathcal{B}}_v \to \mathcal{B}_w$. Note that $Range(I_g \tilde{C}_\varphi) = Range(I_g C_\varphi)$, since for each $f \in \mathcal{B}_v$ we have $f - f(0) \in \widetilde{\mathcal{B}}_v$ and

$$I_g C_\varphi(f) = I_g C_\varphi(f - f(0)) = I_g \tilde{C}_\varphi(f - f(0)).$$

Also, it is worth noting that if φ is a nonconstant self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$ is nonzero, then the operator $I_g \tilde{C}_\varphi : \widetilde{\mathcal{B}}_v \to \mathcal{B}_w$ is one-to-one. This is an immediate consequence of the facts that $\varphi : \mathbb{D} \to \mathbb{D}$ is an open mapping and the zero sets of φ' and g in $\mathbb{D}$ are at most countable.

As a consequence of the above discussions we get the following useful result:

Theorem 2.2 *Let φ be a nonconstant self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ be nonzero, and the operator $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ be bounded. Then $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range if and only if $I_g \tilde{C}_\varphi : \widetilde{\mathcal{B}}_v \to \mathcal{B}_w$ is bounded below.*

A subset G of $\mathbb{D}$ is said to be a *sampling set* for $\mathcal{B}_v$ if there exists $k > 0$ such that

$$\|f\|_v \leq k \sup\{v(z)|f'(z)| : z \in G\},$$

for all $f \in \mathcal{B}_v$. The concept of sampling set for $\mathcal{B}_{v_\alpha}$ is defined in [3, 9, 32]. Ghatage et al. in [9] used sampling sets to study closed range composition operators on the Bloch spaces. The concept of sampling sets has also been used to investigate closed range composition operators on other spaces of analytic functions [3, 32].

For the weights v and w, $\varepsilon > 0$, $g \in H(\mathbb{D})$, and a self-map φ of $\mathbb{D}$, let

$$\Omega_{\varphi,g}^{v,w}(\varepsilon) = \{z \in \mathbb{D} : \tau_{\varphi,g}^{v,w}(z) \geq \varepsilon\}, \quad \text{and} \quad G_{\varphi,g}^{v,w}(\varepsilon) = \varphi(\Omega_{\varphi,g}^{v,w}(\varepsilon)).$$

In the next theorem, we use Theorem 2.1 and Theorem 2.2 to characterize closed range operators $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ in terms of existence of sampling sets for $\mathcal{B}_v$. See [3, Theorem 3.1] for a similar result about closed range bounded composition operators $C_\varphi : \mathcal{B}_{v_\alpha} \to \mathcal{B}_{v_\alpha}$ when $\alpha \geq 1$.

Theorem 2.3 *For arbitrary weight w and essential weight v, let $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ be a bounded operator. Then $I_g C_\varphi$ has closed range if and only if for some $\varepsilon > 0$, $G^{v,w}_{\varphi,g}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$.*

Proof Suppose that $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range. Then, by Theorem 2.2, there exists $m > 0$ such that

$$\|I_g \tilde{C}_\varphi(f)\|_{\mathcal{B}_w} \geq m\|f\|_{\mathcal{B}_v}, \quad \text{for all } f \in \widetilde{\mathcal{B}}_v. \tag{2.2}$$

Also, note that for each $f \in \widetilde{\mathcal{B}}_v$ we have $\|I_g \tilde{C}_\varphi(f)\|_{\mathcal{B}_w} = \|I_g \tilde{C}_\varphi(f)\|_w$. Hence, by (2.2), for each $f \in \widetilde{\mathcal{B}}_v$ with $\|f\|_v > 0$, there exists $z_f \in \mathbb{D}$ such that

$$\tau^{v,w}_{\varphi,g}(z_f)\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| = w(z_f)|(I_g C_\varphi(f))'(z_f)| \geq (m/2)\|f\|_v.$$

On the other hand, by Theorem 2.1,

$$m^{v,w}_{\varphi,g} = \sup_{z\in\mathbb{D}} \tau^{v,w}_{\varphi,g}(z) < \infty,$$

which implies that

$$m^{v,w}_{\varphi,g}\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| \geq \tau^{v,w}_{\varphi,g}(z_f)\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| \geq (m/2)\|f\|_v. \tag{2.3}$$

Note that by the definition of $\tilde{v}$ in (1.2), we have $\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| \leq \|f'\|_{H^\infty_{\tilde{v}}} = \|f\|_v$, and therefore, by (2.3), we get

$$\tau^{v,w}_{\varphi,g}(z_f) \geq (m/2), \quad \text{and} \quad \tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| \geq \frac{m}{2m^{v,w}_{\varphi,g}}\|f\|_v. \tag{2.4}$$

Hence, by letting $\varepsilon = m/2$, (2.4) implies that $z_f \in \Omega^{v,w}_{\varphi,g}(\varepsilon)$, and also since the weight v is essential, using the positive constant C given in (1.3), we obtain

$$\|f\|_v \leq \frac{2Cm^{v,w}_{\varphi,g}}{m} \sup\{v(z)|f'(z)| : \ z \in \varphi(\Omega^{v,w}_{\varphi,g}(\varepsilon))\}, \tag{2.5}$$

for each $f \in \widetilde{\mathcal{B}}_v$. Since for each $f \in \mathcal{B}_v$ we have $f - f(0) \in \widetilde{\mathcal{B}}_v$ and $\|f\|_v = \|f - f(0)\|_v$, (2.5) also holds for each $f \in \mathcal{B}_v$. This implies that $G^{v,w}_{\varphi,g}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$.

Conversely, suppose that for some $\varepsilon > 0$, $G^{v,w}_{\varphi,g}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$. Therefore, using the fact that $v \leq \tilde{v}$, there exists $k > 0$ such that

$$\|f\|_v \leq k \sup\{\tilde{v}(z)|f'(z)| \ : \ z \in G^{v,w}_{\varphi,g}(\varepsilon)\},$$

for all $f \in \mathcal{B}_v$. Therefore, for each $f \in \mathcal{B}_v$, there exists $z_f \in \mathbb{D}$ such that $\tau^{v,w}_{\varphi,g}(z_f) \geq \varepsilon$, $\varphi(z_f) \in G^{v,w}_{\varphi,g}(\varepsilon)$, and

$$\|f\|_v \leq 2k\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))|.$$

Consequently, for each $f \in \mathcal{B}_v$, we have

$$\begin{aligned}\|I_g C_\varphi(f)\|_w &\geq w(z_f)|\varphi'(z_f)||g(z_f)||f'(\varphi(z_f))| \\ &= \tau^{v,w}_{\varphi,g}(z_f)\tilde{v}(\varphi(z_f))|f'(\varphi(z_f))| \\ &\geq \frac{\varepsilon\|f\|_v}{2k}.\end{aligned}$$

This implies that for each $f \in \widetilde{\mathcal{B}}_v$ we have

$$\|I_g\tilde{C}_\varphi(f)\|_{\mathcal{B}_w} \geq \|I_g\tilde{C}_\varphi(f)\|_w = \|I_g C_\varphi(f)\|_w \geq \frac{\varepsilon\|f\|_v}{2k} = \frac{\varepsilon\|f\|_{\mathcal{B}_v}}{2k}.$$

Therefore, $I_g\tilde{C}_\varphi : \widetilde{\mathcal{B}}_v \to \mathcal{B}_w$ is bounded below and hence, by Theorem 2.2, $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range. □

As mentioned before, (1.1) shows that if $\varphi(0) = 0$ then $I_{g\circ\varphi}C_\varphi = C_\varphi I_g$. This, along with Theorem 2.3, implies the following result:

For arbitrary weight w and essential weight v, let $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ be a bounded operator and $\varphi(0) = 0$. Then $C_\varphi I_g$ has closed range if and only if for some $\varepsilon > 0$, $G^{v,w}_{\varphi,g\circ\varphi}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$.

By analyzing the proof of Theorem 2.3 one can see that the condition $\varphi(0) = 0$ is not used in "if part". Hence, we have the following result:

Theorem 2.4 *For the weights v and w, let $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ be a bounded operator.*

- *(i) If there exists $\varepsilon > 0$ such that $G^{v,w}_{\varphi,g\circ\varphi}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$, then $C_\varphi I_g$ has closed range.*
- *(ii) If the weight v is essential, $\varphi(0) = 0$, and $C_\varphi I_g$ has closed range, then there exists $\varepsilon > 0$ such that $G^{v,w}_{\varphi,g\circ\varphi}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$.*

Let $\rho(z, w) = |\varphi_z(w)|$ denote the *pseudo-hyperbolic distance* on $\mathbb{D}$, where φ_z is the disk automorphism of $\mathbb{D}$ defined by

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in \mathbb{D}.$$

For $r \in (0, 1)$ we say that a subset G of $\mathbb{D}$ is an r-*net* for $\mathbb{D}$ if for every $z \in \mathbb{D}$ there exists $w \in G$ such that $\rho(z, w) < r$; see [3, 31, 32]. Closed range composition operators between Bloch-type spaces have been characterized in terms of existence of r-nets. See, for example, [3, 31, 32]. By applying the next lemma and using Theorem 2.2, we next show that if $G^{v,w}_{\varphi,g}(\varepsilon)$ is an r-net for some small enough $r > 0$, then $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range (see [21, Theorem 3.7(b)]).

Lemma 2.5 *[2, Lemma 1] Let v be a weight satisfying condition $(L1)$, which is continuously differentiable with respect to $|z|$. Then there exists $M_v > 0$ such that for all $f \in H^\infty_v$ and $z_1, z_2 \in \mathbb{D}$ we have*

$$|v(z_1)f(z_1) - v(z_2)f(z_2)| \leq M_v\|f\|_{H^\infty_v}\rho(z_1, z_2).$$

Theorem 2.6 *Let v be a weight satisfying condition $(L1)$, which is continuously differentiable with respect to $|z|$. For an arbitrary weight w let $I_gC_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ be a bounded operator. If there exist $\varepsilon > 0$ and $r \in (0, M_v^{-1})$ such that $G_{\varphi,g}^{v,w}(\varepsilon)$ is an r-net, then I_gC_φ has closed range.*

Proof Let $f \in \widetilde{\mathcal{B}}_v$ with $\|f\|_v = 1$. Since $\delta = \frac{1}{2}(1 + rM_v) < 1$, there exists $a \in \mathbb{D}$ such that $|f'(a)|v(a) \geq \delta$. On the other hand, since $f' \in H_v^\infty$, using Lemma 2.5, we have

$$|f'(a)v(a) - f'(\varphi(z_a))v(\varphi(z_a)))| \leq M_v \|f'\|_{H_v^\infty} \rho(a, \varphi(z_a)) \leq M_v r.$$

Therefore, using the fact $v \leq \tilde{v}$, we conclude that

$$\begin{aligned}
\|(I_g\tilde{C}_\varphi)(f)\|_{\mathcal{B}_w} \geq \|(I_gC_\varphi)(f)\|_w &\geq |f'(\varphi(z_a))||\varphi'(z_a)||g(z_a)|w(z) \\
&= |f'(\varphi(z_a))|\tilde{v}(\varphi(z_a))\tau_{\varphi,g}^{v,w}(z_a) \\
&\geq |f'(\varphi(z_a))|v(\varphi(z_a))\tau_{\varphi,g}^{v,w}(z_a) \\
&\geq \varepsilon(|f'(a)v(a)| - M_v r) \\
&\geq \varepsilon(\delta - M_v r) = \varepsilon(\frac{1 - M_v r}{2}).
\end{aligned}$$

This implies that $I_g\tilde{C}_\varphi : \widetilde{\mathcal{B}}_v \to \mathcal{B}_w$ is bounded below and hence, by Theorem 2.2, $I_gC_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range. □

In order to prove the converse of Theorem 2.6 we first need to prove the next lemma, which is a generalization of [3, Theorem 3.2] for the standard weights v_α. The types of weights v considered in the next lemma are introduced in [2, Theorem 2(b)]. We will later mention examples of weights satisfying the assumptions of the next lemma.

Lemma 2.7 *Let v be a weight such that $\mu(z) = \frac{v(z)}{(1-|z|^2)^p}$ is a weight on $\mathbb{D}$ for some $0 < p < \infty$ and $\mu = \tilde{\mu}$. Then any sampling set for $\mathcal{B}_v$ is an r-net for some $r \in (0,1)$.*

Proof Let $a \in \mathbb{D}$. Then there exists $g_a \in H_\mu^\infty$ such that $\|g_a\|_{H_\mu^\infty} = 1$ and $g_a(a)\mu(a) = g_a(a)\tilde{\mu}(a) = 1$. Define

$$f_a(z) = \int_0^z g_a(\zeta) \left(\frac{1 - |a|^2}{(1 - \overline{a}\zeta)^2} \right)^p d\zeta,$$

for all $z \in \mathbb{D}$. Then, as shown in [2, Theorem 2(b)], we have $f_a \in \mathcal{B}_v$ and $\|f_a\|_v = 1$.

Now let G be a sampling set for $\mathcal{B}_v$. Therefore, there exists a constant $k > 1$ such that

$$\|f\|_v \leq k \sup\{v(z)|f'(z)| : z \in G\}, \quad \text{for all } f \in \mathcal{B}_v. \tag{2.6}$$

Applying (2.6) for f_a and using the sup property one can find $z_a \in G$ such that

$$\begin{aligned} 1 = \|f_a\|_v \leq & 2kv(z_a)|f_a'(z_a)| \\ = & 2kv(z_a)|g_a(z_a)| \left(\frac{1-|a|^2}{|1-\bar{a}z_a|^2} \right)^p \\ = & 2k \frac{v(z_a)|g_a(z_a)|}{(1-|z_a|^2)^p} \left(\frac{(1-|a|^2)(1-|z_a|^2)}{|1-\bar{a}z_a|^2} \right)^p \\ = & 2k\mu(z_a)|g_a(z_a)|(1-|\varphi_a(z_a)|^2)^p \\ \leq & 2k\|g_a\|_{H_\mu^\infty}(1-|\varphi_a(z_a)|^2)^p \\ \leq & 2k(1-|\varphi_a(z_a)|^2)^p. \end{aligned}$$

Therefore, $\rho(a, z_a) = |\varphi_a(z_a)| \leq r = \sqrt{1 - \left(\frac{1}{2k}\right)^{\frac{1}{p}}}$, and hence G is an r-net. □

Theorem 2.8 *Let v be an essential weight such that $\mu(z) = \frac{v(z)}{(1-|z|^2)^p}$ is a weight on $\mathbb{D}$ for some $0 < p < \infty$ and $\mu = \tilde{\mu}$. Let w be an arbitrary weight such that $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ is a bounded operator. If $I_g C_\varphi$ has closed range, then $G_{\varphi,g}^{v,w}(\varepsilon)$ is an r-net for some $\varepsilon > 0$ and $r \in (0,1)$.*

Proof Suppose that $I_g C_\varphi : \mathcal{B}_v \to \mathcal{B}_w$ has closed range. Then, by Theorem 2.3, there exists $\varepsilon > 0$ such that $G_{\varphi,g}^{v,w}(\varepsilon)$ is a sampling set for $\mathcal{B}_v$. Therefore, by Lemma 2.7, $G_{\varphi,g}^{v,w}(\varepsilon)$ is an r-net for some $r \in (0,1)$. □

Example 2.9 *The following weights satisfy the assumptions of Lemma 2.7 and Theorem 2.8. See [2, Example 3].*

(i) For all $0 < \alpha < \infty$, the standard weight $v_\alpha(z) = (1-|z|^2)^\alpha$.

(ii) For all $0 < \alpha, \beta < \infty$, the weight v given by $v(z) = (1-|z|^2)^\alpha(1-\log(1-|z|^2))^{-\beta}$.

Recalling the argument before Theorem 2.4, we have the next result for the operator $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$.

Theorem 2.10 *For the weights v and w, let $C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ be a bounded operator.*

(i) Let v be a weight satisfying condition $(L1)$, which is continuously differentiable with respect to $|z|$. If there exist $\varepsilon > 0$ and $r \in (0, M_v^{-1})$ such that $G_{\varphi,g\circ\varphi}^{v,w}(\varepsilon)$ is an r-net, then $C_\varphi I_g$ has closed range.

(ii) Let v be an essential weight such that $\mu(z) = \frac{v(z)}{(1-|z|^2)^p}$ is a weight on $\mathbb{D}$ for some $0 < p < \infty$ and $\mu = \tilde{\mu}$. If $C_\varphi I_g$ has closed range and $\varphi(0) = 0$, then $G_{\varphi,g\circ\varphi}^{v,w}(\varepsilon)$ is an r-net for some $\varepsilon > 0$ and $r \in (0,1)$.

We conjecture that the results of Theorem 2.4(ii) and Theorem 2.10(ii) are valid for the operator $C_\varphi I_g$ without the assumption $\varphi(0) = 0$.

3. Essential norms of Li–Stević integral type operators

Recall that for the Banach spaces X and Y, the norm of a bounded operator $T : X \to Y$ is denoted by $\|T\|_{X\to Y}$. Also, the essential norm of a bounded operator $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined as the distance from T to $\mathcal{K}(X,Y)$, the space of compact operators from X into Y. Clearly, a bounded operator $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. Therefore, essential norm estimates of bounded operators lead to necessary and/or sufficient conditions for the compactness of such operators. Essential norm estimates of different types of operators have been intensively studied between certain classes of Banach function spaces. See, for example, [7, 18–20, 22, 23] and the references therein.

In this section we give the essential norm of Li–Stević integral-type operators I_gC_φ and $C_\varphi I_g$ between Bloch-type spaces. Indeed, more generally, we give the essential norm of a bounded operator T between general classes of Banach spaces. Consequently, we get the essential norm of the operators I_gC_φ and $C_\varphi I_g$ between different Banach spaces of analytic functions including Bloch-type spaces. We next introduce the form of general Banach spaces of analytic functions that we consider.

Let X be a Banach space of analytic functions on the open unit disk $\mathbb{D}$ containing the constant functions, with norm satisfying

$$\|f\|_X = |f(0)| + \|f\|_{sX}, \quad f \in X, \tag{3.1}$$

where $\|\cdot\|_{sX}$ is a seminorm on X. Moreover, assume that for all $f \in X$ and for any constant (function) c,

$$\|f + c\|_{sX} = \|f\|_{sX}. \tag{3.2}$$

See [24] for the above mentioned definitions and note that Bloch-type spaces $\mathcal{B}_v$ are examples of Banach spaces X with norm satisfying conditions (3.1) and (3.2). At the end of this section we will give more examples of such Banach spaces X. For a Banach space X, satisfying conditions (3.1) and (3.2), the *derivative space* is defined by $Y = \{f' : f \in X\}$ equipped with the norm

$$\|f'\|_Y := \|f\|_{sX}, \quad f' \in Y. \tag{3.3}$$

Indeed, Y is a Banach space since the Banach space X satisfies conditions (3.1) and (3.2) and $\|f'\|_Y = \|f - f(0)\|_X$ for all $f' \in Y$. The spaces (X,Y) are called *derivative pair spaces* or briefly D*-pair spaces*.

Note that $(\mathcal{B}_v, H_v^\infty)$ is a D-pair Banach space for any weight v. At the end of this section we will introduce several other well-known D-pair Banach spaces. Next we give our essential norm results for general D-pair Banach spaces. We use the notation $\widetilde{X}$ to denote the closed subspace of X containing those f with $f(0) = 0$.

Note that for general Banach spaces X, Y, and Z, if $U : X \to Y$ is an invertible bounded operator and $T : Y \to Z$ is a bounded operator, then

$$\frac{1}{\|U\|}\|TU\|_{e,X\to Z} \le \|T\|_{e,Y\to Z} \le \|U^{-1}\|\|TU\|_{e,X\to Z}.$$

In particular, if $U : X \to Y$ is a surjective isometry, then

$$\|T\|_{e,Y\to Z} = \|TU\|_{e,X\to Z}. \tag{3.4}$$

Theorem 3.1 *Let X and Y be Banach spaces of analytic functions on $\mathbb{D}$ containing the constant functions. Let X satisfy (3.1) and (3.2). If $T : X \to Y$ is a bounded operator, then*

$$\|T\|_{e,X\to Y} = \|T\|_{e,\widetilde{X}\to Y}.$$

Proof We also denote the operator $T|_{\widetilde{X}} : \widetilde{X} \to Y$ by $T : \widetilde{X} \to Y$. For every compact operator $K : X \to Y$, the operator $K|_{\widetilde{X}} : \widetilde{X} \to Y$ is also compact and therefore

$$\begin{aligned}\|T\|_{e,\widetilde{X}\to Y} &\leq \|T - K|_{\widetilde{X}}\|_{\widetilde{X}\to Y} \\ &= \sup_{f\in\widetilde{X},\|f\|_X\leq 1} \|Tf - Kf\|_Y \\ &\leq \sup_{f\in X,\|f\|_X\leq 1} \|Tf - Kf\|_Y \\ &= \|T - K\|_{X\to Y},\end{aligned}$$

which implies that

$$\|T\|_{e,\widetilde{X}\to Y} \leq \|T\|_{e,X\to Y}. \tag{3.5}$$

In order to prove the converse of (3.5), for an arbitrary compact operator $\widetilde{K} : \widetilde{X} \to Y$, define $K : X \to Y$, the extension of $\widetilde{K}$ to X, by

$$K(f) = \widetilde{K}(f - f(0)1) + f(0)1,$$

for all $f \in X$. Note that the extension $K : X \to Y$ is indeed a compact operator, since X and Y contain the constant functions and X satisfies (3.1) and (3.2). Also, define compact operators $T_0, K_0 : X \to Y$ by

$$T_0(f) = T(f(0)1) = f(0)T1,$$

$$K_0(f) = K(f(0)1) = f(0)K1,$$

for all $f \in X$. Then, applying a similar argument as in the proof of [22, Theorem 1], one can see that

$$\|T\|_{e,X\to Y} \leq \|T - \widetilde{K}\|_{e,\widetilde{X}\to Y},$$

and since $\widetilde{K} : \widetilde{X} \to Y$ was an arbitrary compact operator, we get the desired result:

$$\|T\|_{e,X\to Y} \leq \|T\|_{e,\widetilde{X}\to Y}.$$

□

Theorem 3.2 *Let X, Y, and Z be Banach spaces of analytic functions on $\mathbb{D}$ containing the constant functions. Let X satisfy (3.1) and (3.2) and (Y, Z) be D-pair Banach spaces. If $T : X \to Y$ is a bounded operator and $D : Y \to Z$ is the derivation operator, then*

$$\|T\|_{e,X\to Y} = \|DT\|_{e,X\to Z}.$$

Proof By Theorem 3.1, it is enough to show that $\|T\|_{e,\widetilde{X}\to Y} = \|DT\|_{e,\widetilde{X}\to Z}$. Let the restriction operator $T|_{\widetilde{X}} : \widetilde{X} \to Y$ be also denoted by $T : \widetilde{X} \to Y$. Then,

$$\begin{aligned}\|T\|_{e,\widetilde{X}\to Y} &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \|T-K\|_{\widetilde{X}\to Y} \\ &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-K)f\|_Y \\ &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-K)f\|_{sY}. \end{aligned} \tag{3.6}$$

To see the last equality, define $T_0 f = (Tf)(0)1$ for all $f \in \widetilde{X}$. Also, for each $K \in \mathcal{K}(\widetilde{X}, Y)$, define $K_0 f = (Kf)(0)1$ for all $f \in \widetilde{X}$. Then $T_0, K_0 \in \mathcal{K}(\widetilde{X}, Y)$, since X and Y contain the constant functions and satisfy (3.1) and (3.2). Hence, by applying (3.2) we have

$$\begin{aligned}\|T\|_{e,\widetilde{X}\to Y} &\le \|T-(K-K_0+T_0)\|_{\widetilde{X}\to Y} \\ &\le \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-T_0)f-(K-K_0)f\|_Y \\ &= \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-T_0)f-(K-K_0)f\|_{sY} \\ &= \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-K)f\|_{sY}.\end{aligned}$$

This, along with the fact that $\|\cdot\|_{sY} \le \|\cdot\|_Y$, implies (3.6). Therefore, by (3.3) we have

$$\begin{aligned}\|T\|_{e,\widetilde{X}\to Y} &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \sup_{\|f\|_{\widetilde{X}}\le 1} \|(T-K)f\|_{sY} \\ &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \sup_{\|f\|_{\widetilde{X}}\le 1} \|D(T-K)f\|_Z \\ &= \inf_{K\in\mathcal{K}(\widetilde{X},Y)} \|DT-DK\|_{\widetilde{X}\to Z}.\end{aligned} \tag{3.7}$$

On the other hand, by considering the (bounded) integration operator $S : Z \to Y$, one can see that for each compact operator $\Lambda \in \mathcal{K}(\widetilde{X}, Z)$ we have $\Lambda = DS\Lambda$, and therefore Λ is of the form DK for some $K \in \mathcal{K}(\widetilde{X}, Y)$. Hence, (3.7) implies

$$\|T\|_{e,\widetilde{X}\to Y} = \inf_{\Lambda\in\mathcal{K}(\widetilde{X},Z)} \|DT-\Lambda\|_{\widetilde{X}\to Z} = \|DT\|_{e,\widetilde{X}\to Z}$$

and completes the proof. □

As an immediate consequence of Theorem 3.2 we get the following result:

Theorem 3.3 *For $i = 1, 2$ let (X_i, Y_i) be D-pair Banach spaces of analytic functions on $\mathbb{D}$ containing the constant functions and $T : X_1 \to X_2$ be a bounded operator. If $D_2 : X_2 \to Y_2$ and $S_1 : Y_1 \to X_1$ respectively denote the derivation operator and the integration operator, then*

$$\|T\|_{e,X_1\to X_2} = \|D_2TS_1\|_{e,Y_1\to Y_2}.$$

Proof Note that the integration operator $S_1 : Y_1 \to X_1$, given by $(S_1 f)(z) = \int_0^z f(\zeta)d\zeta$, is indeed an isometry between Y_1 and $\widetilde{X}_1$. Therefore, by (3.4), we have

$$\|D_2 T\|_{e,\widetilde{X}_1 \to Y_2} = \|D_2 T S_1\|_{e,Y_1 \to Y_2}.$$

On the other hand, by Theorem 3.1 and Theorem 3.2, we have

$$\|D_2 T\|_{e,\widetilde{X}_1 \to Y_2} = \|D_2 T\|_{e,X_1 \to Y_2} = \|T\|_{e,X_1 \to X_2},$$

which implies the desired result. □

As an immediate consequence of Theorem 3.3, we get the following result for the essential norm of Li–Stević integral-type operators $I_g C_\varphi, C_\varphi I_g : X_1 \to X_2$ in terms of the essential norm of weighted composition operators $D_2 I_g C_\varphi S_1 = (\varphi' \cdot g) C_\varphi : Y_1 \to Y_2$ and $D_2 C_\varphi I_g S_1 = (\varphi' \cdot g \circ \varphi) C_\varphi : Y_1 \to Y_2$.

Corollary 3.4 *For $i = 1, 2$, let (X_i, Y_i) be D-pair Banach spaces of analytic functions on $\mathbb{D}$ containing the constant functions.*

(i) If $I_g C_\varphi : X_1 \to X_2$ is a bounded operator, then

$$\|I_g C_\varphi\|_{e,X_1 \to X_2} = \|(\varphi' \cdot g) C_\varphi\|_{e,Y_1 \to Y_2}.$$

(ii) If $C_\varphi I_g : X_1 \to X_2$ is a bounded operator, then

$$\|C_\varphi I_g\|_{e,X_1 \to X_2} = \|(\varphi' \cdot g \circ \varphi) C_\varphi\|_{e,Y_1 \to Y_2}.$$

Remark 3.5 *It is worth mentioning that unlike Theorem 2.4(ii) and Theorem 2.10(ii), for the essential norm of the operator $C_\varphi I_g$, in Corollary 3.4(ii), there is no need to assume that $\varphi(0) = 0$.*

Remark 3.6 *Recall that, for the real scalars A and B, the notation $A \asymp B$ means that A and B are "almost" equal; that is, $cB \le A \le CB$ for some positive constants c and C. In many papers, the results concerning essential norms are given in terms of "almost" equalities, not "exact" equalities. See, for example, [18, 19, 22, 23]. It is worth mentioning here that the results of Theorems 3.1, 3.2, and 3.3 and Corollary 3.4 are given in terms of "exact" equalities, not "almost" equalities.*

We next apply Corollary 3.4 to several well-known D-pair Banach spaces and consequently we get the essential norm of the operators $I_g C_\varphi$ and $C_\varphi I_g$ between such spaces.

As mentioned before, $(\mathcal{B}_v, H_v^\infty)$ is a D-pair Banach space for each weight v. Therefore, using [20, Theorem 2.1] and applying Corollary 3.4 to the operators $I_g C_\varphi, C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ we get the following result for the essential norm of such operators.

Theorem 3.7 *For the weights v and w, let $I_g C_\varphi, C_\varphi I_g : \mathcal{B}_v \to \mathcal{B}_w$ be bounded operators. Then:*

(i)

$$\|I_g C_\varphi\|_{e,\mathcal{B}_v \to \mathcal{B}_w} = \lim_{r \to 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)|\varphi'(z)||g(z)|}{\tilde{v}(\varphi(z))}.$$

(*ii*)

$$\|C_\varphi I_g\|_{e,\mathcal{B}_v\to\mathcal{B}_w} = \lim_{r\to 1^-}\sup_{|\varphi(z)|>r}\frac{w(z)|\varphi'(z)||g(\varphi(z))|}{\tilde{v}(\varphi(z))}.$$

We next introduce some other well-known D-pair Banach spaces (X, Y) for which Theorem 3.3 can be applied to give the essential norm of $I_gC_\varphi, C_\varphi I_g : X \to Y$.

For each $0 < \alpha < \infty$, the *Zygmund-type space* $\mathcal{Z}_{v_\alpha}$ consists of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} v_\alpha(z)|f''(z)| < \infty.$$

The Zygmund-type space $\mathcal{Z}_{v_\alpha}$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_{v_\alpha}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} v_\alpha(z)|f''(z)|, \quad f \in \mathcal{Z}_{v_\alpha}.$$

Note that the Zygmund-type space $\mathcal{Z}_{v_\alpha}$ consists of those analytic functions on $\mathbb{D}$ whose derivatives belong to the Bloch-type space $\mathcal{B}_{v_\alpha}$. Indeed, $(\mathcal{Z}_{v_\alpha}, \mathcal{B}_{v_\alpha})$ is a D-pair Banach space. Boundedness, compactness, and essential norm estimates of different types of operators between Zygmund-type spaces have been studied by many authors. See, for example, [10, 11, 13–15, 22, 23, 25] and the references therein.

Essential norm estimates of weighted composition operators between Bloch-type spaces are given in [18, Theorems 3 and 4] and in the proof of [19, Theorem 8]. Therefore, by applying Corollary 3.4 we get the following estimates for the essential norm of the operators $I_gC_\varphi, C_\varphi I_g : \mathcal{Z}_{v_\alpha} \to \mathcal{Z}_{v_\beta}$ in different cases of $\alpha, \beta > 0$. In order to simplify the notations in the statement of the next theorem, we use the following simplifications (see [18, 22]):

$$A(g,\varphi,\alpha,\beta) = \limsup_{|\varphi(z)|\to 1}\frac{v_\beta(z)}{v_\alpha(\varphi(z))}|g(z)|,$$

$$B(g,\varphi,\beta) = \limsup_{|\varphi(z)|\to 1} v_\beta(z)|g(z)|\log\frac{1}{1-|\varphi(z)|^2}.$$

Theorem 3.8 *Let $\alpha, \beta > 0$ and $I_gC_\varphi, C_\varphi I_g : \mathcal{Z}_{v_\alpha} \to \mathcal{Z}_{v_\beta}$ be bounded operators.*

(*i*) *If $0 < \alpha < 1$, then*

$$\|I_gC_\varphi\|_{e,\mathcal{Z}_{v_\alpha}\to\mathcal{Z}_{v_\beta}} \asymp A(\varphi'^2\cdot g,\varphi,\alpha,\beta),$$

$$\|C_\varphi I_g\|_{e,\mathcal{Z}_{v_\alpha}\to\mathcal{Z}_{v_\beta}} \asymp A(\varphi'^2\cdot g\circ\varphi,\varphi,\alpha,\beta).$$

(*ii*)

$$\|I_gC_\varphi\|_{e,\mathcal{Z}\to\mathcal{Z}_{v_\beta}} \asymp \max\left\{A(\varphi'^2\cdot g,\varphi,1,\beta), B\left((\varphi'\cdot g)',\varphi,\beta\right)\right\},$$

$$\|C_\varphi I_g\|_{e,\mathcal{Z}\to\mathcal{Z}_{v_\beta}} \asymp \max\left\{A(\varphi'^2\cdot g\circ\varphi,\varphi,1,\beta), B\left((\varphi'\cdot g\circ\varphi)',\varphi,\beta\right)\right\}.$$

(*iii*) *If $1 < \alpha < \infty$, then*

$$\|I_gC_\varphi\|_{e,\mathcal{Z}_{v_\alpha}\to\mathcal{Z}_{v_\beta}} \asymp \max\left\{A(\varphi'^2\cdot g,\varphi,\alpha,\beta), A\left((\varphi'\cdot g)',\varphi,\alpha-1,\beta\right)\right\},$$

$$\|C_\varphi I_g\|_{e,\mathcal{Z}_{v_\alpha}\to\mathcal{Z}_{v_\beta}} \asymp \max\left\{A(\varphi'^2\cdot g\circ\varphi,\varphi,\alpha,\beta), A\left((\varphi'\cdot g\circ\varphi)',\varphi,\alpha-1,\beta\right)\right\}.$$

Let A denote the area measure on $\mathbb{D}$ normalized by the condition $A(\mathbb{D}) = 1$. For $1 < p < \infty$ and $-1 < \alpha < \infty$, the *Besov-type space* $B_{p,\alpha}$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty.$$

The Besov-type space $B_{p,\alpha}$ is a Banach space with the norm

$$\|f\|_{B_{p,\alpha}} = |f(0)| + \|f\|_{p,\alpha}, \quad f \in B_{p,\alpha}.$$

For $1 \leq p < \infty$ and $-1 < \alpha < \infty$, the standard *weighted Bergman space* is defined as

$$A_\alpha^p = \left\{ f \in H(\mathbb{D}) : \ \|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty \right\}.$$

Note that $B_{p,\alpha}$ consists of those analytic functions on $\mathbb{D}$ whose derivatives belong to the weighted Bergman space A_α^p. Indeed, for $1 < p < \infty$ and $-1 < \alpha < \infty$, $(B_{p,\alpha}, A_\alpha^p)$ is a D-pair Banach space. By [5, Theorem 3], for each $1 < q < p < \infty$ and $-1 < \alpha, \beta < \infty$, every bounded weighted composition operator between A_α^p and A_β^q is compact. Therefore, by Corollary 3.4, when $1 < q < p < \infty$ and $-1 < \alpha, \beta < \infty$, bounded operators $I_g C_\varphi, C_\varphi I_g : B_{p,\alpha} \to B_{q,\beta}$ are compact; that is, $\|I_g C_\varphi\|_{e, B_{p,\alpha} \to B_{q,\beta}} = \|C_\varphi I_g\|_{e, B_{p,\alpha} \to B_{q,\beta}} = 0$. In order to give the essential norms of $I_g C_\varphi, C_\varphi I_g : B_{p,\alpha} \to B_{q,\beta}$ in the case of $1 < p \leq q < \infty$, we use the following simplification (see [5]):

$$I_{\varphi,\alpha,\beta}(u)(a) = \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \bar{a}\varphi(z)|^2} \right)^{(2+\alpha)q/p} |u(z)|^q (1 - |z|^2)^\beta dA(z).$$

Finally, by applying [5, Theorem 2] and Corollary 3.4, we get the following result:

Theorem 3.9 *Let $1 < p \leq q < \infty$, $-1 < \alpha, \beta < \infty$, and $I_g C_\varphi, C_\varphi I_g : B_{p,\alpha} \to B_{q,\beta}$ be bounded operators. Then,*

$$\|I_g C_\varphi\|_{e, B_{p,\alpha} \to B_{q,\beta}} \asymp \left(\limsup_{|a| \to 1} I_{\varphi,\alpha,\beta}(\varphi' \cdot g)(a) \right)^{1/q},$$

$$\|C_\varphi I_g\|_{e, B_{p,\alpha} \to B_{q,\beta}} \asymp \left(\limsup_{|a| \to 1} I_{\varphi,\alpha,\beta}(\varphi' \cdot g \circ \varphi)(a) \right)^{1/q}.$$

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References

[1] Basallote M, Contreras MD, Hernandez-Mancera C, Martin MJ, Paul PJ. Volterra operators and semigroups in weighted Banach spaces of analytic functions. Collect Math 2014; 65: 233-249.

[2] Bonet J, Lindström M, Wolf E. Isometric weighted composition operators on weighted Banach spaces of type H^∞. P Am Math Soc 2008; 136: 4267-4273.

[3] Chen H, Gauthier P. Boundedness from below of composition operators on α-Bloch spaces. Canad Math Bull 2008; 51: 195-204.

[4] Contreras MD, Hernandez-Diaz AG. Weighted composition operators in weighted Banach spaces of analytic functions. J Austral Math Soc Ser A 2000; 69: 41-60.

[5] Cuckovic Z, Zhao R. Weighted composition operators between different weighted Bergman spaces and different Hardy spaces. Illinois J Math 2007; 51: 479-498.

[6] Domanski P, Lindström M. Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions. Ann Polon Math 2002; 89: 233-264.

[7] Galindo P, Lindström M, Stević S. Essential norm of operators into weighted-type spaces on the unit ball. Abstr Appl Anal 2011; 2011: 939873.

[8] Ghatage P, Yan J, Zheng D. Composition operators with closed range on the Bloch space. P Am Math Soc 2001; 129: 2039-2044.

[9] Ghatage P, Zheng D, Zorboska N. Sampling sets and closed range composition operators on the Bloch space. P Am Math Soc 2005; 133: 1371-1377.

[10] Li S, Stević S. Volterra-type operators on Zygmund spaces. J Inequal Appl 2007; 2007: 32124.

[11] Li S, Stević S. Weighted composition operators from Zygmund spaces into Bloch spaces. Appl Math Comput 2008; 206: 825-831.

[12] Li S, Stević S. Products of composition and integral type operators from H^∞ to the Bloch space. Complex Var Elliptic Equ 2008; 53: 463-474.

[13] Li S, Stević S. Generalized composition operators on Zygmund spaces and Bloch type spaces. J Math Anal Appl 2008; 338: 1282-1295.

[14] Li S, Stević S. Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces. J Math Anal Appl 2008; 345: 40-52.

[15] Li S, Stević S. Integral-type operators from Bloch-type spaces to Zygmund-type spaces. Appl Math Comput 2009; 215: 464-473.

[16] Li S, Stević S. Products of integral-type operators and composition operators between Bloch-type spaces. J Math Anal Appl 2009; 349: 596-610.

[17] Liu Y, Yu Y. On a Li-Stević integral-type operator from Bloch-type spaces into logarithmic Bloch spaces. Integral Transform Spec Funct 2010; 21: 93-103.

[18] MacCluer B, Zhao R. Essential norms of weighted composition operators between Bloch-type spaces. Rocky Mountain J Math 2003; 33: 1437-1458.

[19] Manhas JS, Zhao R. New estimates of essential norms of weighted composition operators between Bloch type spaces. J Math Anal Appl 2012; 389: 32-47.

[20] Montes-Rodriguez A. Weighted composition operators on weighted Banach spaces of analytic functions. J Lond Math Soc 2000; 61: 872-884.

[21] Palmberg N. Weighted composition operators with closed range. Bull Austral Math Soc 2007; 75: 331-354.

[22] Sanatpour AH. Estimates of essential norm of the Li-Stević integral type operator between Zygmund type spaces. Math Inequal Appl 2016; 19: 685-696.

[23] Sanatpour AH, Hassanlou M. Essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces. Turk J Math 2014; 38: 872-882.

[24] Shabazz M, Tjani M. Isometries among composition operators on Besov type spaces. Complex Var Elliptic Equ 2016; 61: 761-771.

[25] Stević S. On an integral operator from the Zygmund space to the Bloch-type space on the unit ball. Glasg J Math 2009; 51: 275-287.

[26] Stević S. Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces. Siberian Math J 2009; 50: 726-736.

[27] Stević S. Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball. Siberian Math J 2009; 50: 1098-1105.

[28] Stević S. On an integral operator between Bloch-type spaces on the unit ball. Bull Sci Math 2010; 134: 329-339.

[29] Stević S, Ueki SI. Integral-type operators acting between weighted-type spaces on the unit ball. Appl Math Comput 2009; 215: 2464-2471.

[30] Ueki SI. On the Li-Stević integral type operators from weighted Bergman spaces into β-Zygmund spaces. Integral Equations Operator Theory 2012; 74: 137-150.

[31] Zorboska N. Isometric and closed-range composition operators between Bloch-type spaces. Int J Math Math Sci 2011; 2011: 132541.

[32] Zorboska N. Univalently induced, closed range, composition operators on the Bloch-type spaces. Canad Math Bull 2012; 55: 441-448.