Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 3117 – 3122
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doi:10.3906/mat-1610-85

Conditional expectation type operators and modular inequalities

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Received: 21.10.2016 • **Accepted/Published Online:** 14.10.2018 • **Final Version:** 27.11.2018

Abstract: In this paper we discuss the connection between conditional expectation type operators and integral operators. A variant of Schur's lemma is established and we obtain modular inequalities for a class of conditional expectation type operators.

Key words: Conditional expectation, modular inequalities, norm inequalities

1. Introduction

Let $(\Omega, \mathcal{S}, \mathbf{P})$ be a probability space and let X be a real-valued random variable on Ω. The expectation EX of X is defined as $\int_\Omega X d\mathbf{P}$ if the integral exists. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{S}$. The conditional expectation of X given $\mathcal{A}$ is defined as a random variable $E(X|\mathcal{A})$, measurable for $\mathcal{A}$, such that for all $A \in \mathcal{A}$, $\int_A E(X|\mathcal{A}) d\mathbf{P} = \int_A X d\mathbf{P}$, if such a $E(X|\mathcal{A})$ exists. For any $X \in L^1(\Omega, \mathcal{S}, \mathbf{P})$ and any sub-σ-algebra $\mathcal{A}$ of $\mathcal{S}$, a conditional expectation $E(X|\mathcal{A})$ exists, and if Y and Z are conditional expectations of X given $\mathcal{A}$, then $Y = Z$ almost everywhere (see [5, 10.1.1 Theorem]). The operator $E(\cdot|\mathcal{A}) : L^1(\Omega, \mathcal{S}, \mathbf{P}) \to L^1(\Omega, \mathcal{A}, \mathbf{P})$ is called the conditional expectation operator induced by $\mathcal{A}$. If X is also $\mathcal{A}$-measurable, then $E(X|\mathcal{A}) = X$ and hence $E(\cdot|\mathcal{A})$ is a projection from $L^1(\Omega, \mathcal{S}, \mathbf{P})$ onto $L^1(\Omega, \mathcal{A}, \mathbf{P})$. It is known that $E(\cdot|\mathcal{A})$ is a bounded linear operator and for each $1 \le p \le \infty$, if $X \in L^p(\Omega, \mathcal{S}, \mathbf{P})$, then $E(X|\mathcal{A}) \in L^p(\Omega, \mathcal{A}, \mathbf{P})$ and $\|E(X|\mathcal{A})\|_p \le \|X\|_p$. For more important properties and detailed discussion, we refer the readers to [1,3–5,16].

Recently, Estaremi and Jabbarzadeh established the boundedness and compactness properties for weighted conditional expectation type operators. Let $(\Omega, \mathcal{S}, \mu)$ be a complete σ-finite measure space and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{S}$. Let $L^0(\Omega, \mathcal{S}, \mu)$ be the vector space of all equivalence classes of almost everywhere finite-valued measurable functions on Ω and $\mathcal{D} = \{f \in L^0(\Omega, \mathcal{S}, \mu) : E(|f||\mathcal{A}) \in L^0(\Omega, \mathcal{A}, \mu)\}$. Take $u, w \in \mathcal{D}$ and define $T = M_w E(\cdot|\mathcal{A}) M_u : L^p(\Omega, \mathcal{S}, \mu) \to L^0(\Omega, \mathcal{S}, \mu)$, where M_u and M_w are multiplication operators. Consider the boundedness of $T : L^p(\Omega, \mathcal{S}, \mu) \to L^q(\Omega, \mathcal{S}, \mu)$. In [9, Theorem 2.1], it was proved that for $1 < p = q < \infty$, T is bounded if and only if

$$\{E(|w|^p|\mathcal{A})\}^{1/p}\{E(|u|^{p^*}|\mathcal{A})\}^{1/p^*} \in L^\infty(\Omega, \mathcal{A}, \mu),$$

where $1/p + 1/p^* = 1$. In the case $p = q = 1$, the condition is

$$uE(|w||\mathcal{A}) \in L^\infty(\Omega, \mathcal{S}, \mu).$$

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2010 *AMS Mathematics Subject Classification:* Primary 47B38; Secondary 26D15
This research was supported by the Ministry of Science and Technology, R.O.C., under Grant MOST 104-2115-M-214-004.

The authors showed in [9, Theorem 2.2] that for $1 < q < p < \infty$, T is bounded if and only if

$$\{E(|w|^q|\mathcal{A})\}^{1/q}\{E(|u|^{p^*}|\mathcal{A})\}^{1/p^*} \in L^r(\Omega, \mathcal{A}, \mu),$$

where $1/r = 1/p^* - 1/q^*$. In the case $1 < p < q < \infty$, the conditions for the boundedness of T were obtained in [9, Theorem 2.3]. The cases $q = 1$ or $p = 1$ were also considered in [9, Theorem 2.4]. Similar results for composition Lambert type operators can be found in [10]. Necessary and sufficient conditions for Lambert type operators and conditional type operators to be compact on L^p spaces can be found in [8,9,11]. The boundedness and compactness properties of operators of the form $E(\cdot|\mathcal{A})M_u$ on Orlicz spaces were established in [7].

The theory of boundedness for integral operators on weighted L^p spaces has been developed well. Let (E, μ) and (T, λ) be two σ-finite measure spaces and let k be a nonnegative measurable function defined on $(E \times T, \mu \times \lambda)$. We define the integral operator T_k for nonnegative measurable functions g on (T, λ) by

$$T_k g(x) = \int_T k(x,t)g(t)d\lambda(t), \quad x \in E. \tag{1.1}$$

Weighted inequalities of the form

$$\left\{\int_E T_k g(x)^q u(x)d\mu(x)\right\}^{1/q} \le C\left\{\int_T g(t)^p v(t)d\lambda(t)\right\}^{1/p}, \tag{1.2}$$

where $1 \le p, q \le \infty$, play an important role in analysis and have been investigated by many authors. Here u and v are nonnegative locally integrable weight functions on (E, μ), (T, λ), respectively. A natural generalization of (1.2) is modular inequalities of the form

$$\left\{\int_E \phi^q(T_k g(x))u(x)d\mu(x)\right\}^{1/q} \le C_1\left\{\int_T \phi^p(C_2 g(t))v(t)d\lambda(t)\right\}^{1/p} \tag{1.3}$$

for all nonnegative functions g defined on (T, λ), where $0 < p, q < \infty$ and $\phi : [0, \infty) \to [0, \infty)$. See [6] and the references given there.

It is known that conditional expectation type operators are closely related to a class of integral operators (see [13, Example 2.4 & Example 4.2] and [5, §10.1 Problem 9]). Since Schur's lemma is a useful way to investigate the boundedness of operators by choosing a suitable function that satisfies certain inequalities (see [2,12,15,17,18]), we are interested in applying this method to obtain modular inequalities of the form (1.3) for conditional expectation type operators. In this paper we discuss the connection between conditional expectation type operators and integral operators. Then a variant of Schur's lemma is established and we obtain sufficient conditions for inequalities of the form (1.3) to hold, where T_k is replaced by a class of conditional expectation type operators and $\phi^{1/s}$ is quasiconvex for some $s \ge \max\{1/p, 1/q\}$.

Throughout this paper all functions are assumed to be measurable on their domains. For constants c_1 and c_2, we write $c_1 \lesssim c_2$ if there exists a constant $d > 0$ such that $c_1 \le dc_2$. We use the convention that $0^0 = \infty^0 = 1$ and $\infty/\infty = 0/0 = 0 \cdot \infty = 0$. For $1 \le p \le \infty$, p^* is defined by $1/p + 1/p^* = 1$.

2. Conditional expectation type operators and integral operators

In this section we show the connection between integral operators and conditional expectation type operators.

Definition 2.1 *([5, §10.1 Problem 5]) Let* $(\Omega, \mathcal{S}, \mathbf{P})$ *be a probability space and let* $\mathcal{A}$ *be a sub-*σ*-algebra of* $\mathcal{S}$*. A random variable* X *on* $(\Omega, \mathcal{S}, \mathbf{P})$ *is said to be independent of* $\mathcal{A}$ *if for every* $A \in \mathcal{A}$ *and measurable set* B *in the range of* X*,*

$$\mathbf{P}(X^{-1}(B) \cap A) = \mathbf{P}(X^{-1}(B))\mathbf{P}(A). \tag{2.1}$$

Note that if X is real-valued and independent of $\mathcal{A}$, with $E|X| < \infty$, then $E(X|\mathcal{A}) = EX$ almost everywhere.

Lemma 2.2 *([5, §10.1 Problem 9]) Let* $(\Omega, \mathcal{S}, \mathbf{P})$ *be a probability space,* $(E, \mathcal{M}_E)$ *and* $(T, \mathcal{M}_T)$ *two measurable spaces,* $\Phi : \Omega \to E$ *and* $\Psi : \Omega \to T$ *measurable functions, and* f *a real-valued measurable function on* $(E \times T, \mathcal{M}_E \bigotimes \mathcal{M}_T)$ *with* $\int_\Omega |f(\Phi, \Psi)| d\mathbf{P}(\omega) < \infty$*. Let* $\mathcal{A}$ *be a sub-*σ*-algebra of* $\mathcal{S}$*. Suppose that* Φ *is measurable for* $\mathcal{A}$ *and* Ψ *is independent of* $\mathcal{A}$*. Let* $\lambda = \mathbf{P} \circ \Psi^{-1}$ *on* $(T, \mathcal{M}_T)$*. Then we have*

$$E(f(\Phi, \Psi)|\mathcal{A}) = \int_T f(\Phi, y) d\lambda(y). \tag{2.2}$$

Choose $(E, \mathcal{M}_E) = (T, \mathcal{M}_T) = (\Omega, \mathcal{S})$ in Lemma 2.2. Suppose that k is a nonnegative measurable function on $(\Omega \times \Omega, \mathcal{S} \bigotimes \mathcal{S})$. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{S}$ and suppose that $\Phi : \Omega \to \Omega$ is measurable for $\mathcal{A}$ and $\Psi : \Omega \to \Omega$ is independent of $\mathcal{A}$. Let g be a nonnegative $\mathcal{S}$-measurable function on Ω with $\int_\Omega k(\Phi, \Psi) g(\Psi) d\mathbf{P}(\omega) < \infty$. Define $\lambda = \mathbf{P} \circ \Psi^{-1}$ on $(\Omega, \mathcal{S})$. By Lemma 2.2 with $f(x,t) = k(x,t)g(t)$, we have

$$E(k(\Phi, \Psi)g(\Psi)|\mathcal{A})(\omega) = \int_\Omega k(\Phi(\omega), y) g(y) d\lambda(y) \quad \text{for } \mathbf{P}\text{-a.e. } \omega \in \Omega. \tag{2.3}$$

Hence, $E(k(\Phi, \Psi)g(\Psi)|\mathcal{A})$ can be written as an integral operator for g.

In the following we show that integral operators of the form (1.1) can also be written as a conditional expectation type. Let $(E, \mathcal{M}_E, \mu)$ and $(T, \mathcal{M}_T, \lambda)$ be two σ-finite measure spaces. Let $\Omega = E \times T$ and let $\mathcal{S} = \mathcal{M}_E \bigotimes \mathcal{M}_T$ be the product σ-algebra. Suppose that k is a nonnegative measurable function on $(\Omega, \mathcal{S})$. For nonnegative measurable functions g on $(T, \mathcal{M}_T, \lambda)$, define $T_k g(x) = \int_T k(x,t) g(t) d\lambda(t)$ for $x \in E$. Suppose that there exist positive measurable functions u and v on E and T, respectively, such that $\int_E u d\mu = \int_T v d\lambda = 1$. Define $d\mathbf{P}_1 = u d\mu$, $d\mathbf{P}_2 = v d\lambda$, and $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$. Then $(\Omega, \mathcal{S}, \mathbf{P})$ is a probability space. Note that $k(x,t)v(t)^{-1}g(t)$ is $\mathcal{S}$-measurable on Ω and we write $T_k g(x) = \int_T k(x,t) v(t)^{-1} g(t) d\mathbf{P}_2(t)$ for $x \in E$. Define $\mathcal{A} = \{A \times T : A \in \mathcal{M}_E\}$. Then $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{S}$. Here we show that $T_k g$ can be written as $E(kv^{-1}g|\mathcal{A})$. Define $\Phi : \Omega \to E$ and $\Psi : \Omega \to T$ by $\Phi(x,t) = x$ and $\Psi(x,t) = t$, respectively. It is easy to show that Φ is measurable for $\mathcal{A}$. On the other hand, for $D \in \mathcal{A}$ and $D_2 \in \mathcal{M}_T$, $D = D_1 \times T$ for some $D_1 \in \mathcal{M}_E$ and $\Psi^{-1}(D_2) = E \times D_2$. Then

$$\mathbf{P}(D \cap \Psi^{-1}(D_2)) = \mathbf{P}(D_1 \times D_2) = \mathbf{P}_1(D_1)\mathbf{P}_2(D_2) = \mathbf{P}(D)\mathbf{P}(\Psi^{-1}(D_2)).$$

By Definition 2.1 we see that Ψ is independent of $\mathcal{A}$. Let $f(x,t) = k(x,t)v(t)^{-1}g(t)$ for $(x,t) \in \Omega$. By the definitions of Φ and Ψ, we have $f(\Phi, \Psi)(\omega) = f(\omega)$ for $\omega = (x,t) \in \Omega$. Since $\mathbf{P} \circ \Psi^{-1} = \mathbf{P}_2$, by Lemma 2.2 we see that if $\int_\Omega |f(\omega)| d\mathbf{P}(\omega) < \infty$ then for $(x,t) \in \Omega$,

$$E(kv^{-1}g|\mathcal{A})(x,t) = E(f(\Phi, \Psi)|\mathcal{A})(x,t) = \int_T f(\Phi(x,t), y) d\mathbf{P}_2(y) = T_k g(x).$$

This result is also given in [13, Example 4.2].

3. Modular inequalities for conditional expectation type operators

In this section we prove modular inequalities for conditional expectation type operators of the form (2.3). Let $(\Omega, \mathcal{S}, \mathbf{P})$ be a probability space and let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{S}$. Suppose that $\Phi : \Omega \to \Omega$ is measurable for $\mathcal{A}$ and $\Psi : \Omega \to \Omega$ is independent of $\mathcal{A}$. Suppose that k is a nonnegative measurable function on $(\Omega\times\Omega, \mathcal{S}\bigotimes\mathcal{S})$ such that $E(k(\Phi,\Psi)) < \infty$ and $E(k(\Phi,\Psi)|\mathcal{A})(\omega) = 1$ for $\mathbf{P}$-a.e. $\omega \in \Omega$. Let g be a nonnegative $\mathcal{S}$-measurable function on Ω with $E(k(\Phi,\Psi)g(\Psi)) < \infty$. Define $\lambda = \mathbf{P} \circ \Psi^{-1}$. We consider modular inequalities of the form

$$\left\{\int_\Omega \phi^q(E(k(\Phi,\Psi)g(\Psi)|\mathcal{A})(\omega))u(\omega)d\mathbf{P}(\omega)\right\}^{1/q} \leq C_1\left\{\int_\Omega \phi^p(C_2 g(y))v(y)d\lambda(y)\right\}^{1/p}, \tag{3.1}$$

where $0 < p, q < \infty$, $\phi : [0,\infty) \to [0,\infty)$, u is a nonnegative function on Ω, and v is a positive and finite function on Ω.

Since $E(k(\Phi,\Psi)) < \infty$, by (2.3) we have $E(k(\Phi,\Psi)|\mathcal{A})(\omega) = \int_\Omega k(\Phi(\omega), y)d\lambda(y)$. By Schur's lemma [12, Appendix I.1 Lemma] and the condition that $E(k(\Phi,\Psi)|\mathcal{A})(\omega) = 1$ for $\mathbf{P}$-a.e. $\omega \in \Omega$, we see that in the case $\phi(x) = x$, $1 < p = q < \infty$, and $u = v = 1$, if $\sup_{y\in\Omega}\int_\Omega k(\Phi(\omega), y)d\mathbf{P}(\omega) = B < \infty$, then (3.1) holds with $C_1 = B^{1/p}$ and $C_2 = 1$. Moreover, by [12, Appendix I.2 Lemma] we see that the same case of (3.1) holds for $0 < C_1 < \infty$ and $C_2 = 1$ if and only if for all $B > C_1$ there is a measurable function w on Ω that satisfies $0 < w < \infty$ λ-a.e., $E(k(\Phi,\Psi)w(\Psi)) < \infty$, $0 < E(k(\Phi,\Psi)w(\Psi)|\mathcal{A}) < \infty$ $\mathbf{P}$-a.e., and such that

$$\int_\Omega k(\Phi(\omega), y)[E(k(\Phi,\Psi)w(\Psi)|\mathcal{A})(\omega)]^{p/p^*}d\mathbf{P}(\omega) \leq B^p w(y)^{p/p^*}. \tag{3.2}$$

Here we apply a variant of Schur's lemma obtained in [15] to establish (3.1).

Definition 3.1 *([14, Definition 1.1.6])We say that $\phi : [0,\infty) \to [0,\infty)$ is quasiconvex if there exist a convex function ψ and a constant $\ell > 0$ such that $\psi(t) \leq \phi(t) \leq \ell\psi(\ell t)$ for $t \geq 0$. Define $s_\phi = \sup\{s > 0 | \phi^{1/s}$ is quasiconvex.$\}$. If $\phi^{1/s}$ is not quasiconvex for any $s > 0$, we define $s_\phi = 0$.*

The following theorem is an extension of [15, Corollary 2.3] and the method that is used to prove [15, Theorem 2.2] can also be applied to prove this theorem.

Theorem 3.2 *Let $0 < p, q < \infty$, $\phi : [0,\infty) \to [0,\infty)$, and $\max\{1/p, 1/q\} < s_\phi \leq \infty$. Let s be a finite constant in the range $\max\{1/p, 1/q\} < s \leq s_\phi$ such that $\phi^{1/s}$ is quasiconvex. Let $1 < \beta \leq \min\{sp, sq\}$. Suppose that there exist $0 \leq m \leq \beta^*$, $0 < D < \infty$, and a positive and finite function w on Ω such that $E(k(\Phi,\Psi)^m w(\Psi)v(\Psi)^{1-(sp)^*}) < \infty$ and*

$$\mathcal{H}^m_{s,\beta}w(y) \leq Dw(y)^{q(s-1/p)} \quad \textit{for } \lambda - a.e. \quad y \in \Omega, \tag{3.3}$$

where

$$\mathcal{H}^m_{s,\beta}w(y) = \int_\Omega k(\Phi(\omega), y)^{(1-m/\beta^*)sq}u(\omega)\bigg(E(k(\Phi,\Psi)^m w(\Psi)v(\Psi)^{1-(sp)^*}|\mathcal{A})(\omega)\bigg)^{sq/\beta^*}d\mathbf{P}(\omega). \tag{3.4}$$

Then (3.1) holds with

$$C_1 \lesssim D^{1/q}E(w(\Psi)v(\Psi)^{1-(sp)^*})^{(sp-\beta)/(\beta p)}. \tag{3.5}$$

Proof If $\phi^{1/s}$ is quasiconvex, then there exist a constant $\ell > 0$ and a convex function ψ such that $\psi(t) \leq \phi(t)^{1/s} \leq \ell\psi(\ell t)$ for $t \geq 0$. Since $E(k(\Phi,\Psi)|\mathcal{A})(\omega) = 1$ for $\mathbf{P}$-a.e. $\omega \in \Omega$, Jensen's inequality and (2.3) imply that

$$\begin{aligned}\phi^{1/s}(E(k(\Phi,\Psi)g(\Psi)|\mathcal{A})(\omega)) \leq & \ell\psi(E(k(\Phi,\Psi)(\ell g)(\Psi)|\mathcal{A})(\omega)) = \ell\psi\left(\int_\Omega k(\Phi(\omega),y)(\ell g)(y)d\lambda(y)\right) \\ \leq & \ell\int_\Omega k(\Phi(\omega),y)\psi(\ell g(y))d\lambda(y)\end{aligned}$$

for $\mathbf{P}$-a.e. $\omega \in \Omega$ and

$$\int_\Omega \phi^q(E(k(\Phi,\Psi)g(\Psi)|\mathcal{A})(\omega))u(\omega)d\mathbf{P}(\omega) \leq \ell^{sq}\int_\Omega\left(\int_\Omega k(\Phi(\omega),y)h(y)d\sigma(y)\right)^{sq}u(\omega)d\mathbf{P}(\omega), \tag{3.6}$$

where $h(y) = \psi(\ell g(y))v(y)^{(sp)^*-1}$ and $d\sigma(y) = v(y)^{1-(sp)^*}d\lambda(y)$. Hölder's inequality with indices β and β^* implies

$$\int_\Omega k(\Phi(\omega),y)h(y)d\sigma(y) \leq \left(\int_\Omega k(\Phi(\omega),y)^{(1-m/\beta^*)\beta}h(y)^\beta w(y)^{1-\beta}d\sigma(y)\right)^{1/\beta}\left(\int_\Omega k(\Phi(\omega),y)^m w(y)d\sigma(y)\right)^{1/\beta^*}.$$

By (3.6), Minkowski's integral inequality with index sq/β, and (3.3), we have

$$\begin{aligned}&\left\{\int_\Omega \phi^q(E(k(\Phi,\Psi)g(\Psi)|\mathcal{A})(\omega))u(\omega)d\mathbf{P}(\omega)\right\}^{1/q} \\ \leq & \ell^s\left\{\int_\Omega h(y)^\beta(\mathcal{H}^m_{s,\beta}w(y))^{\beta/(sq)}w(y)^{1-\beta}d\sigma(y)\right\}^{s/\beta} \\ \leq & \ell^s D^{1/q}\left\{\int_\Omega h(y)^\beta w(y)^{1-\beta/(sp)}d\sigma(y)\right\}^{s/\beta} \\ \leq & \ell^s D^{1/q}\left\{\int_\Omega w(y)v(y)^{1-(sp)^*}d\lambda(y)\right\}^{(sp-\beta)/(\beta p)}\left\{\int_\Omega h(y)^{sp}d\sigma(y)\right\}^{1/p}.\end{aligned}$$

The last inequality is based on Hölder's inequality with indices sp/β and $(sp/\beta)^*$. By the equality

$$\int_\Omega w(y)v(y)^{1-(sp)^*}d\lambda(y) = \int_\Omega w(\Psi(\omega))v(\Psi(\omega))^{1-(sp)^*}d\mathbf{P}(\omega) = E(w(\Psi)v(\Psi)^{1-(sp)^*})$$

and the inequality

$$\int_\Omega h(y)^{sp}d\sigma(y) \leq \int_\Omega \phi^p(\ell g(y))v(y)d\lambda(y),$$

we have (3.1) with (3.5). □

In the case $u = v = 1$ and $m = sq\beta^*/(sq+\beta^*)$, we have $m = (1-m/\beta^*)sq$ and conditions (3.3)–(3.4) can be reduced to

$$\int_\Omega k(\Phi(\omega),y)^m\left(E(k(\Phi,\Psi)^m w(\Psi)|\mathcal{A})(\omega)\right)^{sq/\beta^*}d\mathbf{P}(\omega) \leq Dw(y)^{q(s-1/p)} \tag{3.7}$$

for λ-a.e. $y \in \Omega$. In particular, in the case $s = 1$, $p = q$, and $\beta = p$, we have $m = 1$ and (3.7) is reduced to the form (3.2).

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