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The cohomological structure of fixed point set for pro-torus actions on compact spaces

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Abstract: In this paper, we study the relationships between the cohomological structure of a space and that of the fixed point set of a finite dimensional pro-torus action on the space.

Key words: Totally nonhomologous to zero, pro-torus, fixed point

1. Introduction

Most of the results in the cohomology theory of transformation groups (based on the Borel construction) concern Lie group actions and results about non-Lie group actions are still quite scarce. However, in the case of topological groups (without assuming smooth structure), other groups exist, e.g., p-adic integers or solenoid. These wild groups play an important role in, for example, Hilbert–Smith conjecture. The main difficulty in generalizing theorems for Lie group actions to topological group actions is how to handle these wild groups.

In this paper, we try to overcome this difficulty by using the fact that for a finite dimensional compact group G, there is a closed normal subgroup N such that the quotient group G/N is a compact Lie group and by applying the cohomological technique of Lie group actions to show the main theorem.

This paper is about finite dimensional compact groups. The concept of the dimension of a compact topological group plays an essential role in transformation group theory. The Lebesgue covering dimension or topological dimension of a compact Hausdorff space is defined as below.

A collection $\mathcal{A}$ of subsets of the space X is said to have order $n+1$ if some point of X lies in $n+1$ members of $\mathcal{A}$, and no point of X lies in more than $n+1$ members of $\mathcal{A}$.

Recall that given a collection $\mathcal{A}$ of subsets of X, a collection $\mathcal{B}$ is said to refine $\mathcal{A}$ (is a refirement of $\mathcal{A}$) if for each element B of $\mathcal{B}$ there is an element A of $\mathcal{A}$ such that $B \subset A$.

A compact Hausdorff space X has dimension $n \geq 0$ if for every finite covering of X by open sets there exists a finite covering by closed sets that refines the given covering and has order $n+1$. A space is said to be finite dimensional if there is some integer n such that the space has dimension n.

It is a well-known fact that a finite dimensional compact group G has a totally disconnected closed normal subgroup N of G such that the factor group G/N is a compact Lie group of the same dimension (see for details [21, Theorem 69]).

Moreover, if G is an n-dimensional compact group, then the group $G = \varprojlim_{N \in \mathcal{N}} G/N$ where $\mathcal{N}$ is a

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filter basis of closed normal zero dimensional (totally disconnnected) subgroups of G such that $\bigcap \mathcal{N} = \{1\}$ and G/N is an n-dimensional Lie group for each $N \in \mathcal{N}$. Besides, group G will be called n-dimensional pro-torus if G is the projective limit of n-dimensional tori, and the 1-dimensional pro-torus will be called solenoid.

Note that pro-tori are compact connected abelian topological groups. Conversely, any compact connected abelian topological group is the projective limit of tori [11,12].

In this paper, by a pro-torus we will always mean a finite dimensional pro-torus.

Unless expressly stated otherwise, k will be a field of characteristic zero, and all spaces will be assumed to be Hausdorff. The proofs of our theorems need a cohomology theory having the continuity property (see Spanier [22]). The Alexander–Spanier cohomology, the Čech cohomology, and the sheaf cohomology are three such theories. Throughout the present paper, the Alexander–Spanier cohomology is used. Indeed, the Alexander–Spanier cohomology and the Čech cohomology are naturally isomorphic for arbitrary topological spaces [9]. Moreover, Godement [10, Theorem 5.10.1] proved that the Čech cohomology coincides with the sheaf cohomology on paracompact spaces (also see [7, Chapter III]). Therefore, all of our results are also valid for these cohomology theories.

2. Preliminaries

In this section, we shall state a few definitions and facts relevant to the study of compact transformation groups. We refer the reader to [1,4,6] for more details.

Let G be a topological group and X be a G-space and N be a closed normal subgroup of G. Then there exists a canonically induced action of the quotient group G/N on the orbit space X/N and a natural homeomorphism from X/G to $(X/N)/(G/N)$ [12, Proposition 10.31].

The next lemma was proved by Ku [15, Lemma 4.2 (ii)].

Lemma 2.1 *Let G be a compact connected group acting on a space X and N be any compact totally disconnected normal subgroup of G. Then the fixed point set X^G is homeomorphic to the fixed point set $(X/N)^{G/N}$ of the action of G/N on X/N.*

Proof For any $x \in X^G$, we have that $G(x) = \{x\}$ and $N(x) = \{x\}$. Thus, $(gN)(N(x)) = N(gx) = N(x)$, which implies $N(x) \in (X/N)^{G/N}$.

Now, for any $N(x) \in (X/N)^{G/N}$, we obtain that $(gN)(N(x)) = N(gx) = N(x)$ for all $g \in G$. Hence, we have that $G(x) = N(x)$. Since $G(x)$ is connected and $N(x)$ is totally disconnected, we get that $G(x) = N(x) = \{x\}$; that is, $x \in X^G$. Thus, $X^G \approx (X/N)^{G/N}$. □

Thus, many problems about the cohomological properties of orbit space and fixed point set of actions of finite dimensional compact groups are reduced to problems about Lie group actions by comparing actions of G on X and G/N on X/N.

Let us recall the Borel construction.

For any topological group G, there exists a universal principal G-bundle $E_G \to B_G$ (see Milnor [18]). B_G is unique up to homotopy equivalence and is called the classifying space of G.

Let X be a G-space. There is the diagonal action on $X \times E_G$ and the Borel construction is defined to be the orbit space $(X \times E_G)/G$ and denoted by X_G. The second projection $X \times E_G \to E_G$ induces a map

$$\pi_2 : X_G = (X \times E_G)/G \to E_G/G = B_G,$$

which is a fibration with fiber X and base space B_G. The fibration $X \xrightarrow{i} X_G \to B_G$ is called the Borel fibration.

Moreover, $H^*(X_G;k)$ is an algebra over $H^*(B_G;k)$ by $\pi_2^*: H^*(B_G;k) \to H^*(X_G;k)$;

$$\begin{array}{rcl} H^*(B_G;k) \times H^*(X_G;k) & \to & H^*(X_G;k) \\ (x,y) & \to & \pi_2^*(x) \cup y := x.y, \end{array}$$

and it is called the equivariant graded cohomology algebra of X with coefficients in k and denoted by $H_G^*(X;k)$ [4].

X is said to be totally nonhomologous to zero (TNHZ) in $X_G \to B_G$ with respect to $H^*(-;k)$ if

$$i^*: H_G^*(X;k) \to H^*(X;k)$$

is surjective, where $i: X \to X_G$ is the inclusion.

The following theorem was proved for torus (compact, connected, abelian Lie group) actions on compact spaces [5, p. 250; 8, Chapter III, Proposition 1.18].

Theorem 2.2 *Let G be a torus, X be a compact G-space, and $\dim_k H^*(X;k) < \infty$. Then*

$$\dim_k H^*\left(X^G;k\right) \leq \dim_k H^*(X;k).$$

Furthermore, $\dim_k H^*\left(X^G;k\right) = \dim_k H^*(X;k)$ if and only if X is TNHZ in $X_G \to B_G$. In the theorem, $\dim$ *denotes the dimension of vector space over k.*

Note that this theorem is true when X is a paracompact space of finite cohomological dimension or finitistic space.

In this short paper, we shall prove this theorem for pro-torus actions on compact spaces. The theorem was generalized for solenoid (1-dimensional pro-torus) actions on compact spaces [20, Theorem 3.2].

Now let us recall the following theorem of Leray-Serre for fibrations, as given in [17, Theorem 5.2]. Note that there is no need for the fiber to be connected, as mentioned there.

Theorem 2.3 *[The cohomology Leray-Serre Spectral sequence] Let R be a commutative ring with unit. Given a fibration $F \hookrightarrow E \xrightarrow{p} B$, where B is path-connected, there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$, with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F;R))$$

and converging to $H^*(E;R)$ as an algebra, where $\mathcal{H}^q(F;R)$ denotes the cohomology of B with local coefficients in the cohomology of F. Furthermore, this spectral sequence is natural with respect to fiber-preserving maps of fibrations.

Note that if k is a field, then the graded commutative algebra $H^*(E;k)$ is isomorphic to the graded commutative algabra $TotE_\infty^{*,*}$, the total complex of $E_\infty^{*,*}$, given by

$$(TotE_\infty^{*,*})^q = \bigoplus_{k+l=q} E_\infty^{k,l}.$$

That is, $H^q(E;k) \cong \bigoplus_{k+l=q} E_\infty^{k,l}$.

The next two lemmas are needed in the proof of our main theorem. The proofs of these lemmas can be found in [14, Theorem 2.1; 15, Theorem 3.3].

Lemma 2.4 *If N is a compact, totally disconnected group and X is a locally compact N-space, then the orbit map $\pi : X \to X/N$ induces an isomorphism*

$$H_c^*(X/N;k) \to (H_c^*(X;k))^N$$

where H_c^* denotes cohomology with compact supports, and $(H_c^*(X;k))^N$ is the fixed point set of the induced action of N on $H_c^*(X;k)$.

Proof It is well known that $(N,X) = \varprojlim_{H\in\mathcal{N}}(N/H;X/H)$, where each N/H is a finite group. By [4, Chapter III, (2.3)], we have that

$$H_c^*(X/N;k) \overset{\cong}{\to} (H_c^*(X/H;k))^{N/H}.$$

Because of the continuity property of Alexander–Spanier cohomology [22], by passage to the direct limit, we obtain that

$$H_c^*(X/N;k) \overset{\cong}{\to} (H_c^*(X;k))^N.$$

□

It is clear that if the space X is compact, we have that

$$H^*(X/N;k) \overset{\cong}{\to} (H^*(X;k))^N.$$

Remark 2.5 *Let G be a finite dimensional compact connected group acting on a compact space X. Let N be a totally disconnected closed normal subgroup of G such that G/N is a compact connected Lie group. Since G is connected, its action (and hence that of N) on $H^*(X;k)$ is trivial [7, Corollary 11.11]. Therefore, Lemma 2.4 implies that*

$$H^*(X/N;k) \overset{\cong}{\to} H^*(X;k).$$

The proof of the following lemma can be found in [15, Proposition 3.9].

Lemma 2.6 *If N is a compact, totally disconnected group, then $H^*(B_N;k) = k$; that is, B_N is acyclic over k.*

Proof It is clear that $N = \varprojlim_{H\in\mathcal{N}}N/H$, where N/H is a finite group for every $H \in \mathcal{N}$. Then we have that $H^i(B_N;k) = \varinjlim_{H\in\mathcal{N}}H^i\left(B_{N/H};k\right)$ by [13, Chapter III, Corollary 1.12]. Because $H^i\left(B_{N/H};k\right) = \{0\}$ for every $i \geq 1$ [4, Chapter IV, Proposition 2.4], we get that

$$H^i(B_N;k) = \varinjlim_{N\in\mathcal{N}}H^i\left(B_{N/H};k\right) = \{0\}$$

for every $i \geq 1$. Moreover, since B_N is path-connected, we obtain $H^*(B_N;k) = k$. □

3. Main results

First we shall state the cohomology algebra structure of the classifying space of a finite dimensional compact group. For this, we need the following theorem about finite dimensional compact groups.

For any closed subgroup $H \subseteq G$, we say that H has a local cross-section if there is a neighborhood U of eH with a map $s : U \to G$ satisfying $p \circ s = 1_U$. If G is a Lie group, and H is a closed subgroup of G, then there is a local cross-section. More generally, Nagami [19] proved the next theorem.

Theorem 3.1 *Any closed subgroup of a locally compact, finite dimensional group has a local cross-section.*

It is well known that the quotient map $G \to G/N$ is a principal N-bundle if and only if it has a local cross-section.

According to the theorem above, if G is a finite dimensional compact group and N is a closed normal subgroup of G, then the quotient map $G \to G/N$ is a principal N-bundle.

When the quotient map $G \to G/N$ is a principal N-bundle, we can take E_G for E_N, and $E_G \to E_G/N = B_N$ to be the universal bundle for N. We thus obtain the next lemma. The proof of this lemma can be found in [2, Theorem 2.4.12].

Lemma 3.2 *If G is a finite dimensional compact group and N is a closed normal subgroup of G, then the sequence $B_N \to B_G \to B_{G/N}$, induced by the inclusion $N \hookrightarrow G$ and the quotient map $G \to G/N$, is a fibration.*

Now we can prove the next lemma.

Lemma 3.3 *Let G be a finite dimensional compact group and N a closed, totally disconnected normal subgroup such that G/N is a Lie group. Then the homomorphism*

$$Bq^* : H^*\left(B_{G/N}; k\right) \to H^*\left(B_G; k\right)$$

is induced by the quotient map $q : G \to G/N$ is an isomorphism.

Proof The quotient map $G \to G/N$ induces the map $B_G \to B_{G/N}$, and the fiber of this map is the acyclic space B_N by Lemma 2.6. There is a Leray-Serre spectral sequence converging to $H^*\left(B_G; k\right)$ such that the second term is

$$E_2^{p,q} = H^p\left(B_{G/N}; \mathcal{H}^q\left(B_N; k\right)\right).$$

On the other hand, the homomorphism $Bq^n : H^n\left(B_{G/N}; k\right) \to H^n\left(B_G; k\right)$ induced by the projection $B_G \to B_{G/N}$ decomposes as follows:

$$H^n\left(B_{G/N}; k\right) = E_2^{n,0} \twoheadrightarrow E_3^{n,0} \twoheadrightarrow ... \twoheadrightarrow E_{n+1}^{n,0} = E_\infty^{n,0} \subseteq H^n\left(B_G; k\right)$$

for each n.

Since the fiber B_N is acyclic, the local coefficient system is simple and we have that

$$E_2^{p,q} = 0 \text{ if } q \neq 0 \text{ and } E_2^{p,0} = H^p\left(B_{G/N}; \mathcal{H}^0\left(B_N; k\right)\right) = H^p\left(B_{G/N}; k\right).$$

Therefore, we obtain that $E_\infty^{p,q} = 0$ for $q \neq 0$, $E_\infty^{n,0} = E_2^{n,0}$ and

$$H^n\left(B_G; k\right) \cong \bigoplus_{k+l=n} E_\infty^{k,l} = E_\infty^{n,0} = E_2^{n,0} = H^n\left(B_{G/N}; k\right).$$

Then the result follows. □

This lemma could also be proven using the Vietoris–Begle mapping theorem.

Now we can prove our main theorem. The first part of this theorem was also proved by Biller [3, Theorem 1.3].

Theorem 3.4 *Let G be a pro-torus, X be a compact G-space, and $\dim_k H^*(X;k) < \infty$. Then*

$$\dim_k H^*\left(X^G;k\right) \leq \dim_k H^*(X;k).$$

Furthermore, $\dim_k H^*\left(X^G;k\right) = \dim_k H^*(X;k)$ *if and only if X is TNHZ in $X_G \to B_G$.*

Proof Let N be a closed totally disconnected subgroup of G such that G/N is a torus. Consider the quotient group G/N on the orbit space X/N. Since $(X/N)^{G/N} \approx X^G$ by Lemma 2.1 and $H^*(X/N;k) \cong H^*(X;k)$ by Remark 2.5, we have that

$$\dim_k H^*\left((X/N)^{G/N};k\right) = \dim_k H^*\left(X^G;k\right)$$

and

$$\dim_k H^*(X/N;k) = \dim_k H^*(X;k).$$

Therefore, we obtain that $\dim_k H^*\left(X^G;k\right) \leq \dim_k H^*(X;k)$ by Theorem 2.2.

Now let us prove the second part of the theorem.

First, suppose that $\dim_k H^*\left(X^G;k\right) = \dim_k H^*(X;k)$. As before we have that

$$\dim_k H^*\left((X/N)^{G/N};k\right) = \dim_k H^*(X/N;k).$$

Therefore, by Theorem 2.2, we have that X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$.

Now, consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_G & \longrightarrow & B_G \\ \downarrow & & \downarrow & & \downarrow \\ X/N & \longrightarrow & (X/N)_{G/N} & \longrightarrow & B_{G/N} \end{array}$$

for the Borel fibrations

$$X \to X_G \to B_G$$

and

$$X/N \to (X/N)_{G/N} \to B_{G/N}.$$

From the induced commutative diagram

$$\begin{array}{ccc} H^*_{G/N}(X/N;k) & \longrightarrow & H^*(X/N;k) \\ \downarrow & & \downarrow \cong \\ H^*_G(X;k) & \longrightarrow & H^*(X;k) \end{array}$$

we have that X is TNHZ in $X_G \to B_G$. Since the map $H^*(X/N;k) \to H^*(X;k)$ is an isomorphism and X is TNHZ in $X_G \to B_G$, it follows that X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$.

Now assume that X is TNHZ in $X_G \to B_G$. We will show that X/N is also TNHZ in $(X/N)_{G/N} \to B_{G/N}$. Let us consider again the above commutative diagram of fibrations.

Suppose that $(E^{*,*})$ and $(\bar{E}^{*,*})$ are Leray-Serre spectral sequences of the fibrations

$$X \to X_G \to B_G$$

and

$$X/N \to (X/N)_{G/N} \to B_{G/N},$$

respectively. Since the quotient group G/N is a connected Lie group, then $B_{G/N}$ is simply connected space, and therefore the local coefficients system of the fibration $X/N \to (X/N)_{G/N} \to B_{G/N}$ is simple; that is,

$$\bar{E}_2^{p,q} = H^p\left(B_{G/N}; \mathcal{H}^q(X/N;k)\right) = H^p\left(B_{G/N}; H^q(X/N;k)\right).$$

On the other hand, since k is a field, from the universal coefficient theorem, we get that

$$\bar{E}_2^{p,q} = H^p\left(B_{G/N}; H^q(X/N;k)\right) = H^p\left(B_{G/N};k\right) \otimes H^q(X/N;k).$$

Moreover, since X is TNHZ in $X_G \to B_G$, then the local coefficents system of the fibration $X \to X_G \to B_G$ is also simple [16, Chapter III, Theorem 4.4]. We get that

$$E_2^{p,q} = H^p(B_G; \mathcal{H}^q(X;k)) = H^p(B_G; H^q(X;k)) = H^p(B_G;k) \otimes H^q(X;k).$$

Because of the isomorphisms

$$H^*(X/N;k) \cong H^*(X;k)$$

and

$$H^*\left(B_{G/N};k\right) \cong H^*(B_G;k),$$

we have that

$$\bar{E}_2^{p,q} = H^p\left(B_{G/N};k\right) \otimes H^q(X/N;k) \cong H^p(B_G;k) \otimes H^q(X;k) = E_2^{p,q}.$$

Since $\bar{E}_2^{0,q} \to E_2^{0,q}$ and $\bar{E}_2^{p,0} \to E_2^{p,0}$ are isomorphisms for all p, q, $\bar{E}_\infty^{p,q} \to E_\infty^{p,q}$ are isomorphisms for all p, q, by Zeeman's comparison theorem [23; 17, Theorem 3.26]. Furthermore, since

$$H^n_{G/N}(X/N;k) \cong \bigoplus_{k+l=n} \bar{E}_\infty^{k,l}$$

and

$$H^n_G(X;k) \cong \bigoplus_{k+l=n} E_\infty^{k,l}$$

for every $n \geq 0$, then we obtain that the map

$$H^*_{G/N}(X/N;k) \to H^*_G(X;k)$$

is an isomorphism. Thus, from the commutative diagram

$$\begin{array}{ccc} H^*_{G/N}(X/N;k) & \longrightarrow & H^*(X/N;k) \\ \cong\downarrow & & \downarrow\cong \\ H^*_G(X;k) & \longrightarrow & H^*(X;k) \end{array}$$

we have that

$$H^*_{G/N}\left(X/N;k\right) \to H^*\left(X/N;k\right)$$

is surjective; that is, X/N is TNHZ in $(X/N)_{G/N} \to B_{G/N}$. By Theorem 2.2, we obtain that

$$\dim_k H^*\left(\left(X/N\right)^{G/N};k\right) = \dim_k H^*\left(X/N;k\right).$$

On the other hand, because of

$$\dim_k H^*\left(\left(X/N\right)^{G/N};k\right) = \dim_k H^*\left(X^G;k\right)$$

and

$$\dim_k H^*\left(X/N;k\right) = \dim_k H^*\left(X;k\right),$$

we have that

$$\dim_k H^*\left(X^G;k\right) = \dim_k H^*\left(X;k\right).$$

□

An immediate consequence of this theorem is the next result.

Corollary 3.5 *Let G be a pro-torus, and let X be a compact G-space. If X is TNHZ in $X_G \to B_G$, and $0 < \dim_k H^*\left(X;k\right) < \infty$, then $X^G \neq \emptyset$.*

Proof Since $\dim_k H^*\left(X;k\right) < \infty$ and X is TNHZ in $X_G \to B_G$, then we have that $0 < \dim_k H^*\left(X^G;k\right) = \dim_k H^*\left(X;k\right)$, which implies $X^G \neq \emptyset$. □

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References

[1] Allday C, Puppe V. Cohomological Methods in Transformation Groups. Cambridge, UK: Cambridge University Press, 1993.

[2] Benson DJ. Representations and Cohomology II. Cohomology Groups and Modules. Cambridge Studies in Advanced Mathematics, Vol. 31. Cambridge, UK: Cambridge University Press, 1991.

[3] Biller H. Fixed points of pro-tori in cohomology spheres. Forum Math 2005; 17: 479-492.

[4] Borel A. Seminar on Transformation Groups. Princeton, NJ, USA: Princeton University Press, 1960.

[5] Bredon GE. Cohomological aspects of transformation groups. In: Mostert, PS, editor. Proceedings of the Conference on Transformation Groups, New Orleans, 1967. Berlin, Germany: Springer-Verlag, 1968, pp. 245-280.

[6] Bredon GE. Introduction to Compact Transformation Groups. New York, NY, USA: Academic Press, 1972.

[7] Bredon GE. Sheaf Theory. 2nd ed. Graduate Texts in Mathematics 170. New York, NY, USA: Springer, 1997.

[8] Dieck T. Transformation Groups. de Gruyter Studies in Mathematics 8. Berlin, Germany: Walter de Gruyter, 1987.

[9] Dowker CH. Homology groups of relations. Ann Math 1952; 56: 84-95.

[10] Godement R. Topologie Algébrique et Théorie des Faisceaux. Paris, France: Editions Hermann, 1958 (in French).

[11] Hofmann KH, Morris SA. Projective limits of finite dimensional Lie groups. P Lond Math Soc 2003; 87: 647-676.

[12] Hofmann KH, Morris SA. The Structure of Compact Groups. 3rd ed. Berlin, Germany: de Gruyter, 2013.

[13] Hofmann KH, Mostert PS. Cohomology Theories for Compact Abelian Groups. Berlin, Germany: Springer, 1973.

[14] Ku MC. On the cohomology of compact groups. In: Mostert, PS, editor. Proceedings of the Conference on Transformation Groups, New Orleans, 1967. Berlin, Germany: Springer-Verlag, 1968, pp. 381-400.

[15] Ku MC. Some topics in compact transformation groups. PhD, Tulane University, New Orleans, LA, USA, 1967.

[16] Mamoru M, Toda H. Topology of Lie Groups, I and II. Tokyo, Japan: American Mathematical Society, 1978.

[17] McCleary J. A User's Guide to Spectral Sequences. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge, UK: Cambridge University Press, 2001.

[18] Milnor J. Construction of universal bundles II. Ann Math 1956; 63: 430-436.

[19] Nagami KR. Cross sections in locally compact groups. J Math Soc Japan 1963; 15: 301-303.

[20] Özkurt AA. On the equivariant cohomology algebra for solenoidal actions. Turk J Math 2014; 38: 1081-1089.

[21] Pontryagin LS. Topological Groups. 3rd ed. New York, NY, USA: Gordon and Breach Science Publishers, 1986.

[22] Spanier EH. Algebraic Topology. New York, NY, USA: McGraw-Hill, 1966.

[23] Zeeman EC. A proof of the comparison theorem for spectral sequence. Proc Cambridge Philos Soc 1957; 53: 57-62.