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Digital topological complexity numbers

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Abstract: The intersection of topological robotics and digital topology leads to us a new workspace. In this paper we introduce the new digital homotopy invariant digital topological complexity number $TC(X, \kappa)$ for digital images and give some examples and results about it. Moreover, we examine adjacency relations in the digital spaces and observe how $TC(X, \kappa)$ changes when we take a different adjacency relation in the digital spaces.

Key words: Topological complexity number, digital topology, digital topological complexity number

1. Introduction

Topological robotics is an area that uses some topological properties to compute the homotopy invariant topological complexity number $TC(X)$, mostly in mechanics and robot motion planning problems. Michael Farber [16] first introduced the topological complexity number $TC(X)$ that measures the degree of the deflection of the contractibility of a given area for a robot . After that, Farber published many works on this subject, such as [17–20]. Besides Farber, Grant [23], Dranishnikov [12], Tabachnikov and Yuzvinsky [22], and other researchers have improved it with their different works. Simply, $TC(X)$ computes the minimum number that is an invariant in topology (and of course in algebraic topology too) for a robot to move from our given initial point to our given final point. This number requires a continuous motion planning algorithm. A motion planning algorithm in a path-connected space X takes two points a and b of X and composes a path α in X such that $\alpha(0) = a$ and $\alpha(1) = b$. Therefore, we observe that robotics are motivated by this motion planning problem and accordingly topological approaches. To see how complex the instabilities of robot motion are, $TC(X)$ is a very important tool and takes its power from topological works [34]. For example, Farber gave an upper bound for $TC(X)$ using the dimension of X and also a lower bound using the structure of the cohomological algebra of X, e.g., a cup-product and tensor product [16]. Later, Grant and Farber [21] generalized this result about the lower bound with the help of cohomology operations.

Digital topology also uses topological properties for examining digital images in some areas such as image processing, computer vision, and robot design. Rosenfeld, a computer image analyst, first used the term 'digital topology' in his publication [32]. Rosenfeld [33], Boxer [3, 8, 9], Karaca [9], and many other researchers had interests in and studies on this topic in the following years. See [27–31] for more studies and details. One of these studies is about homotopy, an important material for algebraic topology, so Kong [26] defined the digital version of the fundamental group of a given digital picture. Boxer [4, 5] had some studies about the digital homotopy

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and its properties, too. Another important tool from algebraic topology is homology and its digital version was introduced in [1] and studied also in [11, 13]. After homotopy and homology, cohomology is sure enough used in digital topology studies and the digital version of the simplicial cup-product with basic properties was given in [14].

In this paper we deal with the digital analog of compact-open topology construction, because the space of all continuous paths PX in the definition of the topological complexity number of Michael Farber [16] has compact-open topology on the space PX.

We first give some preliminaries about the digital topology. In Section 3, we define the digital topological complexity number and give some results and examples about it. Later we mention adjacency relations because we observe that when we take a smaller κ-adjacency relation in any digital space, the digital topological complexity number remains the same or increasing. We prove the main result, that the digital topological complexity number is a homotopy invariant in digital images. We also give bounds for the digital topological complexity number using the relation with $cat(X,\kappa)$ that was introduced by Borat and Vergili [2]. Finally, we explain the future research about this subject in the conclusion section.

2. Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space. We say that (X,κ) is a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation for the members of the digital image X [4]. For a positive integer l with $1 \leq l \leq n$ and two distinct points p and q in $\mathbb{Z}^n$, p and q are c_l*-adjacent* if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$ [4]. For example, if $n = 1$, we have only $c_1 = 2-$ adjacency. If $n = 2$, then we have $c_1 = 4$ and $c_2 = 8$ adjacencies in $\mathbb{Z}^2$. If we study in $\mathbb{Z}^3$, we have $c_1 = 6$, $c_2 = 18$, and $c_3 = 26$ adjacencies, too.

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A κ*-neighbor* of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p [24]. Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is κ*-connected* if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$, and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, ..., r-1$ [24].

Let $X \subset \mathbb{Z}^{n_0}$, $Y \subset \mathbb{Z}^{n_1}$. Let $f : X \to Y$ be a function. Let κ_i be an adjacency relation defined on $\mathbb{Z}^{n_i}$, $i = \{0,1\}$ respectively. We say that f is (κ_0,κ_1)*-continuous* if the image under f of every κ_0-connected subset of X is κ_1-connected [4].

Proposition 2.1 *([4]). Let (X,κ_0) and (Y,κ_1) be digital images. Then the function $f : X \to Y$ is (κ_0,κ_1)-continuous if and only if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are κ_0-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent.*

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. A function $f : X \to Y$ is *digital* (κ_0,κ_1)*-isomorphism* if f is bijective and digital (κ_0,κ_1)-continuous and also $f^{-1} : Y \to X$ is digital (κ_1,κ_0)-continuous [7].

Let X be digital image with κ-adjacency. If

$$f : [0,m]_{\mathbb{Z}} \to X$$

is a $(2,\kappa)$-continuous function such that $f(0) = x$ and $f(m) = y$, then f is called a *digital path* from x to y

in X where the digital interval is defined by $[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} : a \leq z \leq b\}$ [6]. If $f(0) = f(m)$, then the κ-path is said to be closed, and the function f is called a $\kappa-loop$.

Let $X \in \mathbb{Z}^{n_0}$ and $Y \in \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1 adjacencies, respectively. For two (κ_0, κ_1)-continuous functions $f, g : X \to Y$, if there is a positive integer m and a function $H : X \times [0,m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $H(x,0) = f(x)$ and $H(x,m) = g(x)$;
- for all $x \in X$, $H_x : [0,m]_{\mathbb{Z}} \to Y$ defined by $H_x(t) = H(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2, \kappa_1)$-continuous;
- for all $t \in [0,m]_{\mathbb{Z}}$, $H_t : X \to Y$ defined by $H_t(x) = H(x,t)$ for all $x \in X$ is (κ_0, κ_1)-continuous,

they are said to be *digitally* (κ_0, κ_1)*-homotopic* in Y and this is denoted by $f \simeq_{(\kappa_0,\kappa_1)} g$. The function H is called a digital (κ_0, κ_1)-homotopy between f and g [4].

Proposition 2.2 *([4]). Digital homotopy is an equivalence relation among digitally continuous functions.*

A digitally continuous function $f : X \to Y$ is *digitally nullhomotopic* in Y if f is digitally homotopic in Y to a constant function [4].

A (κ_0, κ_1)-continuous function f from a digital image X to another digital image Y with κ_0 and κ_1 adjacencies respectively is a (κ_0, κ_1)*-homotopy equivalence* if there exists a (κ_0, κ_1)-continuous function g from Y to X such that $g \circ f$ is (κ_0, κ_0)-homotopic to the identity function 1_X and $f \circ g$ is (κ_1, κ_1)-homotopic to the identity function 1_Y [5].

A digital image (X, κ) is said to be κ*-contractible* if its identity map is (κ, κ)-homotopic to a constant function c for some $c_0 \in X$ where the constant function $c : X \to X$ is defined by $c(x) = c_0$ for all $x \in X$ [4].

Lemma 2.3 *([2]). If X is a κ-contractible digital image, then it is κ-connected.*

If $f : [0, m_1]_{\mathbb{Z}} \to X$ and $g : [0, m_2]_{\mathbb{Z}} \to X$ are digital κ-paths with the condition $f(m_1) = g(0)$, then define *the product* $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ [25] by

$$(f * g)(t) = \begin{cases} f(t) & , t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1) & , t \in [m_1, m_1 + m_2]. \end{cases}$$

Adjacency for the Cartesian product of (X_i, κ_i), $i \in \{0, 1\}$, is defined in [10]. Given points x_i, $y_i \in (X_i, \kappa_i)$, (x_0, x_1) and (y_0, y_1) are *adjacent* in $X_0 \times X_1$ if and only if one of the following is satisfied:

- $x_0 = x_1$ and $y_0 = y_1$; or
- $x_0 = x_1$ and y_0 and y_1 are κ_1-adjacent; or
- x_0 and x_1 are κ_0-adjacent and $y_0 = y_1$; or
- x_0 and x_1 are κ_0-adjacent and y_0 and y_1 are κ_1-adjacent.

A simple closed κ-curve of $m \geq 4$ points in a digital image X is a sequence $\{f(0), f(1), ..., f(m-1)\}$ of images of the κ-path $f : [0, m-1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ are κ*-adjacent* if and only if $j = i \pm mod\ m$ [5].

Theorem 2.4 *([4]). Let $X \subset \mathbb{Z}^2$ be a digital simple closed 4-curve such that $\mathbb{Z}^2 - X$ is 8-disconnected. Then X is not digitally 4-contractible.*

3. The digital topological complexity number

We now study digital images of the form (X, κ) and compute the digital topological complexity $TC(X, \kappa)$ number of the digital images. For this purpose, we regard PX as a space of all digitally continuous digital paths. In topological robotics, Farber uses the compact-open topology on the space PX for the continuity. We now have a different construction on our new PX space to guarantee the digital continuity.

We define a digital continuity on the digital motion planning algorithm $s : X \times X \to PX$ as follows. Let us take two points (A, B) and (C, D) in $X \times X$. If (A, B) and (C, D) are λ-connected on the Cartesian product space $X \times X$, then $s(A, B)(t)$ and $s(C, D)(t)$ are γ-connected on the space PX. That the two digital paths are γ-connected here means that, for all 't' times, these paths are γ-connected where $[0, m]_{\mathbb{Z}}$.

Note that we have to explain the situation when we have digital paths having different steps (or t times). For example, consider two digital paths $\alpha = abcd$ and $\beta = ab$ respectively with 4 and 2 steps in any digital images having the k-neighbor points a, b, c, and d. How can we say that they are k-adjacent or not? The answer is to increase the number of steps of β to equalize with the path α while making no headway. This means that we think of the path β as $abbb$. Now we can examine whether α and β are k-adjacent.

Theorem 3.1 *Let (X, κ) be a digital image. A digitally continuous motion planning $s : X \times X \to PX$ exists if and only if the space X is κ-contractible.*

Proof Suppose that a digitally continuous section $s : X \times X \to PX$ exists.

Fix a point $A_0 \in X$ and define the digital homotopy function

$$H : X \times [0, m]_{\mathbb{Z}} \to X$$

$$(B, t) \longmapsto H(B, t) = s(A_0, B)(t),$$

where $B \in X$, $t \in [0, m]_{\mathbb{Z}}$. It is easy to see that

- $H(B, 0) = s(A_0, B)(0) = A_0$, $H(B, m) = s(A_0, B)(m) = B$.
- $H_x : [0, m]_{\mathbb{Z}} \to X$ defined by $H_x(t) = H(x, t)$ is $(2, \kappa)$-continuous.
- $H_t : X \to X$ defined by $H_t(x) = H(x, t)$ is (κ, κ)-continuous.

Conversely, assume that there is a continuous digital homotopy $H : X \times [0, m]_{\mathbb{Z}} \to X$ such that for all $A \in X$, $m \in \mathbb{Z}$, $H(A, 0) = A$, $H(A, m) = A_0$.

Given a pair $(A, B) \in X \times X$, we may compose the $(2, \kappa)$-continuous path $(A, t) \longmapsto H(A, t)$ with the inverse of $(B, t) \longmapsto H(B, t)$, which gives a $(2, \kappa)$-continuous motion planning in X.

We move A into the base point A_0 along the digital contraction. Later we move A_0 into B with the inverse of the digital path above. Therefore, we have a digital path from A to B. It means that we get a digitally continuous motion planning in a digitally contractible space X. □

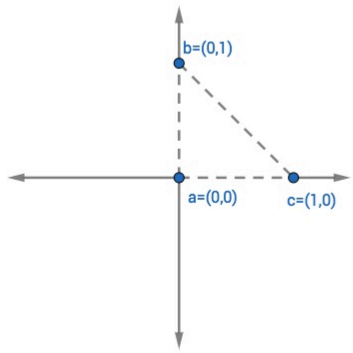

Figure 3.1. Digital image X.

Definition 3.2 *Digital topological complexity number $TC(X,\kappa)$ is the minimal number k such that*

$$X \times X = U_1 \cup U_2 \cup ... \cup U_k$$

with the property that π admits a digitally continuous map $s_j : U_j \to PX$ such that $\pi \circ s_j = 1_{U_j}$ over each $U_j \subset X \times X$.

If no such k exists, we will set $TC(X,\kappa) = \infty$.

Corollary 3.3 *$TC(X,\kappa) = 1$ if and only if (X,κ) is κ-contractible.*

Example 3.4 *Let $X = \{a = (0,0), b = (0,1), c = (1,0)\} \subset \mathbb{Z}^2$ be a digital space with an adjacency relation $\kappa = 4$ in Figure 3.1.*

We need to show that X is 4-contractible. We can define the digital homotopy function

$$H : X \times [0,2]_{\mathbb{Z}} \to X$$

by, for $i = 0,1,2$,

- *if $t = 0$, then $H(a,t) = a$, $H(b,t) = b$, $H(c,t) = c$;*
- *if $t = 1$, then $H(a,t) = a$, $H(b,t) = b$, $H(c,t) = a$;*
- *if $t = 2$, then $H(a,t) = b$, $H(b,t) = b$, $H(c,t) = b$.*

By Corollary 3.3 we conclude that $TC(X,4) = 1$.

Theorem 3.5 *Let $MSC_8 = \{a = (0,0), b = (-1,1), c = (-1,2), d = (0,3), e = (1,2), f = (1,1)\}$. Then $TC(MSC_8, 8) = 2$.*

Proof $TC(MSC_8, 8) \neq 1$ since MSC_8 is not 8-contractible [4]. Thus, the digital topological complexity number of the digital space MSC_8 with 8-adjacency must be greater than 1. We show that $TC(MSC_8, 8) = 2$.

Let $MSC_8 \times MSC_8 = U_1 \cup U_2$ where

$U_1 = \{(x, y) \in MSC_8 : x, y \in \{a, b, c\}$ *or* $x \in \{a, b, c\}, y \in \{d, e, f\}$ *or* $x \in \{d, e, f\}, y \in \{a, b, c\}\}$ and $U_2 = \{(x, y) \in MSC_8 : x, y \in \{d, e, f\}\}$.

Since we choose the number 2 as a minimal number, then we conclude that $TC(MSC_8, 8) = 2$.

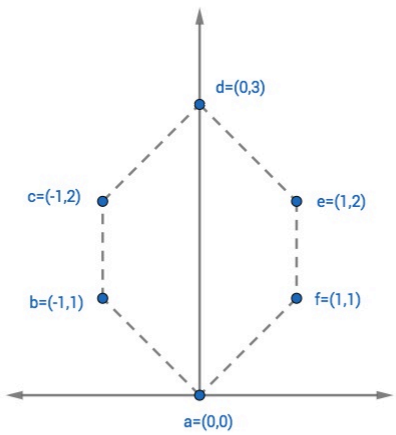

Figure 3.2. MSC_8.

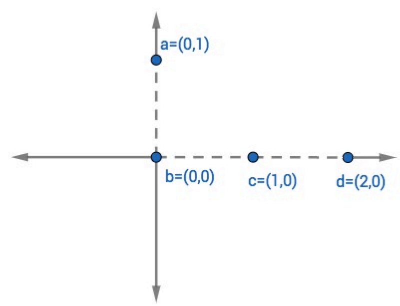

Figure 3.3. Digital projective line $\mathbb{Z}P^1$.

□

Example 3.6 *A real projective line is one of the important specific spaces in topology. The digital version of the real projective line* $\mathbb{R}P^1$ *via quotient map from* $MSC_4 = \{a = (0, 1), b = (0, 0), c = (1, 0), d = (2, 0), e = (2, 1), f = (2, 2), g = (1, 2), h = (0, 2)\}$ *with antipodal points is* $\mathbb{Z}P^1$ *defined in [15].*

We can easily check that $\mathbb{Z}P^1$ *is* 4*-contractible when we choose*

$$H : X \times [0, 2]_{\mathbb{Z}} \to X$$

- *for* $t = 0$, $H(a, 0) = a$, $H(b, 0) = b$, $H(c, 0) = c$, $H(d, 1) = d$;
- *for* $t = 1$, $H(a, 1) = b$, $H(b, 1) = b$, $H(c, 1) = c$, $H(d, 1) = c$;
- *for* $t = 2$, $H(a, 2) = b$, $H(b, 2) = b$, $H(c, 2) = b$, $H(d, 2) = b$;

as a digital homotopy function. It means that the digital topological complexity number of the digital projective line with 4*-adjacency is equal to* 1.

Corollary 3.7 *Let* (X, κ_1) *and* (X, κ_2) *be digital images with different adjacency relations and let* $\kappa_1 < \kappa_2$. *Then*

$$TC(X, \kappa_1) \geq TC(X, \kappa_2).$$

Proof Let $\kappa_1 < \kappa_2$ and assume that $TC(X, \kappa_1) = m$. By the definition of the digital topological complexity, we have $X \times X = U_1 \cup U_2 \cup ... \cup U_m$ with the property that $\pi \circ s_j = 1_{U_j}$ over each $U_j \subset X \times X$, and all of s_j are digitally continuous for all $1 \leq j \leq m$.

The digital continuity of s_j is equivalent to the following statement: That the map $s_j : U_j \to PX$ is digitally continuous means that for all (p_1, p_2) and (q_1, q_2) in U_j, if (p_1, p_2) and (q_1, q_2) are digitally connected for the Cartesian product, then $s_j(p_1, p_2)(t)$ and $s_j(q_1, q_2)(t)$ are also digitally connected for all t times. Since p_1 and q_1 or p_2 and q_2 are κ_1-adjacent, then they are also κ_2-adjacent because of $\kappa_1 < \kappa_2$. We can thus say, for all the same j, that the maps $s_j : U_j \to PX$ are digitally continuous according to κ_2-adjacencies too. Hence $TC(X, \kappa_2)$ must be at least equal to or greater than the number m. Therefore, these required result holds. □

4. A homotopy invariance property in digital images

Theorem 4.1 *Let (X, κ) be a digital image. Then $TC(X, \kappa)$ depends only on the homotopy type of (X, κ).*

Proof Let (X, κ_0) and (Y, κ_1) be two digital images and suppose that there exist digitally continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is (κ_1, κ_1)-homotopic to 1_Y. We will show that $TC(Y, \kappa_1) \leq TC(X, \kappa_0)$.

Assume that $U \subset X \times X$ is a subset such that there exists a digitally continuous motion planning $s : U \to PX$ over U. Define $V = (g \times g)^{-1}(U) \subset Y$. We need to find a digitally continuous motion planning $\alpha : V \to PY$ over V explicitly. Since $f \circ g$ is digitally homotopic to the identity function on Y, we have a digital homotopy $H : Y \times [0, m]_{\mathbb{Z}} \to Y$ with $H(A, 0) = 1_Y(A) = A$ and $H(A, m) = f \circ g(A)$.

For $(A, B) \in V$ and $t \in [0, m]_{\mathbb{Z}}$, take

$$\alpha(A,B)(t) = \begin{cases} H(A,t), & 0 < t < m \\ f(s(gA, gB)(t-m)), & m < t < 2m \\ H(B, 3m-t), & 2m < t < 3m. \end{cases}$$

α is a continuous motion planning for the digital image Y because the digital homotopy H, digitally continuous motion planning s for the digital image X, and digitally continuous functions f and g take care of adjacency relations with digital connectedness and do not allow dissolution of digital continuity.

We see that for $k = TC(X, \kappa)$, decomposition $X \times X = U_1 \cup U_2 \cup ... \cup U_k$ with a digitally continuous motion planning over each U_i defines a new decomposition $V_1 \cup V_2 \cup ... \cup V_k$ of $Y \times Y$ with the similar properties. Thus, we get $TC(Y, \kappa_1) \leq TC(X, \kappa_0)$.

Similarly, $TC(X, \kappa_0) \leq TC(Y, \kappa_1)$ can be proved easily whenever there exist digitally continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is (κ_0, κ_0)-homotopic to 1_X.

Therefore, we now conclude that $TC(X, \kappa)$ is independent of the choice of the digital image and depends only on the homotopy type of a given digital image. □

5. Bounds for the number $TC(X)$

Theorem 5.1 *If (X, κ) is a digitally path-connected space, then*

$$cat_\kappa(X) \leq TC(X, \kappa) \leq cat_\lambda(X \times X)$$

where λ is an adjacency relation for the Cartesian product space $X \times X$.

Proof Let $TC(X,\kappa) = m$. Then $X \times X$ is the union of m digital images $U_1, U_2, ..., U_m$ with a digitally continuous motion planning $s_j : U_j \to PX$, $1 \leq j \leq m$, and $\pi \circ s_j = 1_{U_i}$. Let A_0 also be a fixed point in the digital image X. Consider $U_i \subset X \times X$, for any $1 \leq i \leq m$, such that $s_i : U_i \to PX$ is digitally continuous, and define the digital image $V_i = \{B \in X \mid (A_0, B) \in U_i\} \subset X$. Then

$$F : V_i \times [0,m]_{\mathbb{Z}} \longrightarrow X$$

$$(B,t) \longmapsto F(B,t) = s_i(A_0,B)(t)$$

is a digital homotopy in $X \times X$ because:
(i) for all $B \in V_i$, $F(B,0) = s_i(A_0,B)(0) = A_0$ and $F(B,m) = s_i(A_0,B)(m) = B$;
(ii) for all $B \in V_i$, $F_B : [0,m]_{\mathbb{Z}} \to X$ is $(2,\kappa)$-continuous;
(iii) for all $t \in [0,m]_{\mathbb{Z}}$, $F_t : V_i \to X$ is (κ,κ)-continuous.
We now conclude that $V_i \subset X$ is digitally contractible for all $i = 1, 2, \ldots, m$. Then the inclusion map $V_i \hookrightarrow X$, for all $i = 1, ..., m$, is digitally κ-nullhomotopic, where $A_0 \times V_i = U_i \cap (A_0 \times X)$, since $TC(X,\kappa) = m$. Hence, $TC(X,\kappa) \geq cat_\kappa(X)$.

On the other hand, digitally κ-nullhomotopic maps are digitally continuous but a digitally continuous map need not to be digitally κ-nullhomotopic. It means that $TC(X,\kappa) \leq cat_\lambda(X \times X)$. □

6. Conclusion

We aimed to introduce the concept of digital topological complexity in this paper. First the digital topological complexity number is defined and secondly it is shown how adjacency relations affect this number when they change. Finally, we prove that $TC(X,\kappa)$ is an homotopy invariant in digital images similarly as the topological complexity number of Farber in topological spaces.

Borat and Vergili [2] gave the notion of the digital Lusternik–Schnirelmann category. We know that the Lusternik–Schnirelmann theory and topological complexity number have a close relation with each other [16, 19]. An upper bound for the homotopy invariant $TC(X)$ of Farber benefits from the number $cat(X)$. That is why our definition of topological complexity in the digital viewpoint can be used and improved with the work of Borat and Vergili.

On the other hand, Farber [16] gave the cohomological lower bound for $TC(X)$. He used a well-known tool from algebraic topology, the cup-product, for this purpose. Ege and Karaca [14] defined a cup-product for digital images and proved some of its main properties. They also introduced the cohomology groups of a digital simplicial complex. This means that our works in this paper can be adapted to the future works on this topic.

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