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Spectrum and scattering function of the impulsive discrete Dirac systems

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Abstract: In this paper, we investigate analytical and asymptotic properties of the Jost solution and Jost function of the impulsive discrete Dirac equations. We also study eigenvalues and spectral singularities of these equations. Then we obtain characteristic properties of the scattering function of the impulsive discrete Dirac systems. Therefore, we find the Jost function, point spectrum, and scattering function of the unperturbed impulsive equations.

Key words: Dirac systems, Jost solution, scattering function, eigenvalues

1. Introduction

Impulsive differential and discrete equations appear as natural descriptions of observed evolution phenomena of several real-world problems. Many physical phenomena involving thresholds, bursting rhythm models in medicine, and mathematical models in economics do exhibit impulsive differential and discrete equations [6,7,17]. Therefore the theory of impulsive equations is a new and important branch of applied mathematics, which has extensive physical and realistic mathematical models. For the general theory of impulsive differential equations, we refer to the monographs [1,2,8]. In the literature, impulsive equations are called different kinds of names. Some of these names are equations with jump condition, equations with interface condition, and equations with transmission condition. In particular, impulsive Sturm–Liouville problems have been investigate in detail in [3,4,10–15,18–22].

In the following, we will use these notations:

$$\begin{aligned}
&\mathbb{N} := \{1,2,3,...\}\,,\\
&\mathbb{N}^0 := \{0,1,2,...\}\,,\\
&\mathbb{N}^*_{m_0} := \mathbb{N}\backslash\{m_0\}\,,\\
&\mathbb{N}^{m_0} := \{m_0+1, m_0+2,...\}\,,\\
&\mathbb{N}_{m_0} := \{1,2,...,m_0-2,m_0-1\}\,,\\
&\mathbb{N}\,(m_0) := \mathbb{N}\backslash\{m_0-1,m_0,m_0+1\}\,,
\end{aligned}$$

where $m_0 \geq 3$ is an integer number.

Now we consider the impulsive boundary value problem generated by the system of difference equations

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of first order

$$\begin{cases} a_{n+1}y_{n+1}^{(2)}+b_ny_n^{(2)}+p_ny_n^{(1)}=\lambda y_n^{(1)}, \\ a_{n-1}y_{n-1}^{(1)}+b_ny_n^{(1)}+q_ny_n^{(2)}=\lambda y_n^{(2)}, \quad n\in\mathbb{N}\left(m_0\right), \end{cases} \tag{1.1}$$

with the boundary condition

$$y_0^{(1)}=0, \tag{1.2}$$

and the impulsive condition

$$\begin{pmatrix} y_{m_0+1}^{(1)} \\ y_{m_0+2}^{(2)} \end{pmatrix}=\mathbb{B}\begin{pmatrix} y_{m_0-1}^{(2)} \\ y_{m_0-2}^{(1)} \end{pmatrix}, \tag{1.3}$$

where $\mathbb{B}=\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ is a real matrix, $\det\mathbb{B}>0$; $\{a_n\}_{n\in\mathbb{N}^0},\{b_n\}_{n\in\mathbb{N}},\{p_n\}_{n\in\mathbb{N}},\{q_n\}_{n\in\mathbb{N}}$, are real sequences such that $a_n\neq 0, n\in\mathbb{N}^0, b_n\neq 0, n\in\mathbb{N}$; and λ is a spectral parameter.

If $a_n\equiv 1, n\in\mathbb{N}_{m_0}^*\cup\{0\}, b_n\equiv 1, n\in\mathbb{N}_{m_0}^*$ then system (1.1) reduces to

$$\begin{cases} \Delta y_n^{(2)}+p_ny_n^{(1)}=\lambda y_n^{(1)}, \\ -\Delta y_{n-1}^{(1)}+q_ny_n^{(2)}=\lambda y_n^{(2)}, \quad n\in\mathbb{N}\left(m_0\right), \end{cases} \tag{1.4}$$

where Δ is a forward difference operator defined by $\Delta y_n=y_{n+1}-y_n$.

System (1.4) is the discrete analog of the well-known canonical Dirac system [9]:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}+\begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and so system (1.4) is called a canonical discrete Dirac system.

This paper is organized as follows. In the next section, we study asymptotic properties of the Jost solution and Jost function of the impulsive boundary value problem (IBVP) (1.1)–(1.3). We investigate the point spectrum and characteristic properties of the scattering function of the impulsive discrete Dirac system in Section 3. Finally, in Section 4, we present the Jost function, eigenvalues, and scattering function of the unperturbed impulsive Dirac equations.

2. The Jost solution

Suppose that the sequences $\{a_n\},\{b_n\},\{p_n\}$, and $\{q_n\}, n\in\mathbb{N}_{m_0}^*$, satisfy

$$\sum_{n\in\mathbb{N}_{m_0}^*} n(|1-a_n|+|1+b_n|+|p_n|+|q_n|)<\infty. \tag{2.1}$$

Let

$$P(z)=\{P_n(z)\}_{n\in\mathbb{N}_{m_0}}=\left\{\begin{pmatrix} P_n^{(1)}(z) \\ P_n^{(2)}(z) \end{pmatrix}\right\}_{n\in\mathbb{N}_{m_0}}$$

and

$$Q(z) = \{Q_n(z)\}_{n\in\mathbb{N}_{m_0}} = \left\{ \begin{pmatrix} Q_n^{(1)}(z) \\ Q_n^{(2)}(z) \end{pmatrix} \right\}_{n\in\mathbb{N}_{m_0}}$$

be the solutions of (1.1) for $\lambda = 2\sin\dfrac{z}{2}$ and $z \in \mathbb{C}$, satisfying the conditions

$$P_0^{(1)}(z) = 0, \qquad P_1^{(2)}(z) = -1 \tag{2.2}$$

and

$$Q_0^{(1)}(z) = \frac{1}{a_0}, \qquad Q_1^{(2)}(z) = 0, \tag{2.3}$$

respectively.

It is clear that

$$\deg\left[P_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-1, \qquad \deg\left[P_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-2, \tag{2.4}$$

for all $n \in \mathbb{N}_{m_0}$.

Under condition (2.1) for $\lambda = 2\sin\dfrac{z}{2}$, system (1.1) has the bounded solution

$$f(z) = \{f_n(z)\}_{n\in\mathbb{N}^{m_0}} = \left\{ \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} \right\}_{n\in\mathbb{N}^{m_0}}, \quad z \in \overline{\mathbb{C}}_+,$$

satisfying the following asymptotic condition:

$$f_n(z) = [I_2 + o(1)] \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{inz}, \quad z \in \overline{\mathbb{C}}_+, \quad n \to +\infty, \tag{2.5}$$

where $I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \quad \Im(z) \geq 0\}$ such that $\Im(z)$ is the imaginary part of z.

It is well known from [5] that the solution $f(z) = \{f_n(z)\}_{n\in\mathbb{N}^{m_0}}$ has the following representation:

$$\begin{cases} f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n \left(I_2 + \displaystyle\sum_{m=1}^{\infty} A_{nm} e^{imz}\right) \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{inz}, \\ f_0^{(1)}(z) = \alpha_0^{11} \left[e^{i\frac{z}{2}} \left(1 + \displaystyle\sum_{m=1}^{\infty} A_{0m}^{11} e^{imz}\right) - i \displaystyle\sum_{m=1}^{\infty} A_{0m}^{12} e^{imz} \right], \end{cases} \tag{2.6}$$

for all $n \in \mathbb{N}^{m_0}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix},$$

and also, $A_{nm}^{ij}, i,j=1,2,$ satisfy

$$|A_{nm}^{ij}| \leq M \sum_{k=n+[m/2]}^{\infty} (|1-a_k| + |1+b_k| + |p_k| + |q_k|),$$

where $[m/2]$ is the integer part of $m/2$ and $M>0$ is a constant.

Note that the function $f(z) = \{f_n(z)\}_{n\in\mathbb{N}^{m_0}}$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \quad \Im(z) > 0\}$, continuous up to the real axis, and $f_n(z+4\pi) = f_n(z)$ for all z in $\mathbb{C}_+$ [5].

Theorem 2.1 *The following equations are satisfied for the functions* $f(z) = \{f_n(z)\}_{n\in\mathbb{N}^{m_0}}$ *and* $P(z) = \{P_n(z)\}_{n\in\mathbb{N}_{m_0}}$:

$$(i) \lim_{|z|\to+\infty} f_n(z)e^{-inz} = -i \begin{pmatrix} \alpha_n^{12} \\ \alpha_n^{22} \end{pmatrix}, \quad z \in \overline{\mathbb{C}}_+,$$

$$(ii) \lim_{|z|\to+\infty} P_n^{(1)}(z)e^{i\left(n-\frac{1}{2}\right)z} = i\,(-1)^n K_1(n), \quad z \in \overline{\mathbb{C}}_+,$$

$$\lim_{|z|\to+\infty} P_n^{(2)}(z)e^{i(n-1)z} = i\,(-1)^n K_2(n), \quad z \in \overline{\mathbb{C}}_+,$$

where

$$K_1(n) := -\left(b_1 \prod_{k=2}^{n} a_k b_k\right)^{-1}, \quad K_2(n) := -\left(\prod_{k=2}^{n} a_k b_{k-1}\right)^{-1}. \tag{2.7}$$

Proof (i) It follows from (2.6) that

$$f_n^{(1)}(z)e^{-inz} + i\alpha_n^{12} = \sum_{m=1}^{\infty} \left[(\alpha_n^{11}A_{nm}^{11} + \alpha_n^{12}A_{nm}^{21})e^{i\frac{z}{2}} - (\alpha_n^{11}A_{nm}^{12} + \alpha_n^{12}A_{nm}^{22})i\right] e^{imz}$$
$$+ \alpha_n^{11}e^{i\frac{z}{2}},$$

$$f_n^{(2)}(z)e^{-inz} + i\alpha_n^{22} = \sum_{m=1}^{\infty} \left[(\alpha_n^{21}A_{nm}^{11} + \alpha_n^{22}A_{nm}^{21})e^{i\frac{z}{2}} - (\alpha_n^{21}A_{nm}^{12} + \alpha_n^{22}A_{nm}^{22})i\right] e^{imz}$$
$$+ \alpha_n^{21}e^{i\frac{z}{2}}.$$

Thus, for all $z \in \overline{\mathbb{C}}_+$ and $n \in \mathbb{N}^{m_0}$, we obtain

$$\lim_{|z|\to+\infty} f_n^1(z)e^{-inz} = -i\alpha_n^{12},$$

$$\lim_{|z|\to+\infty} f_n^2(z)e^{-inz} = -i\alpha_n^{22},$$

or

$$\lim_{|z|\to+\infty} f_n(z)e^{-inz} = -i \begin{pmatrix} \alpha_n^{12} \\ \alpha_n^{22} \end{pmatrix}.$$

(ii) Using (1.1) and (2.2), we see that

$$P_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right) = K_1(n)\lambda^{2n-1} + \sum_{m=0}^{2n-2} k_m\lambda^m, \quad k_m \in \mathbb{R}, \tag{2.8}$$

$$P_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right) = K_2(n)\lambda^{2n-2} + \sum_{m=0}^{2n-3} l_m\lambda^m, \quad l_m \in \mathbb{R}, \tag{2.9}$$

are satisfied for $n \in \mathbb{N}_{m_0}$, where

$$K_1(n) := -\left(b_1\prod_{k=2}^{n} a_k b_k\right)^{-1}, \quad K_2(n) := -\left(\prod_{k=2}^{n} a_k b_{k-1}\right)^{-1}.$$

It follows from (2.8) and (2.9) that

$$\lim_{|z|\to+\infty} P_n^{(1)}(z)e^{i\left(n-\frac{1}{2}\right)z} = i\left(-1\right)^n K_1, \quad z \in \overline{\mathbb{C}}_+,$$
$$\lim_{|z|\to+\infty} P_n^{(2)}(z)e^{i(n-1)z} = i\left(-1\right)^n K_2, \quad z \in \overline{\mathbb{C}}_+,$$

hold. □

Definition 2.2 *The Wronskian of two solutions* $\{Y_n(z)\}_{n\in\mathbb{N}^*_{m_0}} = \left\{\begin{pmatrix} y_n^{(1)}(z) \\ y_n^{(2)}(z)\end{pmatrix}\right\}_{n\in\mathbb{N}^*_{m_0}}$ *and* $\{U_n(z)\}_{n\in\mathbb{N}^*_{m_0}} = \left\{\begin{pmatrix} u_n^{(1)}(z) \\ u_n^{(2)}(z)\end{pmatrix}\right\}_{n\in\mathbb{N}^*_{m_0}}$ *of system* (1.1) *is defined by*

$$W\left[Y_n(z), U_n(z)\right] := a_n\left[y_n^{(1)}(z)u_{n+1}^{(2)}(z) - y_{n+1}^{(2)}(z)u_n^{(1)}(z)\right]. \tag{2.10}$$

Using (2.2), (2.3), (2.5), and (2.10) we obtain

$$W\left[P(z), Q(z)\right] = 1, \quad z \in \mathbb{C},$$
$$W\left[f(z), \overline{f(z)}\right] = 2i\cos\frac{z}{2}, \quad z \in \mathbb{R}.$$

The set $\left\{\{f_n(z)\}_{n\in\mathbb{N}^{m_0}}, \left\{\overline{f_n(z)}\right\}_{n\in\mathbb{N}^{m_0}}\right\}$ forms a fundamental system of the solutions of the system (1.1) for $\lambda = 2\sin\frac{z}{2}$, $z \in \mathbb{R}\backslash\{(2k+1)\pi, k \in \mathbb{Z}\}$, and similarly, the set $\left\{\{P_n(z)\}_{n\in\mathbb{N}_{m_0}}, \{Q_n(z)\}_{n\in\mathbb{N}_{m_0}}\right\}$ forms a fundamental system of the solutions of system (1.1) for $\lambda = 2\sin\frac{z}{2}$, $z \in \mathbb{C}$.

We consider the following vector sequences:

$$E_n(z) = \begin{cases} a(z)P_n(z) + b(z)Q_n(z), & n \in \mathbb{N}_{m_0} \\ f_n(z), & n \in \mathbb{N}^{m_0}, \end{cases} \tag{2.11}$$

for $z \in \overline{\mathbb{C}}_+$ and

$$F_n(z) = \begin{cases} P_n(z), & n \in \mathbb{N}_{m_0} \\ c(z)f_n(z) + d(z)\overline{f_n(z)}, & n \in \mathbb{N}^{m_0}, \end{cases} \tag{2.12}$$

for $z \in \mathbb{R} \backslash \{(2k+1)\pi, k \in \mathbb{Z}\}$.

It is clear that $E(z) = \{E_n(z)\}_{n \in \mathbb{N}^*_{m_0}}$ and $F(z) = \{\{F_n(z)\}_{n \in \mathbb{N}^*_{m_0}}$ are the solutions of system (1.1) for $\lambda = 2\sin\dfrac{z}{2}$. Using the impulsive condition (1.3), we have

$$a(z) = \frac{a_{m_0-2}}{\det \mathbb{B}} \left[x(z) f^{(2)}_{m_0+2}(z) - y(z) f^{(1)}_{m_0+1}(z) \right], \tag{2.13}$$

$$b(z) = -\frac{a_{m_0-2}}{\det \mathbb{B}} \left[u(z) f^{(2)}_{m_0+2}(z) - v(z) f^{(1)}_{m_0+1}(z) \right], \tag{2.14}$$

for $z \in \overline{\mathbb{C}}_+$ and

$$c(z) = \frac{a_{m_0+1}}{2i\cos\dfrac{z}{2}} \left[u(z) \overline{f^{(2)}_{m_0+2}(z)} - v(z) \overline{f^{(1)}_{m_0+1}(z)} \right], \tag{2.15}$$

$$d(z) = -\frac{a_{m_0+1}}{2i\cos\dfrac{z}{2}} \left[u(z) f^{(2)}_{m_0+2}(z) - v(z) f^{(1)}_{m_0+1}(z) \right], \tag{2.16}$$

for $z \in \mathbb{R} \backslash \{(2k+1)\pi, k \in \mathbb{Z}\}$ where

$$\begin{cases} x(z) = \gamma_{11} Q^{(2)}_{m_0-1}(z) + \gamma_{12} Q^{(1)}_{m_0-2}(z), \\ y(z) = \gamma_{21} Q^{(2)}_{m_0-1}(z) + \gamma_{22} Q^{(1)}_{m_0-2}(z), \\ u(z) = \gamma_{11} P^{(2)}_{m_0-1}(z) + \gamma_{12} P^{(1)}_{m_0-2}(z), \\ v(z) = \gamma_{21} P^{(2)}_{m_0-1}(z) + \gamma_{22} P^{(1)}_{m_0-2}(z). \end{cases} \tag{2.17}$$

Vector sequence $E(z) = \{E_n(z)\}_{n \in \mathbb{N}^*_{m_0}} = \left\{ \begin{pmatrix} E^{(1)}_n(z) \\ E^{(2)}_n(z) \end{pmatrix} \right\}_{n \in \mathbb{N}^*_{m_0}}$ defined by (2.11) is called the Jost solution of the IBVP (1.1)–(1.3). From (2.11), we see that

$$E^{(1)}_0(z) = \frac{b(z)}{a_0}, \quad z \in \overline{\mathbb{C}}_+.$$

The function $b(z)$ (or $E^{(1)}_0(z)$) defined by (2.14) is called the Jost function of the impulsive Dirac system (1.1)–(1.3). Note that the Jost solution and Jost function are analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$ and also

$$E(z + 4\pi) = E(z), \quad z \in \overline{\mathbb{C}}_+,$$
$$b(z + 4\pi) = b(z), \quad z \in \overline{\mathbb{C}}_+.$$

Using (2.17) and Theorem 2.1, we obtain the following theorem:

Theorem 2.3 *The functions $u(z)$ and $v(z)$ satisfy*

$$(i)\ \lim_{|z|\to+\infty} u(z)e^{i(m_0-2)z} = i(-1)^{m_0}\gamma_{11}K_2(m_0-1), \quad z\in\overline{\mathbb{C}}_+,$$

$$(ii)\ \lim_{|z|\to+\infty} v(z)e^{i(m_0-2)z} = i(-1)^{m_0}\gamma_{21}K_2(m_0-1), \quad z\in\overline{\mathbb{C}}_+,$$

where the function $K_2(n)$ was defined by (2.7).

It follows from (2.13)–(2.17) that

$$\begin{cases} \overline{c(z)} = d(z) = \dfrac{a_{m_0+1}\det\mathbb{B}}{2ia_{m_0-2}\cos\dfrac{z}{2}}b(z), \\ x(z)=\overline{x(z)}, \quad y(z)=\overline{y(z)}, \quad u(z)=\overline{u(z)}, \quad v(z)=\overline{v(z)}, \\ a(z+4\pi)=a(z), \quad b(z+4\pi)=b(z), \end{cases} \tag{2.18}$$

for $z\in\mathbb{R}\backslash\{(2k+1)\pi, k\in\mathbb{Z}\}$.

Theorem 2.4 *For $z\in\mathbb{R}\backslash\{(2k+1)\pi, k\in\mathbb{Z}\}$, the following equation holds:*

$$W\left[E_n(z),\overline{E_n(z)}\right] = \begin{cases} -2i\dfrac{a_{m_0-2}}{a_{m_0+1}\det\mathbb{B}}\cos\dfrac{z}{2}, & n\in\mathbb{N}_{m_0} \\ 2i\cos\dfrac{z}{2}, & n\in\mathbb{N}^{m_0}. \end{cases} \tag{2.19}$$

Proof (i) Let $n\in\mathbb{N}_{m_0}$ and $z\in\mathbb{R}\backslash\{(2k+1)\pi, k\in\mathbb{Z}\}$. In this case, from Definition 2.2, we obtain

$$W\left[E_n(z),\overline{E_n(z)}\right] = -2i\Im\left[\overline{a(z)}b(z)\right], \tag{2.20}$$

where $a(z)$ and $b(z)$ were defined by (2.13) and (2.14). If we define

$$T(z) := \left[\overline{f^{(2)}_{m_0+2}(z)}x(z) - \overline{f^{(1)}_{m_0+1}(z)}y(z)\right]\left[f^{(2)}_{m_0+2}(z)u(z) - f^{(1)}_{m_0+1}(z)v(z)\right],$$

then

$$\overline{a(z)}b(z) = -\left(\frac{a_{m_0-2}}{\det\mathbb{B}}\right)^2 T(z).$$

Furthermore, using (2.17), we obtain

$$\begin{aligned} x(z)v(z) =& \gamma_{11}\gamma_{21}P^{(2)}_{m_0-1}Q^{(2)}_{m_0-1} + \gamma_{11}\gamma_{22}P^{(1)}_{m_0-2}Q^{(2)}_{m_0-1} \\ &+ \gamma_{12}\gamma_{21}P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2} + \gamma_{12}\gamma_{22}P^{(1)}_{m_0-2}Q^{(1)}_{m_0-2} \end{aligned}$$

$$\begin{aligned} y(z)u(z) =& \gamma_{11}\gamma_{21}P^{(2)}_{m_0-1}Q^{(2)}_{m_0-1} + \gamma_{12}\gamma_{21}P^{(1)}_{m_0-2}Q^{(2)}_{m_0-1} \\ &+ \gamma_{11}\gamma_{22}P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2} + \gamma_{12}\gamma_{22}P^{(1)}_{m_0-2}Q^{(1)}_{m_0-2} \end{aligned}$$

and so

$$\begin{aligned}\Phi(z) =&2\left(\gamma_{11}\gamma_{21}P^{(2)}_{m_0-1}Q^{(2)}_{m_0-1}+\gamma_{12}\gamma_{22}P^{(1)}_{m_0-2}Q^{(1)}_{m_0-2}\right)\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\\&+\gamma_{11}\gamma_{22}\left(P^{(1)}_{m_0-2}Q^{(2)}_{m_0-1}\overline{f^{(2)}_{m_0+2}}f^{(1)}_{m_0+1}+P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2}f^{(2)}_{m_0+2}\overline{f^{(1)}_{m_0+1}}\right)\\&+\gamma_{12}\gamma_{21}\left(P^{(1)}_{m_0-2}Q^{(2)}_{m_0-1}f^{(2)}_{m_0+2}\overline{f^{(1)}_{m_0+1}}+P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2}\overline{f^{(2)}_{m_0+2}}f^{(1)}_{m_0+1}\right),\end{aligned}$$

where $\Phi(z):=x(z)v(z)\overline{f^{(2)}_{m_0+2}}f^{(1)}_{m_0+1}+y(z)u(z)f^{(2)}_{m_0+2}\overline{f^{(1)}_{m_0+1}}$ and $\Re(z)$ is the real part of z. Therefore,

$$T(z)=x(z)u(z)|f^{(2)}_{m_0+2}|^2+y(z)v(z)|f^{(1)}_{m_0+1}|^2-\Phi(z)$$

is obtained. Using the equation

$$P^{(1)}_{m_0-2}Q^{(2)}_{m_0-1}=\frac{1}{a_{m_0-2}}+P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2},$$

we can write

$$\begin{aligned}\Phi(z) =&2\left(\gamma_{11}\gamma_{21}P^{(2)}_{m_0-1}Q^{(2)}_{m_0-1}+\gamma_{12}\gamma_{22}P^{(1)}_{m_0-2}Q^{(1)}_{m_0-2}\right)\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\\&+\gamma_{11}\gamma_{22}\left[\frac{\overline{f^{(2)}_{m_0+2}}f^{(1)}_{m_0+1}}{a_{m_0-2}}+2P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2}\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\right]\\&+\gamma_{12}\gamma_{21}\left[\frac{\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}}{a_{m_0-2}}+2P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2}\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\right].\end{aligned}$$

Thus, we get

$$\begin{aligned}\Phi(z) =&2\left(\gamma_{11}\gamma_{21}P^{(2)}_{m_0-1}Q^{(2)}_{m_0-1}+\gamma_{12}\gamma_{22}P^{(1)}_{m_0-2}Q^{(1)}_{m_0-2}\right)\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\\&+2\left(\gamma_{11}\gamma_{22}+\gamma_{12}\gamma_{21}\right)P^{(2)}_{m_0-1}Q^{(1)}_{m_0-2}\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)\\&+2\frac{\gamma_{11}\gamma_{22}}{a_{m_0-2}}\Re\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right)-\frac{\det\mathbb{B}}{a_{m_0-2}}\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}.\end{aligned}$$

Since $\Im T(z)=\Im\Phi(z)$, we can write

$$\Im\left[\overline{a(z)}b(z)\right]=\frac{a_{m_0-2}}{\det\mathbb{B}}\Im\left(\overline{f^{(1)}_{m_0+1}}f^{(2)}_{m_0+2}\right),$$

and we have for $n\in\mathbb{N}_{m_0}$

$$W\left[E_n(z),\overline{E_n(z)}\right]=-2i\frac{a_{m_0-2}}{a_{m_0+1}\det\mathbb{B}}\cos\frac{z}{2},$$

by (2.20).

(ii) Let $n\in\mathbb{N}_{m_0}$. The proof is obvious while Definition 2.2 is used. □

Theorem 2.5 *The Jost function of IBVP (1.1)–(1.3) satisfies:*

$$b(z) = e^{3iz}\left[A + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty,$$

where A is a real constant such that $A \neq 0$.

Proof Using (2.14), we can write

$$b(z)e^{-3iz} = \frac{a_{m_0-2}}{\det \mathbb{B}}\left[f^{(1)}_{m_0+1}e^{-i(m_0+1)z}v(z)e^{i(m_0-2)z} - f^{(2)}_{m_0+2}e^{-i(m_0+2)z}u(z)e^{i(m_0-1)z}\right].$$

Thus, we get

$$b(z)e^{-3iz} = \frac{a_{m_0-2}}{\det \mathbb{B}}\gamma_{21}\alpha^{12}_{m_0+1}(-1)^{m_0+1}\left(\prod_{k=1}^{m_0-2} a_{k+1}b_k\right)^{-1} + o(1), \quad |z| \to +\infty,$$

for $z \in \overline{\mathbb{C}}_+$ by Theorem 2.2 and Theorem 2.3. If we define

$$A := \frac{a_{m_0-2}}{\det \mathbb{B}}\gamma_{21}\alpha^{12}_{m_0+1}(-1)^{m_0+1}\left(\prod_{k=1}^{m_0-2} a_{k+1}b_k\right)^{-1},$$

then we find

$$b(z) = e^{3iz}\left[A + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty.$$

□

3. The scattering function

Let us define the semi strips

$$\Pi_0 := \{z : \quad z \in \mathbb{C}, \quad 0 \leq \Re(z) \leq 4\pi, \quad \Im(z) > 0\}$$
$$\Pi := \Pi_0 \cup [0, 4\pi].$$

We will denote the set of all eigenvalues and spectral singularities of the system (1.1)–(1.3) by σ_d and σ_{ss}, respectively. It is obvious that [5,16]

$$\sigma_d := \left\{\lambda : \quad \lambda = 2\sin\frac{z}{2}, \quad z \in \Pi_0, \quad b(z) = 0\right\},$$
$$\sigma_{ss} := \left\{\lambda : \quad \lambda = 2\sin\frac{z}{2}, \quad z \in [0, 4\pi], \quad b(z) = 0\right\} \setminus \{0\}.$$

Theorem 3.1 *For all $z \in (0, 2\pi) \cup (2\pi, 4\pi), b(z) = 0$.*

Proof Let us assume that there exists z_0 in $(0, 2\pi) \cup (2\pi, 4\pi)$ such that $b(z) = 0$. It follows from (2.18) that $c(z_0) = c(z_0) = 0$. Thus, $F_n(z_0) \equiv 0$ for all $n \in \mathbb{N}_{m_0} \cup \{0\}$. It is a contradiction, so $b(z) \neq 0$ for all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$. □

Remark 3.2 *Because $b(z) \neq 0$ for all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$, we can obtain $\sigma_{ss} = \emptyset$.*

Theorem 3.3 *For all* $n \in \mathbb{N}_{m_0}$ *and all* $z \in (0, 2\pi) \cup (2\pi, 4\pi)$, *the equation*

$$2i \cos \frac{z}{2} \frac{a_{m_0-2}}{a_{m_0+1} \det \mathbb{B}} \frac{P_n(z)}{b(z)} = \overline{E_n(z)} - S(z) E_n(z) \tag{3.1}$$

is valid, where $S(z) = \dfrac{\overline{b(z)}}{b(z)}$.

Proof Since $E(z) = \{E_n(z)\}_{n \in \mathbb{N}^*_{m_0}}$ and $\overline{E(z)} = \left\{\overline{E_n(z)}\right\}_{n \in \mathbb{N}^*_{m_0}}$ form the fundamental system of (1.1) for $\lambda = 2 \sin \dfrac{z}{2}$, we have

$$P_n(z) = c_1 E_n(z) + c_2 \overline{E_n(z)}, \quad n \in \mathbb{N}^*_{m_0}, \quad z \in (0, 2\pi) \cup (2\pi, 4\pi), \tag{3.2}$$

where $|c_1| + |c_2| \neq 0$. Using (2.10) and (2.19) we obtain

$$\begin{aligned} W\left[E_n(z), \overline{E_n(z)}\right] &= -2i \frac{a_{m_0-2}}{a_{m_0+1} \det \mathbb{B}} \cos \frac{z}{2}, \\ W\left[E_n(z), P_n(z)\right] &= -b(z), \\ W\left[\overline{E_n(z)}, P_n(z)\right] &= -\overline{b(z)}, \end{aligned}$$

for $n \in \mathbb{N}^*_{m_0}$ and all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$. Thus, we get

$$c_1 = -\frac{a_{m_0+1} \det \mathbb{B}}{2i a_{m_0-2} \cos \frac{z}{2}} \overline{b(z)}, \quad c_2 = -\frac{a_{m_0+1} \det \mathbb{B}}{2i a_{m_0-2} \cos \frac{z}{2}} b(z),$$

and if we take into account these relations,

$$P_n(z) = -\frac{a_{m_0+1} \det \mathbb{B}}{2i a_{m_0-2} \cos \frac{z}{2}} \overline{b(z)} E_n(z) + \frac{a_{m_0+1} \det \mathbb{B}}{2i a_{m_0-2} \cos \frac{z}{2}} b(z) \overline{E_n(z) b(z)}$$

is obtained by (3.2). Finally, since $b(z) \neq 0$ for all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$, we can divide both sides of the last expression by $b(z)$ and obtain equation (3.1). □

The function

$$S(z) = \frac{\overline{b(z)}}{b(z)} = \frac{\overline{E^{(1)}_0(z)}}{E^{(1)}_0(z)}, \quad z \in (0, 2\pi) \cup (2\pi, 4\pi), \tag{3.3}$$

is called the scattering function of the impulsive Dirac system (1.1)–(1.3). It is evident from (2.14) and (3.3) that

$$S(z) = \frac{u(z) \overline{f^{(2)}_{m_0+2}(z)} - v(z) \overline{f^{(1)}_{m_0+1}(z)}}{u(z) f^{(2)}_{m_0+2}(z) - v(z) f^{(1)}_{m_0+1}(z)} = \frac{\overline{E^{(1)}_0(z)}}{E^{(1)}_0(z)}, \quad z \in (0, 2\pi) \cup (2\pi, 4\pi), \tag{3.4}$$

Note that the characteristic properties of the scattering function of the Sturm–Liouville and Dirac equations were investigated in [9,16].

Theorem 3.4 *For all* $z \in (0, 2\pi) \cup (2\pi, 4\pi)$, *the scattering function is continuous and satisfies*

$$S(z) = \left[\overline{S(z)}\right]^{-1}, |S(z)| = 1.$$

The proof of theorem is the direct consequence of (3.3) and (3.4).

4. Unperturbed Dirac systems

Now we consider the following unperturbed discrete Dirac system:

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} = \lambda y_n^{(1)}, \\ y_{n-1}^{(1)} - y_n^{(1)} = \lambda y_n^{(2)}, \quad n \in \mathbb{N}(3), \end{cases} \tag{4.1}$$

with the boundary condition

$$y_0^{(1)}(z) = 0, \tag{4.2}$$

and the impulsive condition

$$\begin{pmatrix} y_4^{(1)} \\ y_5^{(2)} \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} y_2^{(2)} \\ y_1^{(1)} \end{pmatrix}, \tag{4.3}$$

where γ_1 and γ_2 are real numbers such that $\gamma_1\gamma_2 \neq 0$.

Let $\{E_n(z)\}_{n\in\mathbb{N}^*(3)}$ denote the Jost solution of (4.1)–(4.3) for $\lambda = 2\sin\frac{z}{2}$. It is clear that

$$E_n(z) = \begin{cases} a(z)P_n(z) + b(z)Q_n(z), & n = 0, 1, 2 \\ f_n(z), & n = 4, 5, 6, ..., \end{cases} \tag{4.4}$$

where

$$f_n(z) = \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{inz}.$$

In the event that solution (4.4) satisfies condition (4.3), we obtain

$$a(z) = -\frac{a_1 e^{9iz/2}}{a_0\gamma_1\gamma_2}\left[\gamma_1 e^{iz} + (\gamma_2 - \gamma_1)\right],$$

$$b(z) = \frac{a_1 e^{4iz}}{\gamma_1\gamma_2 i}\left[\gamma_1 e^{2iz} + (\gamma_2 - \gamma_1)e^{iz} - (\gamma_2 - \gamma_1)\right],$$

and also, $b(z) \neq 0$ for all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$. On the contrary, there exists a real number z_0 in $(0, 2\pi) \cup (2\pi, 4\pi)$ such that $b(z_0) = 0$. Then

$$\gamma_1 e^{2iz} + (\gamma_2 - \gamma_1)e^{iz} - (\gamma_2 - \gamma_1) = 0,$$

and so, for $a := \dfrac{\gamma_2 - \gamma_1}{2\gamma_1}$

$$z_0 = -i\ln\left(a \pm \sqrt{a^2 - 2a}\right) + 2k\pi, \quad k \in \mathbb{Z},$$

is obtained, but there is a contradiction as $a \pm \sqrt{a^2 - 2a} \neq 1$. Thus, the assumption cannot be true.

The scattering function $S(z)$ of the problem (4.1)–(4.3) is

$$S(z) = -e^{-8iz}\left[\frac{\gamma_1 e^{-2iz} + (\gamma_2-\gamma_1)e^{-iz} - (\gamma_2-\gamma_1)}{\gamma_1 e^{2iz} + (\gamma_2-\gamma_1)e^{iz} - (\gamma_2-\gamma_1)}\right].$$

Since $b(z) \neq 0$ for all $z \in (0, 2\pi) \cup (2\pi, 4\pi)$, there is not spectral singularity of this problem, whereas there are the eigenvalues of this problem if $\left|a \pm \sqrt{a^2-2a}\right| < 1$ such that $a := \dfrac{\gamma_2-\gamma_1}{2\gamma_1}$.

Case 1 $a^2 - 2a < 0$ *:*

In this case, $0 < a < 2$ *must be. If and only if* $\left|a \pm \sqrt{a^2-2a}\right| < 1$ *holds,* $0 < a < \dfrac{1}{2}$ *or* $0 < \dfrac{\gamma_2}{\gamma_1} < 1$ *is obtained. Therefore, the system (4.1)–(4.3) has the eigenvalues if* $0 < \dfrac{\gamma_2}{\gamma_1} < 1$.

Case 2 $a^2 - 2a \geq 0$ *:*

In this case, $a \in (-\infty, 0] \cup [2, +\infty)$ *must be. If and only if* $\left|a \pm \sqrt{a^2-2a}\right| < 1$ *holds,* $\dfrac{3}{2} < \dfrac{\gamma_2}{\gamma_1}$ *is obtained. Therefore, the system (4.1)–(4.3) has the eigenvalues if* $\dfrac{\gamma_2}{\gamma_1} > \dfrac{3}{2}$.

Finally, the unperturbed discrete Dirac system has the eigenvalues if $0 < \dfrac{\gamma_2}{\gamma_1} < 1$ or $\dfrac{\gamma_2}{\gamma_1} > \dfrac{3}{2}$.

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