Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 3204 – 3210
© TÜBİTAK
doi:10.3906/mat-1806-22

Ellipses and similarity transformations with norm functions

Nihal Yılmaz ÖZGÜR*

Department of Mathematics, Faculty of Arts and Sciences, Balıkesir University, Balıkesir, Turkey

Received: 04.06.2018 • **Accepted/Published Online:** 20.10.2018 • **Final Version:** 27.11.2018

Abstract: In this paper, we deal with a conjecture related to the images of ellipses (resp. circles) under similarities that are the special Möbius transformations. We consider ellipses (resp. circles) corresponding to any norm function (except in the Euclidean case) on the complex plane and examine some conditions to confirm this conjecture. Some illustrative examples are also given.

Key words: Möbius transformation, ellipse, norm

1. Introduction

Images of circles and ellipses (corresponding to the Euclidean norm or to another norm function on the complex plane $\mathbb{C}$) have been studied extensively under some special transformations such as Möbius transformations or harmonic Möbius transformations (see [1–16] and the references therein). Let us consider the real linear space structure of the complex plane $\mathbb{C}$. In [14], the present author proved that the image of any ellipse $E_r(F_1, F_2) = \{z \in \mathbb{C} : \|z - F_1\| + \|z - F_2\| = r\}$ corresponding to any norm function $\|.\|$ (except in the Euclidean case) on $\mathbb{C}$ under the similarity transformation $w = f(z) = \alpha z + \beta;\ \alpha \neq 0,\ \alpha, \beta \in \mathbb{C}$ (which is a special Möbius transformation) is an ellipse corresponding to the same norm function or corresponding to the norm function

$$\|z\|_\phi = \left\|e^{-i\phi} z\right\|, \tag{1.1}$$

where $\phi = \arg(\alpha)$. For a given norm function $\|.\|$, the functions $\|z\|_\phi$ define new norms for every real number ϕ. Clearly, for the Euclidean norm, all of the norm functions $\|.\|_\phi$ are equal to the Euclidean norm. For any other norm function, we have $\|.\|_{k\pi} = \|.\|$ where $k \in \mathbb{Z}$, but we do not know the exact values of ϕ for which $\|.\|_\phi = \|.\|$ for any norm except in the Euclidean case. This was left an open problem in [12, 14] regarding determination of the images of circles (resp. ellipses) corresponding to any norm function $\|.\|$ except the Euclidean norm. If $\|.\|_\phi = \|.\|$, then the transformation $f(z) = \alpha z + \beta$ maps circles (resp. ellipses) to circles (resp. ellipses) corresponding to this norm function.

The rotation transformation $f(z) = e^{i\phi} z$ does not map circles (resp. ellipses) to circles (resp. ellipses) corresponding to the same norm function in general. For example, let $\|.\|$ be any norm with $\|1\| \neq \|i\|$ and $\phi = \frac{\pi}{2}$. From [12, 14] we know that the transformation $z \to e^{\frac{\pi}{2} i} z$ maps circles (resp. ellipses) corresponding to

*Correspondence: nihal@balikesir.edu.tr

2010 *AMS Mathematics Subject Classification:* Primary 30C35; Secondary 51F15

this norm function $\|.\|$ to circles (resp. ellipses) corresponding to the norm function $\|.\|_{\frac{\pi}{2}}$ (see [12, 14] for more details). In [14], the following conjecture was posed on the images of ellipses (resp. circles) under the rotation transformation $z \to e^{\frac{\pi}{2}i}z$.

Conjecture 1.1 *[14] Let $\|.\|$ be any norm on $\mathbb{C}$ with $\|1\| = \|i\|$. Assume that $\|z\| = \|\overline{z}\|$ for all $z \in \mathbb{C}$. Then we have $\|.\|_{\frac{\pi}{2}} = \|.\|$ and hence the transformation $z \to e^{\frac{\pi}{2}i}z$ maps ellipses to ellipses corresponding to this norm function.*

In this paper, we determine some conditions to confirm Conjecture 1.1. We exactly solve the problem for which as values of ϕ we have $\|.\|_{\phi} = \|.\|$ for a given norm function $\|.\|$ (except in the Euclidean case) on $\mathbb{C}$. Consequently, we finish the determination of images of ellipses (resp. circles) under the similarity transformations. We also give some illustrative examples and some figures drawn with Mathematica [17].

2. Proof of Conjecture 1.1 and related results

Let $r > 0$ be any fixed real number. Notice that the function

$$\|z\| = r\,|z| \tag{2.1}$$

defines a new norm on $\mathbb{C}$ for every $r > 0$. Clearly, circles (resp. ellipses) of this new norm are the Euclidean circles (resp. ellipses). We will call these cases the trivial cases.

From now on we assume that $\|.\|$ is any norm function except the trivial cases and $\phi \neq k\pi$, $k \in \mathbb{Z}$. We note that the equations

$$\left\|e^{i\phi}\right\| = \|1\| \tag{2.2}$$

and

$$\left\|e^{i\phi}\right\| = \left\|e^{-i\phi}\right\| \tag{2.3}$$

should be satisfied if $\|.\|_{\phi} = \|.\|$. For $\phi = \frac{\pi}{2}$ we have $\|1\| = \|i\|$ by (2.2) and this is a necessary condition in Conjecture 1.1.

For a fixed norm function, there exists at least one value of ϕ such that equation (2.2) is not satisfied and so $\|.\|_{\phi} \neq \|.\|$. Otherwise, this norm function is reduced to a trivial norm.

We recall the following definition.

Definition 2.1 *[15] We say that a norm function $\|.\|$ defined on $\mathbb{C}$ has Property $\mathcal{C}$ if it satisfies the property $\|z\| = \|\overline{z}\|$ for all $z \in \mathbb{C}$.*

At first, we begin by the following example.

Example 2.2 *Let us consider the norm function*

$$\|z\| = |x + 2y| + |x - 2y| + 2\,|x| \tag{2.4}$$

on $\mathbb{C}$. Clearly we have $\|1\| = \|i\| = 4$ and this norm function has Property $\mathcal{C}$. The image of the ellipse $E_{10}(-1, 1)$ under the transformation $w = e^{\frac{\pi}{2}i}z$ is not an ellipse corresponding to the same norm but it is the *ellipse* $E_{10}(-i,i)$ *corresponding to the norm function* $\|z\|_{\frac{\pi}{2}} = |2x+y| + |2x-y| + 2|y|$ (*see Figure 1*). *We have* $\|.\|_{\frac{\pi}{2}} \neq \|.\|$ *and hence the transformation* $z \to e^{\frac{\pi}{2}i}z$ *does not map ellipses to ellipses corresponding to the norm function defined in* (2.4).

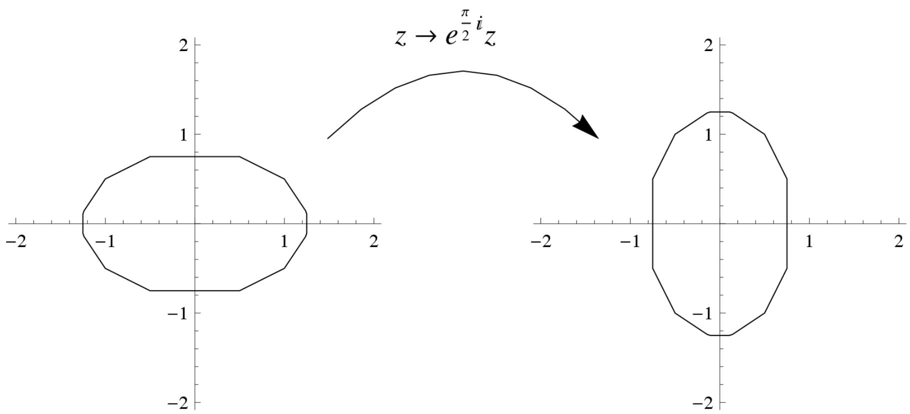

Figure 1.

We have seen that the norm function defined in (2.4) does not satisfy Conjecture 1.1 and so Property $\mathcal{C}$ and the condition $\|1\| = \|i\|$ are not sufficient conditions for $\|.\|_{\frac{\pi}{2}} = \|.\|$.

Let us consider the following property.

Definition 2.3 *We say that a norm function* $\|.\|$ *defined on* $\mathbb{C}$ *has Property* $\mathcal{D}$ *if it satisfies the property*

$$\|x+iy\| = \|y+ix\|,$$

for all $x, y \in \mathbb{R}$.

Clearly, Property $\mathcal{D}$ implies the condition $\|1\| = \|i\|$. In the following example, we see that Property $\mathcal{D}$ is not a sufficient condition for $\|.\|_{\frac{\pi}{2}} = \|.\|$.

Example 2.4 *Let us consider the norm function*

$$\|z\| = |x| + |y| + |x+y| \tag{2.5}$$

on $\mathbb{C}$. *Clearly, this norm function has Property* $\mathcal{D}$. *Also, this norm function does not satisfy Property* $\mathcal{C}$. *The image of the ellipse* $E_6(-1,1)$ *under the transformation* $w = e^{\frac{\pi}{2}i}z$ *is not an ellipse corresponding to the same norm but it is the ellipse* $E_6(-i,i)$ *corresponding to the norm function* $\|z\|_{\frac{\pi}{2}} = |x| + |y| + |y-x|$ (*see Figure 2*). *We have* $\|.\|_{\frac{\pi}{2}} \neq \|.\|$ *and hence the transformation* $z \to e^{\frac{\pi}{2}i}z$ *does not map ellipses to ellipses corresponding to the norm function defined in* (2.5).

Now let us give the following theorem.

Theorem 2.5 *Let* $\|.\|$ *be any norm on* $\mathbb{C}$ *with Property* $\mathcal{D}$ (*resp. Property* $\mathcal{C}$). *Then* $\|.\| = \|.\|_{\frac{\pi}{2}}$ *if and only if the norm function* $\|.\|$ *satisfies Property* $\mathcal{C}$ (*resp. Property* $\mathcal{D}$).

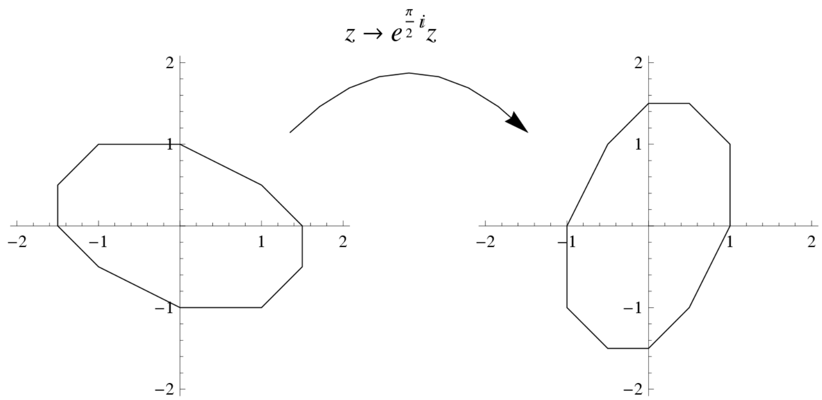

Figure 2.

Proof Let $\|.\| = \|.\|_{\frac{\pi}{2}}$. By (1.1), we have

$$\|x+iy\| = \|i(x+iy)\|,$$

for all $z = x+iy \in \mathbb{C}$. Then, using Property $\mathcal{D}$ and this last equation, we have

$$\|x+iy\| = \|y+ix\| = \|i(x-iy)\| = \|x-iy\|.$$

This shows that the norm function $\|.\|$ satisfies Property $\mathcal{C}$.

Conversely, let the norm function $\|.\|$ satisfy Property $\mathcal{C}$. Using Property $\mathcal{D}$ and Property $\mathcal{C}$ we have

$$\|x+iy\| = \|y+ix\| = \|i(x-iy)\| = \|-i(x+iy)\| = \|i(x+iy)\|$$

and so $\|z\| = \|z\|_{\frac{\pi}{2}}$ for all $z = x+iy \in \mathbb{C}$. □

Corollary 2.6 *Let $\|.\|$ be any norm on $\mathbb{C}$. If the norm function $\|.\|$ satisfies Property $\mathcal{C}$ and Property $\mathcal{D}$ then we have $\|.\| = \|.\|_{\frac{\pi}{2}}$.*

The converse question of this corollary is not always true, as we have seen in the following example.

Example 2.7 *Let us consider the norm function*

$$\|z\| = |2x-y| + |x+2y| \tag{2.6}$$

on $\mathbb{C}$. Clearly this norm function does not satisfy both Property $\mathcal{D}$ and Property $\mathcal{C}$, but we have $\|.\|_{\frac{\pi}{2}} = \|.\|$ and hence the transformation $z \to e^{\frac{\pi}{2}i}z$ maps ellipses to ellipses corresponding to this norm function.

Now we consider the general case for any real number ϕ and define a new property.

Definition 2.8 *For a fixed real number ϕ, we say that a norm function $\|.\|$ defined on $\mathbb{C}$ has Property $\mathcal{E}^{\phi}$ if it satisfies the property*

$$\left\|e^{i(\theta-\phi)}\right\| = \left\|e^{i\theta}\right\|, \tag{2.7}$$

for all $\theta \in \mathbb{R}$.

Now let us give the following theorem.

Theorem 2.9 *Let $\|.\|$ be any norm on $\mathbb{C}$. Then $\|.\| = \|.\|_{\phi}$ for a real number ϕ if and only if the norm function $\|.\|$ satisfies Property $\mathcal{E}^{\phi}$.*

Proof Let $\|.\| = \|.\|_{\phi}$. By (1.1), we have

$$\left\|e^{-i\phi}z\right\| = \|z\|,$$

for all $z = x + iy \in \mathbb{C}$. If we use the polar representation $z = |z|\, e^{i\theta}$ where $\theta = \arg(z)$, we have

$$\left\|e^{-i\phi}\,|z|\,e^{i\theta}\right\| = \left\||z|\,e^{i\theta}\right\|$$

and hence

$$\left\|e^{i(\theta-\phi)}\right\| = \left\|e^{i\theta}\right\|.$$

This shows that the norm function $\|.\|$ satisfies Property $\mathcal{E}^{\phi}$.

Conversely, let the norm function $\|.\|$ satisfies Property $\mathcal{E}^{\phi}$. For any complex number $z = |z|\, e^{i\theta}$ $(\theta = \arg(z))$, from (2.7) we have

$$\left\|e^{i(\theta-\phi)}\right\| = \left\|e^{i\theta}\right\|$$

and

$$\left\|e^{-i\phi}\,|z|\,e^{i\theta}\right\| = \left\||z|\,e^{i\theta}\right\|.$$

Then we obtain $\left\|e^{-i\phi}z\right\| = \|z\|$ for all $z = x + iy \in \mathbb{C}$ and so $\|z\|_{\phi} = \|z\|$. □

Combining Theorem 2.9 and Theorem 2.1 given in [14] on page 191, we can give the following corollaries.

Corollary 2.10 *Let $w = f(z) = \alpha z + \beta$; $\alpha \neq 0$, $\alpha, \beta \in \mathbb{C}$ be a similarity transformation and $\phi = \arg(\alpha)$. If the norm function $\|.\|$ satisfies Property $\mathcal{E}^{\phi}$ then the similarity transformation $f(z) = \alpha z + \beta$ maps ellipses (resp. circles) to ellipses (resp. circles) corresponding to this norm function.*

Corollary 2.11 *Let $\|.\|$ be any norm on $\mathbb{C}$ satisfying Property $\mathcal{E}^{\phi}$ for some real number ϕ. Then the transformation $f(z) = e^{i\phi}z$ maps ellipses (resp. circles) to ellipses (resp. circles) corresponding to this norm function.*

Remark 2.12 *For $\phi = \frac{\pi}{2}$, equation (2.7) becomes*

$$\left\|ie^{i\theta}\right\| = \left\|e^{i\theta}\right\|, \tag{2.8}$$

for all $\theta \in \mathbb{R}$. In Example 2.7 we have seen an example of a norm that does not satisfy both Property $\mathcal{D}$ and Property $\mathcal{C}$ but satisfies Property $\mathcal{E}^{\frac{\pi}{2}}$ (and hence we have $\|.\|_{\frac{\pi}{2}} = \|.\|$).

Example 2.13 *For $\phi = \frac{\pi}{4}$, both of the norm functions defined by*

$$\|z\|_1 = \sqrt{2}\,|2x - y| + |3x + y| + \sqrt{2}\,|x + 2y| + |x - 3y| \tag{2.9}$$

and

$$\|z\|_2 = \left|\frac{1}{\sqrt{2}}(x-y)\right| + \left|\frac{1}{\sqrt{2}}(x+y)\right| + |x| + |y| \tag{2.10}$$

have Property $\mathcal{E}^{\phi}$ *and hence the transformation* $f(z) = e^{i\frac{\pi}{4}}z$ *maps ellipses (resp. circles) to ellipses (resp. circles) corresponding to these norm functions. We note that the norm function defined in (2.9) does not satisfy both Property* $\mathcal{C}$ *and Property* $\mathcal{D}$ *while the norm function defined in (2.10) has both Property* $\mathcal{C}$ *and Property* $\mathcal{D}$*.*

For a given norm function, the values of ϕ such that $\|z\|_{\phi} = \|z\|$ can be easily determined using equations (2.2) and (2.3). Finally, we give some illustrative examples.

Example 2.14 *Let us consider the norm function defined by*

$$\|z\| = \sqrt{\frac{(x+y)^2}{9} + 4(x-y)^2} \tag{2.11}$$

(see Example 2.1 on page 192 in [15]). By equation (2.2), we have

$$\left\|e^{i\phi}\right\| = \|1\| \Rightarrow -70\cos\phi\sin\phi = 0$$

and so $\cos\phi = 0$ *or* $\sin\phi = 0$*. It can be easily checked that* $\left\|e^{i(\theta-\phi)}\right\| = \left\|e^{i\theta}\right\|$ *for all* $\theta \in \mathbb{R}$ *if and only if* $\phi = k\pi$*,* $k \in \mathbb{Z}$*. Hence, this norm function does not satisfy Property* $\mathcal{E}^{\phi}$ *for any* $\phi \neq k\pi$*,* $k \in \mathbb{Z}$ *and so the transformation* $f(z) = e^{i\phi}z$ *does not map ellipses (resp. circles) to ellipses (resp. circles) corresponding to this norm function. From [12, 14], we know that the transformation* $f(z) = e^{i\phi}z$ *maps ellipses (resp. circles) corresponding to the norm function defined in (2.11) to the ellipses (resp. circles) corresponding to the norm function*

$$\begin{aligned} \|z\|_{\phi} &= \left\|e^{-i\phi}z\right\| \\ &= \sqrt{\frac{\left[(x+y)\cos\phi + (y-x)\sin\phi\right]^2}{9} + 4\left[(x-y)\cos\phi + (x+y)\sin\phi\right]^2}. \end{aligned}$$

Example 2.15 *Let us consider the norm function defined by*

$$\|z\| = |x| + |y|. \tag{2.12}$$

By equation (2.2), we have

$$|\cos\phi| + |\sin\phi| = 1 \Rightarrow 2\left|\cos\phi\sin\phi\right| = 0$$

and so we find $\cos\phi = 0$ *or* $\sin\phi = 0$*. Then it can be easily checked that equation (2.7) is satisfied for* $\phi = \frac{(2k+1)\pi}{2} (k \in \mathbb{Z})$*. Hence, we deduce that the rotation transformation* $f(z) = e^{i\phi}z$ *maps ellipses (resp. circles) to ellipses (resp. circles) corresponding to this norm function if and only if* $\phi = \frac{(2k+1)\pi}{2}$ *or* $\phi = k\pi$ $(k \in \mathbb{Z})$*.*

References

[1] Aczel J, McKiernan MA. On the characterization of plane projective and complex Moebius-transformations. Math Nachr 1967; 33: 315-337.

[2] Blair D. Inversion Theory and Conformal Mapping. Student Mathematical Library, 9. Providence, RI, USA: American Mathematical Society, 2000.

[3] Caratheodory C. The most general transformations of plane regions which transform circles into circles. B Am Math Soc 1937; 43: 573-579.

[4] Chuaqui M, Duren P, Osgood B. The Schwarzian derivative for harmonic mappings. J Anal Math 2003; 91: 329-351.

[5] Chuaqui M, Duren P, Osgood B. Ellipses, near ellipses, and harmonic Möbius transformations. P Am Math Soc 2005; 133: 2705-2710.

[6] Coffman A, Frantz M. Ellipses in the inversive plane. In: MAA Indiana Section Meeting, 2003.

[7] Coffman A, Frantz M. Möbius transformations and ellipses. Pi Mu Epsilon Journal 2007; 12: 339-345.

[8] Haruki H. A proof of the principle of circle-transformations by use of a theorem on univalent functions. Lenseign Math 1972; 18: 145-146.

[9] Haruki H, Rassias TM. A new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping. J Math Anal Appl 1994; 181: 320-327.

[10] Jones GA, Singerman D. Complex Functions. An Algebraic and Geometric Viewpoint. Cambridge, UK: Cambridge University Press, 1987.

[11] Ledet A, Solynin AY. Conformal mapping and ellipses. P Am Math Soc 2006; 134: 3507-3513.

[12] Özgür NY. On some mapping properties of Möbius transformations. Aust J Math Anal Appl 2009; 6: 13.

[13] Özgür NY. Ellipses and harmonic Möbius transformations. An Şt Univ Ovidius Constanta 2010; 18: 201-208.

[14] Özgür NY. On the images of ellipses under similarities. An Şt Univ Ovidius Constanta 2013; 21: 189-194.

[15] Özgür NY. On the circle preserving property of Möbius transformations. In: Rassias TM, editor. Mathematics Without Boundaries. Surveys in Pure Mathematics. New York, NY, USA: Springer, 2014, pp. 397-413.

[16] Sönmez N. Hyperbolic ellipses and Möbius transformations. Ars Combin 2012; 107: 289-294.

[17] Wolfram Research, Inc. Mathematica, Trial Version. Champaign, IL, USA: Wolfram, 2016.