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Linear methods of summing Fourier series and approximation in weighted Orlicz spaces

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Abstract: In the present work, we investigate estimates of the deviations of the periodic functions from the linear operators constructed on the basis of its Fourier series in reflexive weighted Orlicz spaces with Muckenhoupt weights. In particular, the orders of approximation of Zygmund and Abel-Poisson means of Fourier trigonometric series were estimated by the $k-th$ modulus of smoothness in reflexive weighted Orlicz spaces with Muckenhoupt weights.

Key words: Boyd indices, weighted Orlicz space, Muckenhoupt weight, modulus of smoothness, Zygmund mean, Abel-Poisson mean

1. Introduction

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of* M *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant c, independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\,dx < \infty.$$

Let N be the complementary Young function of M. It is well known [16,29] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int_{\mathbb{T}} N(|g(x)|)\,dx \leq 1\right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [16]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$.

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If we choose $M(u) = u^p/p$ $(1 < p < \infty)$, then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p} \|u\|_{L_p(\mathbb{T})} \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q} \|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p \, dx \right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})$ space.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [29].

Let $M^{-1} : [0, \infty) \to [0, \infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices*

$$\alpha_M := \lim_{t \to +\infty} -\frac{\log h(t)}{\log t}, \ \beta_M := \lim_{t \to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0, \infty) \to (0, \infty], \quad h(t) := \lim_{y \to \infty} \sup \frac{M^{-1}(y)}{M^{-1}(ty)}, \quad t > 0,$$

first considered by Matuszewska and Orlicz [24], are called the *Boyd indices* of the Orlicz spaces $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0 \leq \alpha_M \leq \beta_M \leq 1, \alpha_N + \beta_M = 1,$ $\alpha_M + \beta_N = 1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \leq \beta_M < 1$. The detailed information about the Boyd indices can be found in [1,2,19,25].

A measurable function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. With any given weight ω, we associate the ω-*weighted Orlicz space* $L_M(\mathbb{T}, \omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1 < p < \infty,$ $1/p + 1/p' = 1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's* A_p*-condition* on $\mathbb{T}$ [4,5,10] if

$$\sup_J \left(\frac{1}{|J|} \int\limits_J \omega^p(t) \, dt \right)^{1/p} \left(\frac{1}{|J|} \int\limits_J \omega^{-p'}(t) \, dt \right)^{1/p'} < \infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us denote by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Note that by [20, Lemma 3.3] and [21, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T}, \omega)$ is also reflexive.

Let $L_M(\mathbb{T}, \omega)$ be a weighted Orlicz space, let $0 < \alpha_M \leq \beta_M < 1$ and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f \in L_M(\mathbb{T}, \omega)$, we set

$$(\nu_h f)(x) := \frac{1}{2h} \int\limits_{-h}^{h} f(x+t) \, dt, \ 0 < h < \pi, \ x \in \mathbb{T}.$$

By reference [10, Lemma 1] the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\omega)} \le C\|f\|_{L_M(\mathbb{T},\omega)}.$$

The function

$$\Omega^k_{M,\omega}(\delta,f) := \sup_{\substack{0<h_i\le\delta \\ 1\le i\le k}} \left\| \prod_{i=1}^{k} (I-\nu_{h_i}) f \right\|_{L_M(\mathbb{T},\omega)}, \ \delta>0, \ k=1,2,...$$

is called k*-th modulus of smoothness* of $f\in L_M(\mathbb{T},\omega)$, where I is the identity operator.

It can easily be shown that $\Omega^k_{M,\omega}(\cdot,f)$ is a continuous, nonnegative, and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega^k_{M,\omega}(\delta,f)=0, \ \Omega^k_{M,\omega}(\delta,f+g)\le \Omega^k_{M,\omega}(\delta,f)+\Omega^k_{M,\omega}(\delta,g), \ \delta>0$$

for $f,g\in L_M(\mathbb{T},\omega)$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x,f) \tag{1.1}$$

be the Fourier series of the function $f\in L_1(\mathbb{T})$, where $A_k(x,f):=(a_k(f)\cos kx+b_k(f)\sin kx)$, $\alpha_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f\in L_1(\mathbb{T})$.

We suppose that $\{\lambda_\nu^{(n)}\}$ $(\lambda_0^{(n)}=1;\ \lambda_\nu^{(n)}=0,\ \nu>n)$ are system of numbers and we consider the sequence of the functions $\{\lambda_\nu(r)\}$ defined in the set $E\subset\mathbb{R}$, satisfying the conditions that

$$\lambda_0(r)=1, \ \lim_{r\longrightarrow r_0}\lambda_\nu(r)=1$$

for an arbitrary fixed $r_0\in E$ and $\nu=0,1,2,....$, we set

$$\begin{aligned} R_n(f,\lambda_\nu^{(n)})(x) &= f(x)-[\frac{a_0}{2}+\sum_{\nu=1}^{n}\lambda_\nu^{(n)}A_\nu(x,f)], \\ R_r(f,\lambda_\nu(r))(x) &= f(x)-[\frac{a_0}{2}+\sum_{\nu=1}^{\infty}\lambda_\nu(r)A_\nu(x,f)]. \end{aligned}$$

The *Zygmund means* and *Abel-Poisson means* of the series (1.1) are defined respectively as [5,32]

$$\begin{aligned} Z_{n,k}(x,f) &= \frac{a_0}{2}+\sum_{\nu=1}^{n}(1-\frac{\nu^k}{(n+1)^k})A_\nu(x,f), \quad n=0,1,2,..., \ k=1,2,..., \\ U_r(x,f) &= \frac{a_0}{2}+\sum_{\nu=1}^{\infty}r^\nu A_\nu(x,f), \ 0\le r<1. \end{aligned}$$

The best approximation of $f \in L_M(\mathbb{T},\omega)$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{M,\omega} := \inf\left\{\|f - T_n\|_{L_M(\mathbb{T},\omega)} : T_n \in \Pi_n\right\}.$$

Note that the existence of $T_n^* \in \Pi_n$ such that

$$E_n(f)_{M,\omega} = \|f - T_n^*\|_{L_M(\mathbb{T},\omega)}$$

follows, for example, from Theorem 1.1 in [7].

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest

We need the following theorem [5]:

Theorem 1.1 *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. Let $\{\lambda_k\}_0^\infty$ be a sequence of numbers such that*

$$|\lambda_k| \le c_1 \quad \text{and} \quad \sum_{k=2^{m-1}}^{2^m-1} |\lambda_k - \lambda_{k+1}| \le c_2, \tag{1.2}$$

where $c_2 > 0$ does not depend on k and m. If $f \in L_M(\mathbb{T},\omega)$ has the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f),$$

then there exists a function $F \in L_M(\mathbb{T},\omega)$ with the Fourier series

$$\frac{\lambda_0 a_0}{2} + \sum_{k=1}^{\infty} \lambda_k A_k(x,f)$$

and

$$\| F \|_{L_M(\mathbb{T},\omega)} \le c_3 \| f \|_{L_M(\mathbb{T},\omega)}.$$

2. Main results

The problems of approximation theory in weighted and nonweighted Lebesgue spaces, as well as weighted and nonweighted Orlicz spaces were investigated by several authors (see, for example, [3–5,7–15,17,18,22,23,26–28,30–34]).

Note that the approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $A_p(\mathbb{T})$ were studied in [4,22,23].

Detailed information on weighted polynomial approximation can be found in [6,26].

In the present paper, we estimate the norms of $R_n(f,\lambda)$ and $R_r(f,\lambda)$ in the weighted Orlicz spaces $L_M(\mathbb{T},\omega)$. Similar problems in different spaces were investigated in [3,5,15,18,27,30,32–34].

Our main results are the following.

Theorem 2.1 *Let $L_M(\mathbb{T},\omega)$ be a reflexive Orlicz space and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. If the system of numbers $\{\lambda_\nu^{(n)}\}$, $(\lambda_0^{(n)} = 1;\ \lambda_\nu^{(n)} = 0,\ \nu > n)$ for some natural number k satisfy the conditions*

$$n^{2k}\left|1-\lambda_\nu^{(n)}\right| \leq c_4\nu^{2k}, \qquad \sum_{\nu=2^{\mu-1}}^{2^\mu-1}\frac{n^{2k}\mid\Delta\lambda_\nu^{(n)}\mid}{\nu^{2k}} \leq c_5,\ (\mu=1,2,...,m;\ n\geq 2^m), \tag{2.1}$$

where $\mid\Delta\lambda_\nu^{(n)}\mid = \left|\lambda_\nu^{(n)} - \lambda_{\nu+1}^{(n)}\right|$, then for $f \in L_M(\mathbb{T},\omega)$ the estimate

$$\begin{aligned}\left\|R_n(f,\lambda_\nu^{(n)})\right\|_{L_M(\mathbb{T},\omega)} &= \| f(x) - [\frac{a_0}{2} + \sum_{\nu=1}^{n}\lambda_\nu^{(n)}A_\nu(x,f)] \|_{L_M(\mathbb{T},\ \omega)} \\ &\leq c_6\Omega_{M,\omega}^k(\frac{1}{n+1},f) \end{aligned}\tag{2.2}$$

holds with a constant $c_6 > 0$ and does not depend on n.

Corollary 2.1 *Let $L_M(\mathbb{T},\omega)$ be a reflexive Orlicz space and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. If $\lambda_\nu^{(n)} = 1 - \frac{\nu^k}{(n+1)^k}$, $k \geq 1$, $\lambda_\nu^{(n)} = 0$, $\nu > n$,* then *for $f \in L_M(\mathbb{T},\omega)$ the estimate*

$$\| f - Z_{n,k}(\ \cdot\ ,f) \|_{L_M(\mathbb{T},\omega)} \leq c_7\Omega_{M,\omega}^k(\frac{1}{n+1},f),$$

holds with a constant $c_7 > 0$ and does not depend on n.
Note that Corollary 2.1 for Zygmund means of order 2 and modulus of continuity $\Omega_{M,\omega}(\frac{1}{n+1},f)$, $(k=1)$ was obtained in [5].

Theorem 2.2 *Let $L_M(\mathbb{T})$ be an Orlicz space and $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. If the sequence of functions $\{\lambda_\nu(r)\}$ defined in the set $E \subset \mathbb{R}$, such that $\lambda_0(r) = 1$, $\lim\limits_{r\to r_0}\lambda_\nu(r) = 1$, for the arbitrary fixed $\nu = 0,1,2,...$ satisfying the conditions*

$$\mid\lambda_\nu(r)\mid \leq c_8, \qquad \sum_{\nu=2^{\mu-1}}^{2^\mu-1}\mid\lambda_\nu(r) - \lambda_{\nu+1}(r)\mid \leq c_9, (\mu = 1,2,...) \tag{2.3}$$

and for some natural number k

$$\frac{[1-\lambda_\nu(r)]}{\mid r-r_0\mid^{2k}} \leq c_{10}\nu^k, \quad \frac{1}{\mid r-r_0\mid^{2k}}\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\frac{|\Delta\lambda_\nu(r)|}{\nu^{2k}} \leq c_{11} \tag{2.4}$$

and further for any fixed $r \in E$, for the arbitrary $f \in L_M(\mathbb{T},\omega)$, the series

$$\frac{a_0}{2} + \sum_{\nu=1}^{\infty}\lambda_\nu(r)A_\nu(x,f)$$

converges in the space $L_M(\mathbb{T},\omega)$*, then the estimate*

$$\begin{aligned}\|R_r(f,\lambda_\nu(r))\|_{L_M(\mathbb{T},\omega)} &= \| f(x)-[\frac{a_0}{2}+\sum_{\nu=1}^{\infty}\lambda_\nu(r)A_\nu(x,f)]\|_{L_M(\mathbb{T},\ \omega)} \\ &\leq c_{12}\Omega^k_{M,\omega}(|\ r-r_0\ |,f). \qquad (2.5)\end{aligned}$$

holds.

Corollary 2.2. *Let* $L_M(\mathbb{T})$ *be an Orlicz space and* $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$*. If* $\lambda_\nu(r)=r^\nu$ $(\nu=0,1,2,...)$*, then for* $f\in L_M(\mathbb{T},\omega)$*, the estimate*

$$\| f-U_r(x,f)\|_{L_M(\mathbb{T},\omega)}\leq c_{13}\Omega^k_{M,\omega}(1-r,f),\ 0\leq r\leq 1,$$

holds with a constant $c_{13}>0$ *and does not depend on* r*.*

Note that similar estimate for modulus of continuity $\Omega_{M,\omega}(\frac{1}{n+1},f)$, $(k=1)$ was proved in [5].

3. Proofs of the theorems

Proof of Theorem 2.1 Let $2^m\leq n<2^{m+1}$. Using the subadditivity of the norm, we get

$$\begin{aligned}&\| \ f-\sum_{\nu=0}^{n}\lambda_\nu^{(n)}A_\nu(\ \cdot\ ,f)\ \|_{L_M(\mathbb{T},\omega)} \\ \leq\ &\|\sum_{\nu=0}^{n}(1-\lambda_\nu^{(n)})A_\nu(\ \cdot\ ,f)\ \|_{L_M(\mathbb{T},\omega)} \\ +\ &\| \sum_{\nu=n+1}^{\infty}A_\nu(\ \cdot\ ,f)\ \|_{L_M(\mathbb{T},\omega)}=I_1+I_2. \qquad (3.1)\end{aligned}$$

We put

$$\mu^{(n)}_{\nu,k}=\frac{1-\lambda_\nu^{(n)}}{\sin^{2k}\frac{\nu}{2n}},\ \nu=1,2,...,n.$$

We show that, for the sequence $\{\mu^{(n)}_{\nu,k}\}$, the conditions (1.2) of Theorem 1.1 are satisfied. Under the assumptions of Theorem 1.1, the inequality

$$|\ \mu^{(n)}_{\nu,k}\ |\leq(2\pi)^{2k}\frac{n^{2k}\left|1-\lambda_\nu^{(n)}\right|}{\nu^{2k}}\leq c_{14}$$

holds.

On the other hand, the following inequality can be written

$$\begin{aligned}
\mid \quad \mu_{\nu,k}^{(n)} - \mu_{\nu+1,k}^{(n)} \mid &=\mid \frac{\lambda_{\nu+1}^{(n)} \sin^{2k} \frac{\nu}{2n} - \lambda_{\nu}^{(n)} \sin^{2k} \frac{\nu+1}{2n}}{\sin^{2k} \frac{\nu}{2n} \sin^{2k} \frac{\nu+1}{2n}} \mid \\
&\leq (2n\pi)^{4k} \mid \frac{(\lambda_{\nu+1}^{(n)} - \lambda_{\nu}^{(n)}) \sin^{2k} \frac{(\nu+1)}{2n} - \lambda_{\nu+1}^{(n)} [\sin^{2k} \frac{(\nu+1)}{2n} - \sin^{2k} \frac{\nu}{2n}]}{\nu^{2k}(\nu+1)^{2k}} \mid \\
&\leq c_{15} \left(\frac{n^{2k} \mid \lambda_{\nu+1}^{(n)} - \lambda_{\nu}^{(n)} \mid}{\nu^{2k}} + \frac{n^{2k} \mid \lambda_{\nu+1}^{(n)} \mid}{\nu^{2k}} \right). \qquad (3.2)
\end{aligned}$$

According to (3.2) and (1.2) we have, for $s \geq m$,

$$\sum_{\nu=2^s}^{2^{s+1}-1} \mid \mu_{\nu,r}^{(n)} - \mu_{\nu+1,r}^{(n)} \mid \leq c_{16},$$

where m such that $2^m \leq n < 2^{m+1}$.

Therefore, the conditions (1.2) of Theorem 1.1 are satisfied. Then using Theorem 1.1, we obtain

$$\begin{aligned}
I_1 &= \| \sum_{\nu=1}^{n} \mu_{\nu,r}^{(n)} A_\nu(\,\cdot\,, f) \sin^{2k} \frac{\nu}{2n} \|_{L_M(\mathbb{T},\omega)} \\
&\leq c_{17} \| \sum_{\nu=1}^{n} A_\nu(\,\cdot\,, f) \sin^{2k} \frac{\nu}{2n} \|_{L_M(\mathbb{T},\omega)} \quad . \qquad (3.3)
\end{aligned}$$

According to [10], the following inequalities hold

$$(2n)^{-2k} \quad \| \quad S_n^{(2k)}(\,\cdot\,, f) \|_{L_M(\mathbb{T},\omega)} \leq c_{18} \Omega_{M,\omega}^{k}(\frac{1}{n+1}, f). \qquad (3.4)$$

$$\| \quad f - S_n(\,\cdot\,, f) \|_{L_M(\mathbb{T},\omega)} \leq c_{19} E_n(f)_{M,\omega}, \qquad (3.5)$$

$$E_n(f)_{M,\omega} \leq c_{20} \Omega_{M,\omega}^{k}(\frac{1}{n+1}, f). \qquad (3.6)$$

From (3.3) and (3.4), we get

$$\begin{aligned}
I_1 &\leq c_{17} \| \sum_{\nu=1}^{n} A_\nu(\,\cdot\,, f) \sin^{2k} \frac{\nu}{2n} \|_{L_M(\mathbb{T},\omega)} \\
&\leq c_{21} \| \sum_{\nu=1}^{n} A_\nu(\,\cdot\,, f) (\frac{\nu}{2n})^{2k} \|_{L_M(\mathbb{T},\omega)} \\
&\leq c_{22} (2n)^{-2k} \| \sum_{\nu=1}^{n} A_\nu(\,\cdot\,, f) \nu^{2k} \|_{L_M(\mathbb{T},\omega)} \\
&\leq c_{23} (2n)^{-2k} \| S_n^{(2k)}(\,\cdot\,, f) \|_{L_M(\mathbb{T},\omega)} \leq c_{24} \Omega_{M,\omega}^{k}(\frac{1}{n+1}, f). \qquad (3.7)
\end{aligned}$$

Using inequality (3.5), we have

$$I_2 = \| \sum_{\nu=n+1}^{\infty} A_\nu(\,\cdot\,,f)\, \|_{L_M(\mathbb{T},\omega)} \leq c_{25} E_n(f)_{M,\omega}. \tag{3.8}$$

Then by (3.8) and (3.6), we obtain

$$I_2 \leq c_{26} \Omega^k_{M,\omega}(\frac{1}{n+1}, f). \tag{3.9}$$

Now combining (3.1), (3.7), and (3.9), we obtain the inequality (2.2) of Theorem 2.1.

Proof of Theorem 2.2. According to the subadditivity of the norm, we get

$$\begin{aligned} \|R_n(f,\lambda_\nu(r))\|_{L_M(\mathbb{T},\omega)} &\leq \| \sum_{\nu=1}^{[\frac{1}{|r-r_0|}]} (1-\lambda_\nu(r))A_\nu(\,\cdot\,,f)]\, \|_{L_M(\mathbb{T},\omega)} + \\ &\| \sum_{\nu=[\frac{1}{|r-r_0|}]+1}^{\infty} (1-\lambda_\nu(r))A_\nu(\,\cdot\,,f)]\, \|_{L_M(\mathbb{T},\omega)} \\ &= I_1 + I_2, \end{aligned} \tag{3.10}$$

where $[\frac{1}{|r-r_0|}]$ is the integer part of real number $\frac{1}{|r-r_0|}$.

The following holds

$$\begin{aligned} I_1 &= \| \sum_{\nu=1}^{[\frac{1}{|r-r_0|}]} (1-\lambda_\nu(r))A_\nu(\,\cdot\,,f)]\, \|_{L_M(\mathbb{T},\omega)} \\ &= \| \sum_{\nu=1}^{[\frac{1}{|r-r_0|}]} \frac{(1-\lambda_\nu(r))}{\sin^{2k}\frac{\nu|r-r_0|}{2}} A_\nu(\,\cdot\,,f) \sin^{2k}\frac{\nu \mid r-r_0 \mid}{2}\, \|_{L_M(\mathbb{T},\omega)}\,. \end{aligned} \tag{3.11}$$

We set

$$\mu_{\nu,r} = \frac{1-\lambda_\nu(r)}{\sin^{2k}\frac{\nu|r-r_0|}{2}},$$

where $1 \leq \nu \leq [\frac{1}{|r-r_0|}]$.

According to relation (2.4), the system of numbers $\{\mu_{\nu,r}\}$ satisfies the conditions (1.2) of Theorem 1.1. Then selecting m, such that $2^m \leq [|\,\frac{1}{|r-r_0|}\,|] < 2^{m+1}$, from (3.11) and (3.4), we have

$$\begin{aligned} I_1 &\leq \| \sum_{\nu=1}^{2^{m+1}} \mu_{\nu,r} A_\nu(\,\cdot\,,f)] \sin^{2k}\frac{\nu \mid r-r_0 \mid}{2}\, \|_{L_M(\mathbb{T},\omega)} \leq \\ &\leq c_{27} \| \sum_{\nu=1}^{\infty} A_\nu(\,\cdot\,,f)] \sin^{2k}\frac{\nu \mid r-r_0 \mid}{2}\, \|_{L_M(\mathbb{T},\omega)} \\ &\leq c_{28} \Omega^k_{M,\omega}(\mid r-r_0 \mid, f). \end{aligned} \tag{3.12}$$

According to (2.3) for the system of numbers $\{1-\lambda_\nu(r)\}$, the conditions of Theorem 1.1 are satisfied. Then applying Theorem 1.1, we obtain

$$\begin{aligned} I_2 &= \| \sum_{\nu=[\frac{1}{|r-r_0|}]+1}^{\infty} (1-\lambda_\nu(r))A_\nu(\,\cdot\,,f)] \|_{L_M(\mathbb{T},\omega)} \\ &\leq c_{29} \| \sum_{\nu=[\frac{1}{|r-r_0|}]+1}^{\infty} A_\nu(\,\cdot\,,f)] \|_{L_M(\mathbb{T},\omega)} . \end{aligned} \tag{3.13}$$

By (3.5) and (3.13), we get

$$I_2 \leq c_{30}\Omega^k_{M,\omega}(|\, r-r_0\,|, f). \tag{3.14}$$

Now combining (3.10), (3.12), and (3.14), we obtain the inequality (2.5) of Theorem 2.2.

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