Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2926 – 2934
© TÜBİTAK
doi:10.3906/mat-1801-6

Description of invariant subspaces in terms of Berezin symbols

Suna SALTAN*
Department of Mathematics, Faculty of Science, Süleyman Demirel University, Isparta, Turkey

Received: 04.01.2018 • **Accepted/Published Online:** 29.08.2018 • **Final Version:** 27.11.2018

Abstract: We consider the stretching operator $(T_w f)(z) = f(wz)$ and the multiple shift operator $S^n f = z^n f$ on the Hardy spaces $H^p(\mathbb{D})$ $(1 \leq p < +\infty)$. We describe in terms of so-called Berezin symbols their lattice of invariant subspaces. We also define a new class of operators on the reproducing kernel Hilbert space $H(\Omega)$, which in a particular case contains all compact operators, and discuss in terms of Berezin symbols their invariant subspaces.

Key words: Invariant subspaces, multiple shift operator, Berezin symbol, stretching operator

1. Introduction

In this article, we describe invariant subspaces of a multiple shift operator on the Hardy space $H^p = H^p(\mathbb{D})$ $(1 \leq p < \infty)$ of all analytic functions f on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\|f\|_p := \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^p dt \right)^{\frac{1}{p}} < +\infty.$$

The usual shift operator S is defined on H^p by

$$(Sf)(z) = zf(z).$$

The multiple shift operator is S^n with $n \geq 2$: $(S^n f)(z) = z^n f$. A subspace (closed) E in H^p is called a nontrivial invariant subspace for an operator $A \in \mathcal{B}(H^p)$, the Banach algebra of all bounded linear operators acting on H^p, if $\{0\} \neq E \neq H^p$ and $AE \subseteq E$, i.e. $Af \in E$ whenever $f \in E$.

The description of invariant subspaces of the shift operator S on some Hilbert or Banach spaces of analytic functions on $\mathbb{D}$ is well investigated and readers can consult, for instance, $[5, 6, 9-11, 17]$ and their references. However, the same question for the multiple shift operator S^n, $n \geq 2$, apparently is not well studied. Here we will describe S^n-invariant subspaces in terms of so-called Berezin symbols for any $n \geq 1$ (Theorem 2). In terms of Berezin symbols we also discuss invariant subspaces of some class of operators on the so-called standard reproducing kernel Hilbert space in the sense of Nordgren and Rosenthal [18] (Theorem 5). Furthermore, we describe invariant subspaces of some stretching operators $(T_w f)(z) = f(wz)$, where $w \in \mathbb{D}$ (Theorem 1).

*Correspondence: sunasaltan@sdu.edu.tr
2000 *AMS Mathematics Subject Classification:* 47B20, 47A12

Recall that a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is the Hilbert space of complex-valued functions on some set Ω, say, in the complex plane $\mathbb{C}$, such that the evaluation functionals $\phi_\lambda(f) := f(\lambda),\ \lambda \in \Omega,$ are continuous on $\mathcal{H}$. Then, by the classical Riesz representation theorem, for each $\lambda \in \Omega$ there exists a unique function $\mathcal{K}_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, \mathcal{K}_\lambda\rangle$ for each $f \in \mathcal{H}$. The collection $\{\mathcal{K}_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel for the space $\mathcal{H}$. $\widehat{\mathcal{K}}_\lambda := \frac{\mathcal{K}_\lambda}{\|\mathcal{K}_\lambda\|}$ is the normalized reproducing kernel of $\mathcal{H}$. Prototypical reproducing kernel Hilbert spaces are Hardy, Bergman, and Fock–Hilbert spaces. A detailed presentation of the theory of reproducing kernel Hilbert spaces and reproducing kernel is given, for instance, in $[1, 9, 18, 19]$. For an operator $A \in \mathcal{B}(\mathcal{H})$, the Berezin symbol A is defined by (see Berezin $[3, 4]$)

$$\widetilde{A}(\lambda) := \left\langle A\widehat{\mathcal{K}}_\lambda(z), \widehat{\mathcal{K}}_\lambda(z)\right\rangle,\ \lambda, z \in \Omega,$$

where the inner product $\langle , \rangle$ is taken in the space $\mathcal{H}$. It is clear that $\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| \le \|A\|$. $Ber(A) := Range(\widetilde{A})$ is called the Berezin set of operator A and it is obvious that $Ber(A)$ is contained in the numerical range $W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\|_{\mathcal{H}} = 1\}$.

It is also clear that

$$ber(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|,$$

which is called the Berezin number of A, satisfies
$ber(A) \le w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H} \text{ and } \|x\| = 1\}$ (the numerical radius)$\le \|A\|$.

Note that these new numerical values of operators on the reproducing kernel space $\mathcal{H}(\Omega)$ were introduced originally by Karaev [12]; see also $[7, 8, 13-16]$ for more fact about Berezin sets and Berezin numbers of operators.

2. Berezin symbols and invariant subspaces of stretching and multiple shift operators

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk in the complex plane and $w \in \mathbb{D}$. Let $T_w : H^p \to H^p$ be the stretching operator defined by $(T_w f)(z) = f(wz)\ \forall z \in \mathbb{D}$. It is well known that T_w is compact, and if $p = 2$, then T_w is self-adjoint if and only if $w \in \mathbb{R}$.

Here we use the Berezin symbols and diagonal operators technique to describe the lattices of invariant subspaces of the stretching operator T_w with $w \in (0, 1)$, and multiple shift operators S^n with $n \ge 1$ acting on the Hardy space $H^p,\ 1 \le p < +\infty$.

For any bounded sequence (a_n) of complex numbers, let $D_{(a_n)}$ be a corresponding diagonal operator defined on the Hardy–Hilbert space H^2 by $D_{(a_n)}z^k = a_k z^k,\ k = 0, 1, 2, ...,$ with respect to the orthonormal basis $(z^n)_{n\ge1}$ in H^2. The following describes $Lat(T_w)$ in terms of Berezin symbols.

Theorem 1 *Let $w \in \mathbb{D} \cap \mathbb{R}_+,\ 1 \le p < +\infty$, and $T_w : H^p \to H^p$ be the stretching operator. Let $E \subset H^p$ be a nontrivial closed subspace. Then $T_w E \subset E$ (i.e. $E \in Lat(T_w)$) if and only if for every $f = \sum_{k=0}^{\infty}\widehat{f}(k)z^k \in E$ there exists a function $g = g_f \in E$ such that*

$$\widetilde{D}_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\left(\sqrt{|z|\, w}\right) = (1 - |z|\, w)\, g(z)$$

for all z in $\mathbb{D}$; here $\widehat{f}(k) = \frac{f^{(k)}_{(0)}}{k!}$ is the kth Taylor coefficient of f.

Proof Indeed, let $z:\mathbb{D}\to\mathbb{D}$ and $u:\mathbb{D}\to\mathbb{D}$ be two arbitrary independent variables in $\mathbb{D}$, and let $k_\lambda(u)=\frac{1}{1-\overline{\lambda}u}$, $\lambda,u\in\mathbb{D}$, be the reproducing kernel of H^2. Since for any $h=\sum\limits_{k=0}^{\infty}\widehat{h}(k)z^k\in H^p$, $(1\le p<+\infty)$, $\left|\widehat{h}(k)\right|\le C_p$ for all $k=0,1,2,...$ and some $C_p>0$ (for $1<p\le 2$, it follows, for example, also from the known Hausdorff–Young theorem [21] that if $g\in H^p$ $(1<p\le 2)$, then $\sum\limits_{k=0}^{\infty}|\widehat{g}(k)|^{\frac{p}{p-1}}<+\infty$) the diagonal operator $D_{\widehat{h}(k)e^{ik\arg(z)}}$ is bounded on H^2 for arbitrary fixed z in $\mathbb{D}$. By considering this, and also by using that $z=|z|\,e^{ik\arg(z)}$, where $0\le\arg(z)<2\pi$, then we have for any $f\in E$ that

$$\begin{aligned}
(T_wf)(z) &= T_w\sum_{k=0}^{\infty}\widehat{f}(k)z^k=\sum_{k=0}^{\infty}\widehat{f}(k)z^kw^k\\
&= \sum_{k=0}^{\infty}\widehat{f}(k)e^{ik\arg(z)}\left(|z|\,w\right)^k\\
&= \frac{\left(1-\left(\sqrt{|z|\,w}\right)^2\right)\sum\limits_{k=0}^{\infty}\widehat{f}(k)e^{ik\arg(z)}\left(\left(\sqrt{|z|\,w}\right)^2\right)^k}{1-\left(\sqrt{|z|\,w}\right)^2}\\
&= \frac{\left(1-\left(\sqrt{|z|\,w}\right)^2\right)\left\langle\sum\limits_{k=0}^{\infty}\left(\sqrt{|z|\,w}\right)^k\left(\widehat{f}(k)e^{ik\arg(z)}\right)u^k,\frac{1}{1-\sqrt{|z|wu}}\right\rangle}{1-\sqrt{|z|\,w}^2}\\
&= \frac{\left(1-\left(\sqrt{|z|\,w}\right)^2\right)\left\langle D_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\sum\limits_{k=0}^{\infty}\left(\sqrt{|z|\,w}\right)^k u^k,\frac{1}{1-\sqrt{|z|wu}}\right\rangle}{1-|z|\,w}\\
&= \frac{\left\langle D_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\widehat{k}_{\sqrt{|z|w}}(u),k_{\sqrt{|z|w}}(u)\right\rangle}{1-|z|\,w}\\
&= \frac{\widetilde{D}_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\left(\sqrt{|z|\,w}\right)}{1-|z|\,w}.
\end{aligned}$$

Hence,

$$(T_wf)(z)=\frac{\widetilde{D}_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\left(\sqrt{|z|\,w}\right)}{1-|z|\,w}\tag{1}$$

for every $z\in\mathbb{D}$. It follows from (1) that $T_wf\in E$ if and only if there exists a function $g=g_f\in E$ such that $\widetilde{D}_{\left(\widehat{f}(k)e^{ik\arg(z)}\right)}\left(\sqrt{|z|\,w}\right)=(1-|z|\,w)\,g(z)$, which proves the theorem. □

The next theorem describes $Lat(S^n)$ in terms of Berezin symbols.

Theorem 2 *Let $E\subset H^p$, $(1\le p<+\infty)$, be a nontrivial closed subspace and $n\ge 1$ be any integer. Then $S^nE\subset E$ if and only if for every $f\in E$ there exists a function $g=g_f\in E$ such that*

$$\widetilde{D}_{\left(\widehat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)=(1-|z|)\,g(z),\quad\forall z\in\mathbb{D}.$$

Proof Let $f \in E$ be arbitrary. Then we have

$$\begin{aligned}(S^n f)(z) &= z^n f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^{k+n} = \sum_{k=n}^{\infty} \widehat{f}(k-n) z^k \\ &= \sum_{k=n}^{\infty} \widehat{f}(k-n) e^{ik\arg(z)} |z|^k = \frac{(1-|z|)\sum_{k=0}^{\infty} \widehat{f}(k-n) e^{ik\arg(z)} |z|^k}{(1-|z|)},\end{aligned}$$

and here we put $\widehat{f}(-n) = \widehat{f}(-(n-1)) = ... = \widehat{f}(-1) = 0$. Then, as in the proof of Theorem 1, we obtain that

$$(S^n f)(z) = \frac{\widetilde{D}_{\left(\widehat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)}{1-|z|}, \quad z \in \mathbb{D}. \tag{2}$$

It follows now from (2) that $S^n f \in E$ if and only if

$$\frac{\widetilde{D}_{\left(\widehat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)}{1-|z|} \in E,$$

and thus $S^n f \in E$ if and only if there exists a function $g = g_f \in E$ such that

$$\widetilde{D}_{\left(\widehat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right) = (1-|z|)\, g(z), \quad \forall z \in \mathbb{D}.$$

This proves the theorem. □

Our next result characterizes S-invariant (usual shift-invariant) subspaces in terms of Berezin sets. Before stating it, let us note the following.

The class $\ell_A^\infty := \ell_A^\infty(\mathbb{D})$ consists of analytic functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ on $\mathbb{D}$ with $\left(\widehat{f}(n)\right)_{n\geq 0} \in \ell^\infty$ (the classical space of all bounded sequences of complex numbers).

Given any subspace $E \subset \ell_A^\infty$, we put

$$B_E := \bigcup_{\substack{f\in E \\ \theta\in[0,2\pi)}} Ber\left(D_{\left(\widehat{f}(n-1)e^{in\theta}\right)}\right)$$

and

$$B_{SE} := \bigcup_{\substack{g\in SE \\ \eta\in[0,2\pi)}} Ber\left(D_{\left(\widehat{g}(n)e^{in\eta}\right)}\right).$$

We need the following lemma due to Ash and Karaev [2].

Lemma 1 *If* $f \in \ell_A^\infty$, *then* $Range\left((1-|z|)\, f\right) = \bigcup_{\theta\in[0,2\pi)} Ber\left(D_{\left(\widehat{f}(n)e^{in\theta}\right)}\right)$, *and here* $\theta \in [0, 2\pi)$ *is an argument of* z *in its polar decomposition* $z = |z|e^{i\arg(z)}$.

Theorem 3 *Let* $E \subset H^p$, $1 \leq p < +\infty$, *be a nontrivial closed subspace. Then* $SE \subset E$ *if and only if* $B_E \subseteq B_{SE}$.

Proof We have from Theorem 2 for $n = 1$ that $SE \subset E$ if and only if for any $f \in E$ there exists a function $g = g_f \in E$ such that

$$\widetilde{D}_{\left(\widehat{f}(k-1)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right) = (1-|z|)\, g(z), \quad (\forall z \in \mathbb{D})\,. \tag{3}$$

By virtue of Lemma 1, we have then that

$$Range\left((1-|z|)\, g(z)\right) = \bigcup_{\theta\in[0,2\pi)} Ber\left(D_{(\widehat{g}(n)e^{in\theta})}\right).$$

Hence, for any $z \in \mathbb{D}$ there exists $\vartheta = \vartheta_z \in \mathbb{D}$ such that

$$(1-|z|)g(z) = \widetilde{D}_{(\widehat{g}(n)e^{in\arg(\vartheta)})}(\sqrt{|\vartheta|}). \tag{4}$$

Since

$$zf = \frac{\widetilde{D}_{(\widehat{g}(n)e^{in\arg(z)})}(\sqrt{|z|})}{1-|z|} = g(z),$$

we have that $g \in zE$, and hence $\widehat{g}(0) = g(0) = 0$. Also, by considering that

$$zf(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^{k+1} = g(z) = \sum_{k=0}^{\infty} \widehat{g}(k) z^k,$$

we have $\widehat{f}(k-1) = \widehat{g}(k)$ for all $k \geq 1$. By considering this and formulas (3) and (4), we have

$$\widetilde{D}_{(\widehat{f}(k-1)e^{ik\arg(z)})}(\sqrt{|z|}) = \widetilde{D}_{(\widehat{g}(k)e^{ik\arg(\vartheta)})}(\sqrt{|\vartheta|}).$$

This shows that $SE \subset E$ if and only if for every $f \in E$ there exists $g \in zE$ such that $Ber(\widetilde{D}_{(\widehat{f}(k-1)e^{ik\theta})}) \subseteq Ber(\widetilde{D}_{(\widehat{g}(k)e^{ik\eta})})$ for al $\theta \in [0, 2\pi)$. Thus, $SE \subset E$ if and only if

$$\bigcup_{\substack{f\in E\\ \theta\in[0,2\pi)}} Ber(D_{(\widehat{f}(k-1)e^{ik\theta})}) \subseteq \bigcup_{\substack{g\in zE\\ \eta\in[0,2\pi)}} Ber(D_{(\widehat{g}(k)e^{ik\eta})}),$$

or equivalently, $B_E \subseteq B_{zE} = B_{SE}$, as desired. □

3. On the invariant subspaces of some class of operators on the reproducing kernel Hilbert space

In this section, we again use the Berezin symbols technique in the study of invariant subspaces of some operator class on the so-called standard in the sense of Nordgren and Rosenthal's [18] reproducing kernel Hilbert space. Following Nordgren and Rosenthal [18], we recall that a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is called standard if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is nonempty and has the property that $\left\{\widehat{K}_{\lambda_n}\right\}$ converges weakly to 0 whenever $\{\lambda_n\}$ is a sequence in Ω that converges to a point in $\partial\Omega$. The common reproducing kernel Hilbert spaces of analytic functions, including H^2(Hardy space), L^2_a(Bergman space), D^2(Dirichlet space), and $F(\mathbb{C})$(Fock space), are standard in this sense.

Definition 1 *We say that an operator* $A:\mathcal{H}(\Omega)\to\mathcal{H}(\Omega)$ *on the reproducing kernel Hilbert space* $\mathcal{H}=\mathcal{H}(\Omega)$ *belongs to the class* $\mathcal{B}_0(\mathcal{H})$ *if* $\lim\limits_{\lambda\to\zeta\in\partial\Omega} A^*\widehat{K}_\lambda=0$ *in strong operator topology for any point* ζ*.*

We remark that if $\mathcal{H}(\Omega)$ is a standard reproducing kernel Hilbert space, then all compact operators are contained in class $\mathcal{B}_0(\mathcal{H})$.

Theorem 4 *Let* $\mathcal{H}=\mathcal{H}(\Omega)$ *be a reproducing kernel Hilbert space and* $A\in\mathcal{B}_0(\mathcal{H})$ *be an operator. If* $E\subset\mathcal{H}$ *is a (closed) subspace and* $AE\subset E$*, then*

$$\lim_{\lambda\to\zeta\in\partial\Omega}\frac{\left|(A\mathcal{K}_{E,\lambda})(\lambda)-\widetilde{A}(\lambda)\mathcal{K}_{E,\lambda}(\lambda)\right|}{\|\mathcal{K}_\lambda\|_{\mathcal{H}}^2}=0,$$

where $\mathcal{K}_{E,\lambda}:=P_E\mathcal{K}_\lambda$ *is the reproducing kernel of the subspace* E*, where* $P_E:\mathcal{H}\to E$ *is an orthogonal projection and* $\widetilde{A}$ *is the Berezin symbol of operator* A*.*

Proof The proof is similar to that of Theorem 4 in the author's previous paper [20], where a similar result is proved for compact operators; however, for completeness reasons, we provide it here. For this aim, let $(e_j(z))_{j\geq 1}$ be an orthonormal basis in E (since $\mathcal{H}$ is a reproducing kernel Hilbert space, it is separable, and since E is a closed subspace of $\mathcal{H}$, E is also separable). Then it is clear that

$$P_Ef=\sum_{j=1}^{\infty}\langle f,e_j\rangle\,e_j$$

for every $f\in\mathcal{H}$. Now by setting

$$g:=A(P_E\mathcal{K}_\lambda)-\widetilde{A}(\lambda)(P_E\mathcal{K}_\lambda),$$

we have that

$$g=(A-\widetilde{A}(\lambda)I_{\mathcal{H}})\sum_{j=1}^{\infty}\langle\mathcal{K}_\lambda,e_j\rangle\,e_j,$$

from which

$$\begin{aligned}g(\lambda)&=\langle g,\mathcal{K}_\lambda\rangle=\left\langle(A-\widetilde{A}(\lambda))\sum_{j=1}^{\infty}\langle\mathcal{K}_\lambda,e_j\rangle\,e_j,\mathcal{K}_\lambda\right\rangle\\&=\left\langle\sum_{j=1}^{\infty}\langle\mathcal{K}_\lambda,e_j\rangle\,e_j,(A^*-\overline{\widetilde{A}}(\lambda))\mathcal{K}_\lambda\right\rangle\\&=\left\langle\sum_{j=1}^{\infty}\langle\mathcal{K}_\lambda,e_j\rangle\,e_j,(A^*-\widetilde{A^*}(\lambda))\mathcal{K}_\lambda\right\rangle.\end{aligned}$$

Therefore, by applying the Cauchy–Schwarz inequality, we obtain from the latter that

$$\begin{aligned} |g(\lambda)| &\leq \left\| \sum_{j=1}^{\infty} \langle \mathcal{K}_\lambda, e_j \rangle e_j \right\|_{\mathcal{H}} \left\| (A^* - \widetilde{A^*}(\lambda))\mathcal{K}_\lambda \right\|_{\mathcal{H}} \\ &\leq \|\mathcal{K}_\lambda\|_{\mathcal{H}} \left\| (A^* - \widetilde{A^*}(\lambda))\mathcal{K}_\lambda \right\|_{\mathcal{H}} \\ &= \|\mathcal{K}_\lambda\|_{\mathcal{H}}^2 \left\| (A^* - \widetilde{A^*}(\lambda)) \frac{\mathcal{K}_\lambda}{\|\mathcal{K}_\lambda\|_{\mathcal{H}}} \right\|_{\mathcal{H}} \\ &= \|\mathcal{K}_\lambda\|_{\mathcal{H}}^2 \left\| (A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}} . \end{aligned}$$

Since $(A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda \perp \widehat{\mathcal{K}}_\lambda$, we obtain

$$\begin{aligned} \left\| (A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 &= \left\langle A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda, A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda \right\rangle \\ &= \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \widetilde{A^*}(\lambda) \left\langle \widehat{\mathcal{K}}_\lambda, A^*\widehat{\mathcal{K}}_\lambda \right\rangle \\ &\quad - \left\langle A^*\widehat{\mathcal{K}}_\lambda, \widetilde{A^*}(\lambda)\widehat{\mathcal{K}}_\lambda \right\rangle \\ &\quad + \widetilde{A^*}(\lambda) \left\langle \widehat{\mathcal{K}}_\lambda, \widetilde{A^*}(\lambda)\widehat{\mathcal{K}}_\lambda \right\rangle \\ &= \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \widetilde{A^*}(\lambda) \left\langle A\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle \\ &\quad - \overline{\widetilde{A^*}}(\lambda) \left\langle A^*\widehat{\mathcal{K}}_\lambda, \widehat{\mathcal{K}}_\lambda \right\rangle + \widetilde{A^*}(\lambda)\overline{\widetilde{A^*}}(\lambda) \\ &= \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \widetilde{A^*}(\lambda)\widetilde{A}(\lambda) - \widetilde{A}(\lambda)\widetilde{A^*}(\lambda) \\ &\quad + \widetilde{A^*}(\lambda)\overline{\widetilde{A^*}}(\lambda) \\ &= \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - 2\left| \widetilde{A^*}(\lambda) \right|^2 + \left| \widetilde{A^*}(\lambda) \right|^2 \\ &= \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \left| \widetilde{A^*}(\lambda) \right|^2, \end{aligned}$$

and hence

$$\left\| (A^* - \widetilde{A^*}(\lambda))\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}} = \sqrt{\left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \left| \widetilde{A^*}(\lambda) \right|^2}$$

for all $\lambda \in \Omega$. Thus, we have

$$|g(\lambda)| \leq \left\| \widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 \sqrt{\left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \left| \widetilde{A^*}(\lambda) \right|^2}$$

for all $\lambda \in \Omega$. Now, by considering that $\left| \widetilde{A^*}(\lambda) \right| \leq \left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}$ and $A \in \mathcal{B}_0(\mathcal{H})$, we obtain that

$$\lim_{\lambda \to \zeta} \left(\left\| A^*\widehat{\mathcal{K}}_\lambda \right\|_{\mathcal{H}}^2 - \left| \widetilde{A^*}(\lambda) \right|^2 \right) = 0$$

for all points $\zeta \in \partial\Omega$, or equivalently

$$\lim_{\lambda \to \zeta} \frac{\left|(A\mathcal{K}_{E,\lambda})(\lambda) - \widetilde{A}(\lambda)\mathcal{K}_{E,\lambda}(\lambda)\right|}{\left\|\widehat{\mathcal{K}}_{\lambda}\right\|_{\mathcal{H}}^{2}} = 0$$

for any point $\zeta \in \partial\Omega$. The proof is completed. □

Since on the standard reproducing kernel Hilbert space every compact operator vanishes on the boundary, the following is an immediate corollary of Theorem 4.

Corollary 1 *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard reproducing kernel Hilbert space on Ω, and let $K \in \mathcal{B}(\mathcal{H})$ be a compact operator. Let $E \subset \mathcal{H}$ be a nontrivial subspace. If $KE \subset E$, then*

$$\left|(K\mathcal{K}_{E,\lambda})(\lambda) - (\widetilde{K}\mathcal{K}_{E,\lambda})(\lambda)\right| = o\left(\left\|\widehat{\mathcal{K}}_{\lambda}\right\|_{\mathcal{H}}^{2}\right)$$

as $\lambda \to \zeta \in \partial\Omega$ for any ζ.

References

[1] Aronzajn N. Theory of reproducing kernels. T Am Math Soc 1950; 68: 337-404.

[2] Ash JM, Karaev MT. On the boundary behavior of special class of C^{∞}-functions and analytic functions. Int Mat Forum 2012; 7: 153-166.

[3] Berezin FA. Covariant and contravariant symbols for operators. Math USSR-Izv 1972; 6: 1117-1151.

[4] Berezin FA. Quantization. Math USSR-Izv 1974; 8: 1109-1165.

[5] Beurling A. On two problems concerning linear transformations in Hilbert space. Acta Math 1949; 81: 239-253.

[6] Gamelin TW. Uniform Algebras. Providence, RI, USA: American Mathematical Society, 2005.

[7] Garayev MT. The Berezin number, norm of a Hankel operator and related topics. In: Bhattacharyya T, Dritschel M, editors. Operator Algebras and Mathematical Physics. Operator Theory: Advances and Applications, Vol. 247. Basel, Switzerland: Springer, 2015, pp. 87-100.

[8] Garayev MT, Gürdal M, Yamancı U. Description of Bloch spaces, weighted Bergman spaces and their invariant subspaces, and related questions. Kuwait Journal of Science 2016; 43: 70-77.

[9] Halmos PR. Hilbert Space Problem Book. Berlin, Germany: Springer, 1982.

[10] Helson H. Lectures on Invariant Subspaces. New York, NY, USA: Academic Press, 1964.

[11] Hoffman K. Banach Spaces of Analytic Functions. New York, NY, USA: Prentice Hall, 1962.

[12] Karaev MT. Berezin set and Berezin number of operators and their applications. In: 8th Workshop on Numerical Ranges and Numerical Radii, WONRA 06. Bremen, Germany: University of Bremen, 2006.

[13] Karaev MT. Berezin symbol and invertibility of operators on the functional Hilbert spaces. J Funct Anal 2006; $23:8181-192$.

[14] Karaev MT. Reproducing kernels and Berezin symbols techniques in various questions of operator theory. Complex Anal Oper Theory 2013; 7: 983-1018.

[15] Karaev MT, Gürdal M, Yamancı U. Some results related with Berezin symbols and Toeplitz operators. Math Inequal Appl 2014; 17: 1031-1045.

[16] Karaev MT, Iskenderov NS. Berezin number of operators and related questions. Methods of Functional Analysis and Topology 2013; 19: 68-72.

[17] Nikolski NK. Treatise on the Shift Operator. Berlin, Germany: Springer, 1986.

[18] Nordgren E, Rosenthal P. Boundary values of Berezin symbols. Oper Th Adv Appl 1994; 73: 362-368.

[19] Saitoh S, Sawano Y. Theory of Reproducing Kernels and Applications. Berlin, Germany: Springer, 2016.

[20] Saltan S. On the weak limit of compact operators on the reproducing kernel Hilbert space and related questions. An Sti U Ovid Co-Mat 2016; 24: 253-260.

[21] Tung J. Taylor coefficients of functions in Fock spaces. J Math Anal Appl 2006; 318: 397-409.