Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 2942 – 2952
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doi:10.3906/mat-1802-28

# Generalization of the Cayley transform in 3D homogeneous geometries

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**Received:** 08.02.2018 • **Accepted/Published Online:** 07.09.2018 • **Final Version:** 27.11.2018

**Abstract:** The Cayley transform maps the unit disk onto the upper half-plane, conformally and isometrically. In this paper, we generalize the Cayley transform in three-dimensional homogeneous geometries which are fiber bundles over the hyperbolic plane. Obtained generalizations are isometries between existing models in corresponding homogeneous geometries. Particularly, constructed isometry between two models of $\widetilde{SL(2,\mathbb{R})}$ geometry is nontrivial and enables comparison and transfer of known and even future results between these two models.

**Key words:** Cayley transform, homogeneous geometry, isometry

## 1. Introduction

Homogeneous geometries came into focus in 1982 when Thurston formulated the geometrization conjecture for three-manifolds. The Thurston conjecture states that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian three-dimensional geometries.

Among these eight 3D homogeneous geometries, $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL(2,\mathbb{R})}$ are specific because they are structured as line bundles over the hyperbolic plane. Particularly, the $\widetilde{SL(2,\mathbb{R})}$ is the least researched and generally, because of its unique features, presents a rich area for future investigation.

The Cayley transform is an isometry (differentiable bijection with a differentiable inverse which preserves distance) between the disk model and the upper half-plane model of the hyperbolic plane. In this paper, we generalize the Cayley transform in $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL(2,\mathbb{R})}$ geometries. In the literature, two models of each of these two homogeneous geometries can be found. For more details about these models, see [11] and [15]. Moreover, the obtained generalizations of the Cayley transform are isometries between existing models in these two geometries.

The isometry between models of $\mathbb{H}^2 \times \mathbb{R}$ geometry is relatively trivial. However, we mention this isometry to expose the nature of the relation between these two existing models.

The main result is the construction of an isometry between the hyperboloid model and the right half-space model of $\widetilde{SL(2,\mathbb{R})}$ geometry. This result enables comparison and transfer of known results between these two models.

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2000 *AMS Mathematics Subject Classification:* 53C30,14M17

This paper is organized as follows. In Section 2, we recall important facts related to the Cayley transform and 2D and 3D homogeneous geometries. In Section 3, a short description of the open cylinder and right half-space model of $\mathbb{H}^2 \times \mathbb{R}$ geometry and isometry between these two models is given. Finally, in Section 4, we first describe the hyperboloid and the right half-space model of $\widetilde{SL(2,\mathbb{R})}$ geometry in detail and then construct an isometry between these two models.

## 2. Preliminaries

### 2.1. The Cayley transform

It is well known that there are several models of the hyperbolic plane: Poincaré half-plane model, Poincaré disk model, Klein model, hyperboloid model, hemisphere model, etc. Each of these models has its own metric, geodesics, isometries, and so on. Moreover, all these models are isometrically equivalent, although some of isometries among them is not easy to determine. For more about this topic, you can see [2] or [10].

In the following sections, we will use two models of the hyperbolic plane $\mathbb{H}^2$: the disk model $\mathcal{D}$ and the upper half-plane model $\mathcal{U}$.

The Poincaré disk model is a unit disk $\mathcal{D} = \{(u,v) : u^2 + v^2 < 1\}$ with the metric

$$(ds)^2_{\mathcal{D}} = \frac{(du)^2 + (dv)^2}{(1-u^2-v^2)^2}. \tag{2.1}$$

The upper half-plane model is an upper half-plane
$\mathcal{U} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$(ds)^2_{\mathcal{U}} = \left(\frac{dx}{2y}\right)^2 + \left(\frac{dy}{2y}\right)^2. \tag{2.2}$$

The Cayley transform is the linear fractional transformation

$$\mathbf{z} \mapsto (-\mathbf{i})\frac{\mathbf{z}+\mathbf{i}}{\mathbf{z}-\mathbf{i}}$$

that maps the disk model $\mathcal{D}$ of the hyperbolic plane isometrically and conformally to the upper half-plane model $\mathcal{U}$ of the hyperbolic plane.

Therefore, the Cayley transform is explicitly given by

$$(u,v) \mapsto (x,y) = \left(\frac{2u}{u^2+(v-1)^2}, \frac{1-u^2-v^2}{u^2+(v-1)^2}\right). \tag{2.3}$$

Converting the rectangular coordinates $(u,v)$ to the polar coordinates $(r,\vartheta)$ by

$$u = \tanh r \cos\vartheta \qquad v = \tanh r \sin\vartheta, \tag{2.4}$$

the Cayley transform between the disk model (using the polar coordinates) and the upper half-plane model of the hyperbolic plane is explicitly given by

$$\begin{aligned} x &= \frac{2\tanh r\cos\vartheta}{\tanh^2 r - 2\tanh r\sin\vartheta + 1}, \\ y &= \frac{1-\tanh^2 r}{\tanh^2 r - 2\tanh r\sin\vartheta + 1}. \end{aligned} \tag{2.5}$$

By pullback of the metric (2.1), using (2.4), we obtain the metric of the disk model in polar coordinates

$$\mathbf{(ds)^2_{\mathcal{D}} = (dr)^2 + \sinh^2 r \cosh^2 r (d\vartheta)^2}. \tag{2.6}$$

This fact is proven later.

**Remark 2.1** *In the next sections, by the Cayley transform we assume the transformation given by formulas in (2.5).*

## 2.2. On homogeneous geometries

First, we recall definitions of isometry, homogeneous manifold, and model geometry.

A diffeomorphism $\Psi : M \to N$ between two Riemannian manifolds $(M, g_M)$ and $(N, g_N)$ is called an **isometry** if $\Psi^* g_N = g_M$, i.e. $g_M(u,v)_p = g_N(\Psi_p(u), \Psi_p(v))$, $\forall p \in M$ and $\forall u, v \in T_pM$.

The Riemannian manifold $(M, g)$ is called **homogeneous** if for any $x, y \in M$ there is an isometry $\Phi : M \to M$ such that $y = \Phi(x)$.

A **model geometry** is a simply connected smooth manifold $X$ together with a transitive action of a Lie group on $X$ with compact stabilizers.

It is known that there are only three 2D complete, simply connected Riemannian manifolds with constant sectional curvature, i.e. every compact connected surface is locally isometric to Euclidean plane $\mathbb{E}^2$, the hyperbolic plane $\mathbb{H}^2$, or sphere $\mathbb{S}^2$. These geometries have constant sectional curvatures $0, -1, 1$, respectively.

In 1982, Thurston formulated a geometrization conjecture for three-manifolds (see [18]) which states that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian three-dimensional geometries:

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \widetilde{SL(2,\mathbb{R})},\ \mathrm{Nil},\ \mathrm{Sol}.$$

The geometrization conjecture implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture.

In 2003, Perelman, a Russian mathematician, sketched a proof of the full geometrization conjecture using the Ricci flow with surgery. He showed that, although the Ricci flow in general produces singularities, it is possible to continue the Ricci flow past the singularity by using surgery to change the topology of the manifold. Roughly speaking, the Ricci flow contracts positive curvature regions (characteristic for $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$ geometries) and expands negative curvature regions (characteristic for $\mathbb{H}^3$ and other homogeneous geometries).

Usually, the 3D homogeneous geometries are classified into the constant curvature geometries ($\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$), the product geometries ($\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$), and the twisted product geometries (Nil, Sol, $\widetilde{SL(2,\mathbb{R})}$).

Two different approaches to the 3D homogeneous geometries can be found in [11] and [15].

# 3. Generalization of the Cayley transform in $\mathbb{H}^2 \times \mathbb{R}$

The space $\mathbb{H}^2 \times \mathbb{R}$ is the Cartesian product of the hyperbolic plane and the real line equipped with the product metric. So, it is a trivial line bundle over the hyperbolic plane.

There are two models of $\mathbb{H}^2 \times \mathbb{R}$ geometry. The first one, which we will call the open cylinder model, represents a slight modification of the model introduced in [11]. We will call the second one the right-half space model which is described in [15] in detail.

### 3.1. Open cylinder model of $\mathbb{H}^2 \times \mathbb{R}$

The open cylinder model of $\mathbb{H}^2 \times \mathbb{R}$ geometry is obtained as a direct product of the Poincaré disk model of the hyperbolic plane and a real line. The metric is product metric given by

$$(\mathbf{ds})^{\mathbf{2}}_{\mathcal{C}} = (\mathbf{dr})^{\mathbf{2}} + \sinh^{\mathbf{2}} \mathbf{r} \cosh^{\mathbf{2}} \mathbf{r} (\mathbf{d}\vartheta)^{\mathbf{2}} + (\mathbf{dt})^{\mathbf{2}}. \tag{3.1}$$

**Remark 3.1** *In the model given in [11], the author also used cylindrical coordinates but gave a slightly different metric:*

$$(\mathbf{ds})^{\mathbf{2}} = (\mathbf{dt})^{\mathbf{2}} + (\mathbf{dr})^{\mathbf{2}} + \sinh^{\mathbf{2}} \mathbf{r} (\mathbf{d}\vartheta)^{\mathbf{2}}.$$

*It is possible to obtain this metric if we substitute* $\tanh r$ *with* $\tanh \frac{r}{2}$ *in the forthcoming formulas (3.3). This metric is used in [16] where geodesic ball packing in* $\mathbb{H}^2 \times \mathbb{R}$ *is considered.*

### 3.2. Right half-space model of $\mathbb{H}^2 \times \mathbb{R}$

The right half-space model of $\mathbb{H}^2 \times \mathbb{R}$ geometry is obtained as a direct product of the Poincaré upper half-plane and a real line. Hence, the metric is a product metric given by

$$(\mathbf{ds})^{\mathbf{2}}_{\mathcal{R}} = \left(\frac{\mathbf{dx}}{\mathbf{2y}}\right)^{\mathbf{2}} + \left(\frac{\mathbf{dy}}{\mathbf{2y}}\right)^{\mathbf{2}} + (\mathbf{dz})^{\mathbf{2}}. \tag{3.2}$$

Dillen and Munteanu used this model in [4] to study surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and in [3], they studied the constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

### 3.3. Isometry between two models of $\mathbb{H}^2 \times \mathbb{R}$ geometry

The isometry between the two models is given in following proposition and represents a relatively trivial generalization of the Cayley transform.

**Proposition 3.2** *An isometry* $\sigma : \mathcal{C} \to \mathcal{R}$ *between the open cylinder model* $\mathcal{C}$ *and the right half-space model* $\mathcal{R}$ *of* $\mathbb{H}^2 \times \mathbb{R}$ *geometry is given by the following formulas:*

$$\begin{aligned} x &= \frac{2\tanh r \cos\vartheta}{\tanh^2 r - 2\tanh r \sin\vartheta + 1}, \\ y &= \frac{1-\tanh^2 r}{\tanh^2 r - 2\tanh r \sin\vartheta + 1}, \\ z &= t. \end{aligned} \tag{3.3}$$

**Proof** The statement follows directly from the fact that the direct product of two isometries is an isometry. Since we did not prove an existence of isometry between the disk model and the upper half-plane model of the

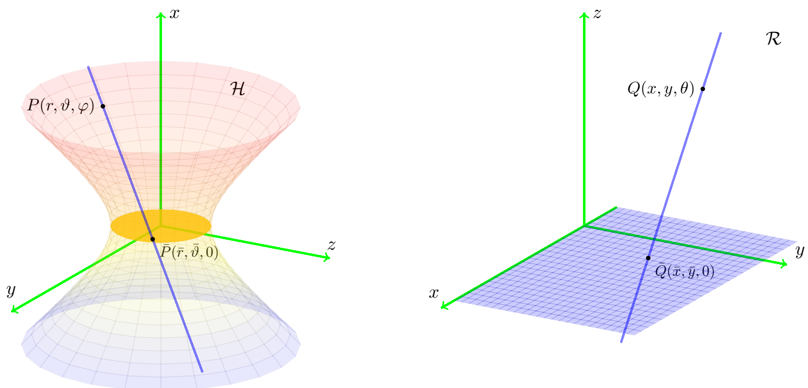


**Figure**. The hyperboloid and the right half-space models of $\widetilde{SL(2,\mathbb{R})}$ geometry.

hyperbolic plane in the previous section, we will here prove an existence of an isometry between two models of $\mathbb{H}^2 \times \mathbb{R}$ geometry.

Differentiation of (3.3) gives

$$\begin{aligned} dx &= \frac{2\cos\vartheta}{\sinh^2 2r(\coth 2r - \sin\vartheta)^2}dr + \frac{1-\coth 2r\sin\vartheta}{(\coth 2r - \sin\vartheta)^2}d\vartheta, \\ dy &= \frac{2(-1+\coth 2r\sin\vartheta)}{\sinh 2r(\coth 2r - \sin\vartheta)^2}dr + \frac{\cos\vartheta}{\sinh 2r(\coth 2r - \sin\vartheta)^2}d\vartheta, \\ dz &= dt. \end{aligned}$$

After substituting these relations in (3.2) and long but straightforward computations, we obtain the metric (3.1). $\square$

## 4. Generalization of the Cayley transform in $\widetilde{SL(2,\mathbb{R})}$ geometry

Generalization of the Cayley transform is more complicated in $\widetilde{SL(2,\mathbb{R})}$ geometry than in $\mathbb{H}^2 \times \mathbb{R}$. In the literature, there are only two models of $\widetilde{SL(2,\mathbb{R})}$ geometry, each of which is useful in certain contexts. The first one is usually called the hyperboloid model and we will call the second one the right-half space model.

As we mentioned before, the $\widetilde{SL(2,\mathbb{R})}$ is the least researched among homogeneous geometries, and thus we will explain both models here in detail. The hyperboloid model is used in the [5, 7, 12, 13, 17] and the right half-space model in [1, 8, 9, 14].

### 4.1. Hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry

The hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry was introduced by Emil Molnár in [11] and described in detail in [5], where geodesics, the fibre translation group, and translation curves (with corresponding spheres) are given. Moreover, E. Molnár proposed a projective spherical model of homogeneous geometries as a unified geometrical model believing that this model could be a starting point for a possible attack on the Thurston conjecture.

In the proposed model of $\widetilde{SL(2,\mathbb{R})}$ geometry, the idea is to start with the collineation group which acts on projective 3-space $\mathcal{P}^3(\mathbb{R})$ and projective sphere $\mathcal{PS}^3(\mathbb{R})$ and preserves a hyperboloid polarity, i.e. a scalar product of signature $(--++)$. Using the one-sheeted solid hyperboloid

$$\mathcal{H}: \; -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0,$$

with an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\tilde{\mathcal{H}}$ of the hyperboloid $\mathcal{H}$ will give the so-called hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry.

As we mentioned, in Molnár's approach, one starts with the one parameter group of matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{4.1}$$

which acts on $\mathcal{P}^3(\mathbb{R})$ and leaves the polarity of signature $(--++)$ and the hyperboloid solid $\mathcal{H}$ invariant. By the right action of this group on the point $(x^0;x^1;x^2;x^3)$, we obtain its orbit

$$(x^0\cos\varphi - x^1\sin\varphi; x^0\sin\varphi + x^1\cos\varphi; x^2\cos\varphi + x^3\sin\varphi; -x^2\sin\varphi + x^3\cos\varphi) \tag{4.2}$$

which is the unique line (fibre) through the given point. This action is called the fibre translation and $\varphi$ is called the fibre coordinate.

The mentioned subgroup of collineations acts transitively on the points of $\tilde{\mathcal{H}}$ and maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3) \in \mathcal{P}^3(\mathbb{R})$. It is represented by the matrix

$$\mathbf{T}: (t_i^j) := \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}. \tag{4.3}$$

Therefore, by using pullback transform on the base differential forms, one obtains the following global quadratic differential form

$$\begin{aligned}(ds)^2 = \Big\{ & \big[-(dx^0)x^1 + (dx^1)x^0 - (dx^2)x^3 + (dx^3)x^2\big]^2 \\ & + \big[-(dx^0)x^2 - (dx^1)x^3 + (dx^2)x^0 + (dx^3)x^1\big]^2 \\ & + \big[-(dx^0)x^3 + (dx^1)x^2 - (dx^2)x^1 + (dx^3)x^0\big]^2 \Big\} \\ & \cdot \big[-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2\big]^{-2}.\end{aligned} \tag{4.4}$$

A bijection between $\mathcal{H}$ and $SL(2,\mathbb{R})$ which maps

$$(x^0;x^1;x^2;x^3) \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

is provided by the following coordinate transformations:

$$a = x^0 + x^3, \; b = x^1 + x^2, \; c = -x^1 + x^2, \; d = x^0 - x^3. \tag{4.5}$$

This is an isomorphism between fiber translations (4.3) and $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with the usual multiplication operations.

Similarly, as the fibre (4.2) is obtained by acting of the group (4.1) on the point $(x^0; x^1; x^2; x^3)$ in $\tilde{\mathcal{H}}$, a fibre in $\widetilde{SL(2,\mathbb{R})}$ is obtained by acting of the group $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ on the "point" $\begin{pmatrix} d & b \\ c & a \end{pmatrix} \in SL(2,\mathbb{R})$.

Next, the transformation formulas which introduce hyperboloid coordinates $(r,\vartheta,\varphi)$

$$\begin{aligned} x^0 &= \cosh r \cos\varphi, \quad x^2 = \sinh r \cos(\vartheta-\varphi), \\ x^1 &= \cosh r \sin\varphi, \quad x^3 = \sinh r \sin(\vartheta-\varphi), \end{aligned} \tag{4.6}$$

are determined by the isometry group of the space $\tilde{\mathcal{H}}$. Notice that

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -\cosh^2 r + \sinh^2 r = -1.$$

In hyperboloid coordinates, $(r,\vartheta)$ are polar coordinates of the intersection point of a fiber and the hyperbolic (disk model) base plane and $\varphi$ is a fiber coordinate.

Substituting the formulas (4.6) in (4.4), the following Riemannian metric in the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry is obtained

$$(\mathbf{ds})^{\mathbf{2}}_{\mathcal{H}} = (\mathbf{dr})^{\mathbf{2}} + \cosh^{\mathbf{2}}\mathbf{r}\sinh^{\mathbf{2}}\mathbf{r}(\mathbf{d}\vartheta)^{\mathbf{2}} + \Big((\mathbf{d}\varphi) + \sinh^{\mathbf{2}}\mathbf{r}(\mathbf{d}\vartheta)\Big)^{\mathbf{2}}. \tag{4.7}$$

One can easily see that the metric (4.7) is invariant under rotations about the fiber through the origin and translations along fibers.

The Euclidean coordinates which correspond to the hyperboloid coordinates $(r,\vartheta,\varphi)$ are given by

$$\begin{aligned} x &= \tan\varphi, \\ y &= \tanh r \cdot \frac{\cos(\vartheta-\varphi)}{\cos\varphi}, \\ z &= \tanh r \cdot \frac{\sin(\vartheta-\varphi)}{\cos\varphi}, \end{aligned} \tag{4.8}$$

where $r \in [0,\infty)$, $\vartheta \in [-\pi,\pi)$ and $\varphi \in (-\frac{\pi}{2},\frac{\pi}{2})$ with extension to $\mathbb{R}$ for the universal covering. These formulas are important for visualization of surfaces in $E^3$.

### 4.2. Right half-space model of $\widetilde{SL(2,\mathbb{R})}$ geometry

In this section, we recall the right half-space model $\mathcal{R}$ of $\widetilde{SL(2,\mathbb{R})}$ geometry.

The right half-space model of $\widetilde{SL(2,\mathbb{R})}$ geometry is explained in detail in [15]. Kokubu in [9] and Inoguchi in [8] used this model to examine some minimal surfaces. This model was also used in [1] where Belkhalfa et al. proved that the only parallel surfaces in $SL(2,\mathbb{R})$ are rotational CMC surfaces. Recently, this model is used in [14] where constant angle surfaces are determined.

Let $G$ denote the real special linear group defined by

$$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

$G$ has the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\},\ A = \left\{ \begin{pmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix} : y > 0 \right\},\ N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

By Iwasawa decomposition, $SL(2,\mathbb{R}) = KAN$, i.e. every element of $SL(2,\mathbb{R})$ can be decomposed uniquely as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{4.9}$$

for some $x \in \mathbb{R}$, $y \in \mathbb{R}^+$, and $\theta \in S^1$, and $(x, y, \theta)$ may be considered as a global coordinate system of the group $SL(2,\mathbb{R})$.

Let us denote by $\mathfrak{g}$ the Lie algebra of $G$

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathfrak{gl}(2,\mathbb{R}) : \mathrm{tr}X = 0\}.$$

If we define an inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}(X^T Y)$, this product will induce a left-invariant metric on $SL(2,\mathbb{R})$.

The right half-space metric of $\widetilde{SL(2,\mathbb{R})}$ geometry is given by

$$(\mathbf{ds})^{\mathbf{2}}_{\mathcal{R}} = \left(\frac{\mathbf{dx}}{\mathbf{2y}}\right)^{\mathbf{2}} + \left(\frac{\mathbf{dy}}{\mathbf{2y}}\right)^{\mathbf{2}} + \left(\frac{\mathbf{dx}}{\mathbf{2y}} + \mathbf{d}\theta\right)^{\mathbf{2}}. \tag{4.10}$$

The coordinates $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$ are the coordinates of the hyperbolic base plane. $\theta \in S^1$ is the fibre coordinate.

The homogeneous space $SL(2,\mathbb{R})/SO(2)$ is diffeomorphic to the upper half-plane $\mathbb{H}^2$ and the upper half-plane metric (2.2) induces the Poincaré metric on the hyperbolic plane $\mathbb{H}^2$. Also, one can see that the right half-space metric is invariant under translations along fibers.

The Iwasawa decomposition (4.9) induces bijection between $SL(2,\mathbb{R})$ and $\mathcal{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{y}}\cos\theta & \frac{x}{\sqrt{y}}\cos\theta + \sqrt{y}\sin\theta \\ -\frac{1}{\sqrt{y}}\sin\theta & -\frac{x}{\sqrt{y}}\sin\theta + \sqrt{y}\cos\theta \end{pmatrix}, \tag{4.11}$$

with the inverse

$$x = \frac{ab + cd}{a^2 + c^2}, \qquad y = \frac{1}{a^2 + c^2}, \qquad \theta = \arctan\left(-\frac{c}{a}\right). \tag{4.12}$$

**Remark 4.1** *The function which maps $g \in SL(2,\mathbb{R})$ to a triple of matrices $(k, a, n)$ from (4.9) is continuous with a continuous inverse, i.e. topologically $K \cong S^1$, $A \cong \mathbb{R}^+ \cong \mathbb{R}$ and $N \cong \mathbb{R}$, and therefore $SL(2,\mathbb{R}) \cong S^1 \times \mathbb{R}^2$. Due to the fact that the plane $\mathbb{R}^2$ is homeomorphic to the open disk $\mathcal{D}$, we conclude that $\widetilde{SL(2,\mathbb{R})}$ is topologically homeomorphic to the interior of a torus solid. Hence, we can imagine $\widetilde{SL(2,\mathbb{R})}$ as a solid torus, fibering by circles over the disk $\mathcal{D}$ (see [6]).*

### 4.3. Isometry between two models of $\widetilde{SL(2,\mathbb{R})}$ geometry

**Theorem 4.2** *The hyperboloid model of* $\widetilde{SL(2,\mathbb{R})}$ *geometry with the metric (4.7) is isometric to the right half-space model of* $\widetilde{SL(2,\mathbb{R})}$ *geometry with the metric (4.10).*

**Proof** First, we explain the construction of bijection between the hyperboloid model $\mathcal{H}$ and the right half-space model $\mathcal{R}$.

By the equations in (4.5) and (4.6), the bijection between $SL(2,\mathbb{R})$ and $\mathcal{H}$ is determined and by the equation in (4.11), the bijection between $SL(2,\mathbb{R})$ and $\mathcal{R}$ is determined. Composing these bijections, we obtain the bijection between $\mathcal{H}$ and $\mathcal{R}$.

Beware that different notations of matrix elements are used in the hyperboloid model and in the Iwasawa decomposition. After proper renaming of the equations (4.12), we obtain the right half-space coordinates $(x, y, \theta)$ expressed by matrix components $d, b, c, a$ in the following way:

$$x = \frac{ac+bd}{c^2+d^2}, \qquad y = \frac{1}{c^2+d^2}, \qquad \theta = \arctan\left(-\frac{c}{d}\right). \tag{4.13}$$

Substituting first (4.6) in (4.5) and then in (4.13), we obtain relations between the hyperboloid and the right half-space coordinates.

The function $\pi : \mathcal{H} \to \mathcal{U}$ which $(r, \vartheta, \varphi) \mapsto (x, y, \theta)$ is given by

$$\begin{aligned} x &= \frac{\cos\vartheta}{\coth 2r - \sin\vartheta}, \\ y &= \frac{1}{\sinh 2r(\coth 2r - \sin\vartheta)}, \\ \theta &= \arctan\left(\frac{\sin\varphi - \tanh r \cdot \cos(\varphi-\vartheta)}{\cos\varphi + \tanh r \cdot \sin(\varphi - \vartheta)}\right). \end{aligned} \tag{4.14}$$

First notice that the first two formulas in (4.14) coincide with the Cayley transform (2.5) and therefore (4.14) presents generalization of (2.5).

The inverse $\pi^{-1}(x, y, \theta) = (r, \vartheta, \varphi)$ is given by

$$\begin{aligned} r &= \operatorname{Artanh}\sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}}, \\ \vartheta &= \arctan\left(\frac{x^2+y^2-1}{2x}\right), \\ \varphi &= \arctan\left(\frac{x+(y+1)\tan\theta}{(y+1)-x\tan\theta}\right). \end{aligned} \tag{4.15}$$

The obtained functions are smooth and therefore $\pi$ is a diffeomorphism.

Next, we prove that $\pi^*$ is an isometry between the right half-space and the hyperboloid model, i.e. $\pi^* g_{\mathcal{R}} = g_{\mathcal{H}}$.

Differentiation of (4.14) gives

$$\begin{aligned} dx &= \frac{2\cos\vartheta}{\sinh^2 2r(\coth 2r - \sin\vartheta)^2} dr + \frac{1 - \coth 2r \sin\vartheta}{(\coth 2r - \sin\vartheta)^2} d\vartheta, \\ dy &= \frac{2(-1 + \coth 2r \sin\vartheta)}{\sinh 2r(\coth 2r - \sin\vartheta)^2} dr + \frac{\cos\vartheta}{\sinh 2r(\coth 2r - \sin\vartheta)^2} d\vartheta, \\ d\theta &= \frac{-\cos\vartheta}{\cosh^2 2r(1 - 2\sin\vartheta\tanh r + \tanh^2 r)} dr + \frac{\tanh r(\sin\vartheta - \tanh r)}{1 - 2\sin\vartheta\tanh r + \tanh^2 r} d\vartheta + d\varphi. \end{aligned}$$

Finally, substituting these expressions in (4.10), after a long but straightforward calculations, we obtain the formula (4.7). □

**Acknowledgments**

The author would like to thank the referees for their careful reading of the manuscript and suggestions for improving this article.

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