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Eta quotients of level 12 and weight 1

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Abstract: We find all the eta quotients in the spaces $M_1\left(\Gamma_0(12), \left(\frac{d}{\cdot}\right)\right)$ ($d = -3, -4$) of modular forms and determine their Fourier coefficients, where $\left(\frac{d}{\cdot}\right)$ is the Legendre–Jacobi–Kronecker symbol.

Key words: Dedekind eta function, eta quotients, Eisenstein series, modular forms, cusp forms, Fourier coefficients, Fourier series

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, and complex numbers, respectively. Let $N \in \mathbb{N}, k \in \mathbb{Z}$, and χ be a Dirichlet character of modulus dividing N. Let $\Gamma_0(N)$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Bigm| a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod N \right\}.$$

Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. It is known that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi); \tag{1.1}$$

see, for example, [9, p. 83]. The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

A product of the form

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z), \tag{1.2}$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an eta quotient.

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Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$ we define $\sigma_{(\chi,\psi)}(n)$ by

$$\sigma_{(\chi,\psi)}(n) = \sum_{1 \leq m | n} \chi(m)\psi(n/m). \tag{1.3}$$

If $n \notin \mathbb{N}$ we set $\sigma_{(\chi,\psi)}(n) = 0$. Let χ_0 denote the trivial character, that is $\chi_0(m) = 1$ for all $m \in \mathbb{Z}$. We note that $\sigma_{(\chi_0,\chi_0)}(n)$ coincides with the number of divisors function $\sigma_0(n) = \sum_{1 \leq m | n} 1$. We define three characters by

$$\chi_1(m) = \left(\frac{-3}{m}\right), \ \chi_2(m) = \left(\frac{-4}{m}\right), \ \chi_3(m) = \left(\frac{12}{m}\right), \quad (m \in \mathbb{Z}), \tag{1.4}$$

which are all the nontrivial characters modulo 12, where $\left(\frac{d}{\cdot}\right)$ is the Legendre–Jacobi–Kronecker symbol.

The cusps of $\Gamma_0(N)$ can be represented by rational numbers a/c, where $a \in \mathbb{Z}$, $c \in \mathbb{N}$, $c|N$, and $\gcd(a,c) = 1$; see [8, p. 320] and [3, p. 103]. We can choose the representatives of cusps of $\Gamma_0(12)$ as

$$1, 1/2, 1/3, 1/4, 1/6, 1/12. \tag{1.5}$$

Let $f(z)$ be an eta quotient given by (1.2). A formula for the order $v_{a/c}(f)$ of $f(z)$ at the cusp a/c (see [8, p. 320] and [7, Proposition 3.2.8]) is given by

$$v_{a/c}(f) = \frac{N}{24\gcd(c^2,N)} \sum_{1 \leq \delta | N} \frac{\gcd(\delta,c)^2 \cdot r_\delta}{\delta}. \tag{1.6}$$

It follows from the dimension formulae [9, Section 6.3] that the only nontrivial modular spaces of level 12 with trivial cuspidal subspaces are $M_2(\Gamma_0(12),\chi_0)$, $M_1(\Gamma_0(12),\chi_1)$, $M_1(\Gamma_0(12),\chi_2)$, and $M_2(\Gamma_0(12),\chi_3)$. We also see that

$$\dim(M_1(\Gamma_0(12),\chi_1)) = \dim(E_1(\Gamma_0(12),\chi_1)) = 3, \tag{1.7}$$

$$\dim(M_1(\Gamma_0(12),\chi_2)) = \dim(E_1(\Gamma_0(12),\chi_2)) = 2. \tag{1.8}$$

All the eta quotients in $M_2(\Gamma_0(12),\chi_0)$ and $M_2(\Gamma_0(12),\chi_3)$ and their Fourier coefficients are given in [10] and [1], respectively. In this paper we find all the eta quotients in $M_1(\Gamma_0(12),\chi_1)$ and $M_1(\Gamma_0(12),\chi_2)$, and determine their Fourier coefficients.

2. Preliminary results

Throughout the remainder of this paper we use the notation $q = q(z) = e^{2\pi i z}$ with $z \in \mathbb{H}$. We define the Eisenstein series $E_{\chi_1,\chi_0}(q)$ and $E_{\chi_2,\chi_0}(q)$ by

$$E\chi_1,\chi_0(q) = \frac{1}{6} + \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_0)}(n)q^n, \quad E_{\chi_2,\chi_0}(q) = \frac{1}{4} + \sum_{n=1}^{\infty} \sigma_{(\chi_2,\chi_0)}(n)q^n.$$

In view of (1.2) for $N = 12$ we define an eta quotient $f(z)$ by

$$f(z) = \eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_3}(3z)\eta^{r_4}(4z)\eta^{r_6}(6z)\eta^{r_{12}}(12z). \tag{2.1}$$

Theorem 2.1 *Let* $f(z) \in M_1(\Gamma_0(12), \chi_1)$ *be an eta quotient given by* (2.1) *, and let* $f(z) = \sum_{n=0}^{\infty} a_n q^n$ *be its Fourier series expansion. Then*

$$f(z) = b_1 E_{\chi_1,\chi_0}(q) + b_2 E_{\chi_1,\chi_0}(q^2) + b_3 E_{\chi_1,\chi_0}(q^4)$$

for unique scalars $b_1, b_2, b_3 \in \mathbb{C}$*, and the Fourier coefficients* a_n *are given by*

$$a_0 = \frac{1}{6}(b_1 + b_2 + b_3), \;\; a_n = b_1 \sigma_{\chi_1,\chi_0}(n) + b_2 \sigma_{\chi_1,\chi_0}(n/2) + b_3 \sigma_{\chi_1,\chi_0}(n/4) \text{ for } n \geq 1.$$

Proof It follows from (1.7) and [9, Theorem 5.9] that the set of Eisenstein series $\{E_{\chi_1,\chi_0}(q),\ E_{\chi_1,\chi_0}(q^2),$ $E_{\chi_1,\chi_0}(q^4)\}$ is a basis for $M_1(\Gamma_0(12), \chi_1)$. Thus,

$$f(z) = b_1 E_{\chi_1,\chi_0}(q) + b_2 E_{\chi_1,\chi_0}(q^2) + b_3 E_{\chi_1,\chi_0}(q^4)$$

for some unique scalars $b_1, b_2, b_3 \in \mathbb{C}$, from which the assertion follows by equating the coefficients of q^n on both sides. □

Similarly to Theorem 2.1, we prove the following theorem.

Theorem 2.2 *Let* $f(z) \in M_1(\Gamma_0(12), \chi_2)$ *be an eta quotient given by* (2.1)*, and let* $f(z) = \sum_{n=0}^{\infty} a_n q^n$ *be its Fourier series expansion. Then*

$$f(z) = b_1 E_{\chi_2,\chi_0}(q) + b_2 E_{\chi_2,\chi_0}(q^3)$$

for unique scalars $b_1, b_2 \in \mathbb{C}$*, and the Fourier coefficients* a_n *are given by*

$$a_0 = \frac{1}{4}(b_1 + b_2), \;\; a_n = b_1 \sigma_{\chi_2,\chi_0}(n) + b_2 \sigma_{\chi_2,\chi_0}(n/3) \text{ for } n \geq 1.$$

We use the following lemma to determine if certain eta quotients are modular forms. See [5, Theorem 5.7, p. 99], [6, Corollary 2.3, p. 37], [4, p. 174], and [7].

Lemma 2.1 *Let* $f(z)$ *be an eta quotient given by* (1.2)*, and let* $k = \frac{1}{2} \sum_{1 \leq \delta | N} r_\delta$ *and* $s = \prod_{1 \leq \delta | N} \delta^{r_\delta}$*. Suppose that the following conditions are satisfied:*

(i) $\sum_{1 \leq \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{1 \leq \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(iii) $\sum_{1 \leq \delta | N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \geq 0$ *for each positive divisor* d *of* N,

(iv) k *is an integer.*

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, *where the character* χ *is given by*

$$\chi(m) = \Big(\frac{(-1)^k s}{m}\Big). \tag{2.2}$$

We take $N = 12$ and $k = 1$ in Lemma 2.1 to obtain the following theorem.

Theorem 2.3 *Let* $f(z)$ *be an eta quotient given by* (2.1), *which satisfies the conditions* (i)–(iv) *in* Lemma 2.1 *with*

$$r_1 + r_2 + r_3 + r_4 + r_6 + r_{12} = 2. \tag{2.3}$$

Then $f(z) \in M_1(\Gamma_0(12), \chi)$, *where the character* χ *is determined by*

$$f(z) \in \begin{cases} M_1(\Gamma_0(12), \chi_1) & \text{if } r_3 + r_6 + r_{12} \equiv 1 \pmod 2, \\ M_1(\Gamma_0(12), \chi_2) & \text{if } r_3 + r_6 + r_{12} \equiv 0 \pmod 2. \end{cases} \tag{2.4}$$

Proof For $N = 12$ we have

$$s = \prod_{1 \le \delta | 12} \delta^{r_\delta} = 1^{r_1} 2^{r_2} 3^{r_3} 4^{r_4} 6^{r_6} 12^{r_{12}} = 2^{r_2 + 2r_4 + r_6 + 2r_{12}} 3^{r_3 + r_6 + r_{12}}. \tag{2.5}$$

The conditions (i) and (ii) in Lemma 2.1 become

$$r_1 + 2r_2 + 3r_3 + 4r_4 + 6r_6 + 12r_{12} \equiv 0 \pmod{24}, \tag{2.6}$$

$$12r_1 + 6r_2 + 4r_3 + 3r_4 + 2r_6 + r_{12} \equiv 0 \pmod{24}, \tag{2.7}$$

respectively. From (2.3), (2.6), and (2.7) we have

$$r_2 + r_6 \equiv 0 \pmod 2. \tag{2.8}$$

Then (2.4) follows from (2.2), (2.5), and (2.8). □

3. Main results

Theorem 3.1 *Let* $f(z)$ *be an eta quotient given by* (2.1). *Then we have* $f(z) \in M_1(\Gamma_0(12), \chi_1)$ *if and only if*

$$r_1 + 2r_2 + 3r_3 + 4r_4 + 6r_6 + 12r_{12} \equiv 0 \pmod{24},$$

$$12r_1 + 6r_2 + 4r_3 + 3r_4 + 2r_6 + r_{12} \equiv 0 \pmod{24},$$

$$0 \le v_{1/c}(f) < 3 \text{ for } c = 1, 2, 3, 4, 6, 12,$$

$$r_1 + r_2 + r_3 + r_4 + r_6 + r_{12} = 2,$$

$$r_3 + r_6 + r_{12} \equiv 1 \pmod 2.$$

Proof Let $f(z) \in M_1(\Gamma_0(12), \chi_1)$ be an eta quotient given by (2.1). By (1.7) we have $\dim(M_1(\Gamma_0(12), \chi_1)) = 3$. We define the eta quotients $f_1(z), f_2(z), f_3(z)$ by

$$f_1(z) = \frac{\eta^{15}(2z)\eta^2(3z)\eta^2(12z)}{\eta^6(z)\eta^6(4z)\eta^5(6z)}, f_2(z) = \frac{\eta^3(z)\eta^3(12z)}{\eta(2z)\eta(3z)\eta(4z)\eta(6z)}, f_3(z) = \frac{\eta(2z)\eta^6(12z)}{\eta^2(4z)\eta^3(6z)}.$$

By Lemma 2.1, we have $f_1(z), f_2(z), f_3(z) \in M_1(\Gamma_0(12),\chi_1)$. One can easily see that the set $\{f_1(z), f_2(z), f_3(z)\}$ is linearly independent, and so it is a basis for $M_1(\Gamma_0(12),\chi_1)$. Appealing to (1.5) and (1.6), we have

$$v_1(f_1) = v_{1/12}(f_1) = 0,\ v_1(f_2) = v_{1/12}(f_2) = 1,\ v_1(f_3) = 0,\ v_{1/12}(f_3) = 2.$$

Thus, for any $b_1, b_2, b_3 \in \mathbb{C}$ we have

$$v_1(b_1f_1 + b_2f_2 + b_3f_3) \in \mathbb{N}_0,\ v_{1/12}(b_1f_1 + b_2f_2 + b_3f_3) \in \mathbb{N}_0.$$

As $f(z)$ can be expressed as a linear combination of $f_1(z), f_2(z)$, and $f_3(z)$, we have

$$v_1(f) \in \mathbb{N}_0,\ v_{1/12}(f) \in \mathbb{N}_0,$$

from which the second and first assertions follow, respectively. The third assertion follows from [6, Corollary 2.3] and the fifth assertion follows from (2.4). The converse follows from Theorem 2.3. □

Theorem 3.2 *Let $f(z)$ be an eta quotient given by* (2.1)*. Then we have $f(z) \in M_1(\Gamma_0(12),\chi_2)$ if and only if*

$$\begin{aligned}
&r_1 + 2r_2 + 3r_3 + 4r_4 + 6r_6 + 12r_{12} \equiv 0 \pmod{24},\\
&12r_1 + 6r_2 + 4r_3 + 3r_4 + 2r_6 + r_{12} \equiv 0 \pmod{24},\\
&0 \le v_{1/c}(f) < 2 \text{ for } c = 1,2,3,4,6,12,\\
&r_1 + r_2 + r_3 + r_4 + r_6 + r_{12} = 2,\\
&r_3 + r_6 + r_{12} \equiv 0 \pmod{2}.
\end{aligned}$$

Proof Let $f(z) \in M_1(\Gamma_0(12),\chi_2)$ be an eta quotient given by (2.1). By (1.8) we have $\dim(M_1(\Gamma_0(12),\chi_2)) = 2$. We define the eta quotients $g_1(z)$ and $g_2(z)$ by

$$g_1(z) := \frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)},\ g_2 := \frac{\eta^3(2z)\eta(6z)\eta^2(12z)}{\eta(z)\eta(3z)\eta^2(4z)}.$$

By Lemma 2.1, we have $g_1(z), g_2(z) \in M_1(\Gamma_0(12),\chi_2)$. One can easily see that the set $\{g_1(z), g_2(z)\}$ is a basis for $M_1(\Gamma_0(12),\chi_2)$. By (1.5) and (1.6) we see that $v_1(b_1g_1 + b_2g_2),\ v_{1/12}(b_1g_1 + b_2g_2) \in \mathbb{N}_0$ for any $b_1, b_2 \in \mathbb{C}$. As $f(z)$ can be expressed as a linear combination of $g_1(z)$ and $g_2(z)$, we have $v_1(f), v_{1/12}(f) \in \mathbb{N}_0$, from which the second and first assertions follow, respectively. The third assertion follows from [6, Corollary 2.3] and the fifth assertion follows from (2.4). The converse follows from Theorem 2.3. □

There are 21 eta quotients in $M_1(\Gamma_0(12),\chi_1)$ and 6 eta quotients in $M_1(\Gamma_0(12),\chi_2)$. We found all the eta quotients with MAPLE using Theorems 3.1 and 3.2. We then determined their Fourier coefficients using Theorems 2.1 and 2.2. All these eta quotients and their Fourier coefficients are listed in Tables 1 and 2 below.

4. Applications and remarks

Theorem 4.1 *Let $f(z) \in M_1(\Gamma_0(12),\chi_1)$ with the Fourier series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

Then for all $m \ge 0$ we have $a_{6m+5} = 0$.

Table 1. $\eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_3}(3z)\eta^{r_4}(4z)\eta^{r_6}(6z)\eta^{r_{12}}(12z) = \frac{1}{6}(b_1+b_2+b_3) + \sum_{n=1}^{\infty}\big(b_1\sigma_{(\chi_1,\chi_0)}(n) + b_2\sigma_{(\chi_1,\chi_0)}(n/2) + b_3\sigma_{(\chi_1,\chi_0)}(n/4)\big)q^n$.

No.	r_1	r_2	r_3	r_4	r_6	r_{12}	b_1	b_2	b_3
1	-6	15	2	-6	-5	2	12	-12	6
2	-4	8	4	-3	-4	1	6	-4	4
3	-3	6	1	0	-2	0	3	0	3
4	-3	8	1	-4	-4	4	3	-4	1
5	-2	1	6	0	-3	0	4	0	2
6	-2	5	-2	-2	5	-2	0	4	2
7	-1	-1	3	3	-1	-1	3	2	1
8	-1	1	3	-1	-3	3	1	-2	1
9	0	-3	0	6	1	-2	3	3	0
10	0	-2	0	1	6	-3	2	4	0
11	0	-1	0	2	-1	2	0	-1	1
12	0	1	0	-2	-3	6	1	-1	0
13	0	6	0	-3	-2	1	-6	12	0
14	1	-4	-3	4	8	-4	3	4	-1
15	1	-2	-3	0	6	0	-1	0	1
16	2	-5	-6	2	15	-6	4	4	-2
17	2	-1	2	0	-1	0	0	8	-2
18	3	-3	-1	3	1	-1	3	6	-3
19	3	-1	-1	-1	-1	3	-3	2	1
20	4	-4	-4	1	8	-3	6	4	-4
21	6	-3	-2	0	1	0	12	0	-6

Table 2. $\eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_3}(3z)\eta^{r_4}(4z)\eta^{r_6}(6z)\eta^{r_{12}}(12z) = \frac{1}{4}(b_1+b_2) + \sum_{n=1}^{\infty}\big(b_1\sigma_{(\chi_2,\chi_0)}(n) + b_2\sigma_{(\chi_2,\chi_0)}(n/3)\big)q^n$.

No.	r_1	r_2	r_3	r_4	r_6	r_{12}	b_1	b_2
1	-4	10	0	-4	0	0	0	4
2	-2	3	2	-1	1	-1	2	2
3	-1	1	-1	2	3	-2	3	1
4	-1	3	-1	-2	1	2	-1	1
5	0	0	-4	0	10	-4	4	0
6	2	1	-2	-1	3	-1	6	-2

Proof Suppose $n \equiv 2 \pmod 3$. Then n is not a perfect square, and

$$\chi_1(n) = \left(\frac{-3}{n}\right) = \left(\frac{n}{3}\right) = -1.$$

Also, for all positive divisors d of n, we have

$$\chi_1(n/d) = \left(\frac{-3}{n/d}\right) = \left(\frac{-3}{nd}\right) = \left(\frac{-3}{n}\right)\left(\frac{-3}{d}\right) = -\left(\frac{-3}{d}\right) = -\chi_1(d).$$

By pairing $\chi_1(d)$ and $\chi_1(n/d)$ for all $d \mid n$ we obtain

$$\sum_{d \mid n} \chi_1(d) = 0. \tag{4.1}$$

The assertion now follows from (4.1), (1.3), and Theorem 2.1. □

Theorem 4.2 *Let* $f(z) \in M_1(\Gamma_0(12), \chi_2)$ *with the Fourier series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

Then for all $m \geq 0$ *we have* $a_{12m+7} = a_{12m+11} = 0$.

Proof Suppose $n \equiv 3 \pmod 4$. Arguing as in the proof of Theorem 4.1 we obtain

$$\sum_{d|n} \chi_2(d) = 0. \tag{4.2}$$

The assertion now follows from (4.2), (1.3), and Theorem 2.2. □

The following corollary follows immediately from Theorems 4.1 and 4.2.

Corollary 4.1 *If an eta quotient* $f(z)$ *given by* (2.1) *is a modular form of weight* 1 *with the Fourier series representation* $f(z) = \sum_{n=0}^{\infty} a_n q^n$, *then for all* $m \geq 0$ *we have*

$$\begin{aligned} &a_{12m+11} = 0, \\ &a_{12m+7} = 0 \text{ if } r_3 + r_6 + r_{12} \equiv 0 \pmod 2, \\ &a_{12m+5} = 0 \text{ if } r_3 + r_6 + r_{12} \equiv 1 \pmod 2. \end{aligned}$$

Remark 4.1 The method used in this paper can also be applied to determine the Fourier series representations of eta quotients in other modular form spaces.

Remark 4.2 Berkovich and Patane [2] recently determined the Fourier coefficients of certain eta quotients of weight 1 and levels 47, 71, 135, 648, 1024, and 1872. They used the theory of binary quadratic forms.

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