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Degenerate maximal hyponormal differential operators for the first order

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Abstract: In this study, all maximal hyponormal extensions are given for the degenerate first order in the Hilbert space of vector-functions on a finite interval. The extensions are defined in terms of the boundary values. The structure of the spectrum of the maximal hyponormal extensions is also investigated.

Key words: Degenerate differential operator, formally hyponormal and hyponormal operator, minimal and maximal operators, extension, spectrum of an operator

1. Introduction

Differential operators theory plays a very important role in mechanics and theoretical physics. Moreover, its spectral analysis is one of the most essential fields of modern mathematical physics. Likewise, nonself-adjoint operator theory has attracted the attention of mathematicians, physicists, and engineers.

A linear closed operator $T : D(T) \subset H \to H$ in a Hilbert space H is called hyponormal if $D(T) \subset D(T^*)$ and $\|T^*x\| \leq \|Tx\|$ for each $x \in D(T)$. This operator class has been studied extensively [14, 15, 17, 18, 21]. A hyponormal operator is called a maximal hyponormal operator iff it has a not nontrivial hyponormal extension.

The general boundary conditions are given in [7–9, 11] when these extensions of the first order are normal operators in L^2 and also with smooth coefficient in [10]. This problem was investigated in the nondegenerate case by Ismailov and Karatash [12]. Moreover, all maximal hyponormal extensions were described with an unbounded operator coefficient in [13]. Instead of a constant operator, we work with a smooth operator function coefficient.

Moreover, the theory of the degenerate Cauchy problem in Banach space has been studied by many authors [1, 5, 19, 20, 22]. This is why the problem has many applications in mathematical physics and in the applied sciences, for example, in the study of the longitudinal oscillations of DNA molecules [3].

Throughout this paper we suppose that H is a Hilbert space with $2 \leq \dim H < +\infty$, $B(H)$ is the linear bounded operators space in H, and $L^2 := L^2(H, (a, b))$ is the H Hilbert valued function space defined on a finite interval $[a, b]$ [4].

2. Maximal hyponormal extensions

Let H be a Hilbert space with $2 \leq \dim H < +\infty$. In the space L^2, consider a linear degenerate differential operator expression for the first order in the form

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$$l(u) = Au'(t) + B(t)u(t), \tag{2.1}$$

where $A : H \to H$ is a self-adjoint positive operator with $1 \le \dim KerA < \dim H$ and $B(t) : [a, b] \to B(H)$ is a strongly continuous self-adjoint operator function.

It is clear that the formally adjoint expression in the space L^2 is in the form of

$$l^+(v) = -(Av(t))' + B(t)v(t). \tag{2.2}$$

Now let us define the operator L'_0 in L^2 on the dense manifold of the vector-functions

$$D'_0 := \{u(t) \in L^2 : u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k,\ \varphi_k(t) \in C_0^\infty(a, b),\ f_k \in H,$$
$$k = 1, 2, \ldots, n,\ n \in \mathbb{N}\}$$

as $L'_0 u := l(u)$.

L'_0 has closure since the domain of L'^*_0 contains the dense linear manifold D'_0. The closure of L'_0 in L^2 is called the minimal operator generated by expression (2.1) and it is denoted by L_0.

Similarly, the minimal operator L_0^+ in L^2 generated by the differential-operator expression (2.2) can be defined. The adjoint operator of L_0^+ (L_0) in L^2 is called the maximal operator generated by (2.1) ((2.2)) and is denoted by L (L^+) [2, 4]. Notice that $L_0 \subset L, L_0^+ \subset L^+$, $D(L_0) = \{u \in L^2 : (Au)' \in L^2$ and $u(a), u(b) \in KerA\}$ and $D(L) = \{u \in L^2 : (Au)' \in L^2\}$.

Also, we abbreviate $H_0 := KerA$, $H_1 := RangeA$ and $A_1 := A|_{H_1} : H_1 \to H_1$.

Theorem 2.1 *Let $\|B'(t)\| \in L^2(a, b)$. The minimal operator L_0 is formally hyponormal on L^2 if and only if the following conditions are satisfied:*

1) $AB(t) = B(t)A\ for\ every\ t \in [a, b]$,

2) $AB'(t) \le 0$ a.e. in $[a, b]$.

Proof Suppose that L_0 is a formally hyponormal operator on L^2; then, for each $u(t) \in D(L_0) \subset D(L_0^*)$, the following equality holds:

$$\|L_0 u\|_{L^2}^2 - \|L^+ u\|_{L^2}^2 = 2\left[((Au(t))', B(t)u(t)) + (B(t)u(t), (Au(t))')\right] \geqslant 0. \tag{2.3}$$

Also, if it is used before the inequation for $e^{i\alpha t}u(t) \in D(L_0)$ and all $\alpha \in \mathbb{R}$, then

$$((Au(t))', B(t)u(t)) + (B(t)u(t), (Au(t))') \geqslant i\alpha\left((B(t)A - AB(t))u(t), u(t)\right).$$

We claim that $((B(t)A - AB(t))u(t), u(t)) = 0$ for all $u(t) \in D(L_0)$. If it is not true, there exists an element $u(t) \in D(L_0)$ where $((B(t)A - AB(t))u(t), u(t))$ is a pure complex number. Since α is an arbitrary real number, then $\|L_0 u\|_{L^2}^2 - \|L^+ u\|_{L^2}^2$ is infinite. This is a contradiction, so our assertion is correct. Moreover, since $D(L_0)$ is dense in L^2 and $B(t)$ is continuous operator function,

$$AB(t) = B(t)A \text{ for } t \in [a, b].$$

Moreover, by substituting this equation in (2.3) for all $u(t) \in D(L_0)$,

$$\|L_0 u\|_{L^2}^2 - \|L^+ u\|_{L^2}^2 = -2\left((AB'(t)u(t), u(t)\right) \geqslant 0.$$

Therefore,

$$AB'(t) \leq 0 \text{ a.e. in } [a, b].$$

The contrary of the theorem can be seen from inequation (2.3). □

Theorem 2.2 *[23] Let T be a densely defined symmetric operator on H. Suppose that $H_+ \subset Ker(T^* - i)$ and $H_- \subset Ker(T^* + i)$ are closed linear subspaces of H such that $\dim H_+ = \dim H_-$ and U is an isometric linear mapping of H_+ onto H_-. Define*

$$D(T_U) = D\left(\bar{T}\right) \dot{+} (E - U) H_+ \text{ and } T_U (x + (E - U) y) = \bar{T}x + iy + iUy$$

for $x \in D\left(\bar{T}\right)$ and $y \in H_+$, where the symbol $\dot{+}$ denotes the direct sum of vector spaces.

Then T_U is a closed symmetric operator such that $T \subset T_U$. Any closed symmetric extension of T on H is of this form.

Theorem 2.3 *Let L_0 be a formally hyponormal operator. Then every maximal hyponormal extension L_h of the minimal operator L_0 has the boundary condition*

$$A^{1/2}u(b) = WA^{1/2}u(a), \tag{2.4}$$

where W is a unitary operator on H_1 and $A^{1/2}\left(W^*B(b)W - B(a)\right)A^{1/2}$ is a positive operator. The unitary operator W is determined uniquely by the extension L_h, i.e. $L_h = L_W$.

On the contrary, the restriction of the maximal operator L that satisfies (2.4) with corresponding property is a maximal hyponormal extension of the minimal operator L_0.

Proof Suppose that L_h is a maximal hyponormal extension of L_0. $B(t)$ is uniformly bounded and L_h is a closed operator; consequently, the operator

$$Im(L_h)u = -i\frac{d}{dt}Au(t), \quad u \in D(L_h)$$

is closed because for each $u \in D(L_h) \subset D(L_h^*)$

$$\begin{aligned}
(L_h u, u)_{L^2} - (u, L_h^* u)_{L^2} &= -i\left[(Im(L_h)u, u)_{L^2} - (u, Im(L_h)u)_{L^2}\right] \\
&= (A^{1/2}u(b), A^{1/2}u(b))_H - (A^{1/2}u(a), A^{1/2}u(a))_H \\
&= \left\|A^{1/2}u(b)\right\|^2 - \left\|A^{1/2}u(a)\right\|^2 = 0,
\end{aligned}$$

$Im(L_h)$ is a symmetric extension of the imaginary part of the minimal operator L_0 in L^2 and there exists an isometric operator $WA^{1/2}u(a) = A^{1/2}u(b)$. Moreover,

$$Ker(\operatorname{Im}(L) - i) = \left\{e^{A_1^{-1}(t-a)}f : f \in H_1\right\} \text{ and } Ker(\operatorname{Im}(L) + i) = \left\{e^{A_1^{-1}(b-t)}f : f \in H_1\right\},$$

and from Theorem 2.2 this extension domain has the form

$$D(L_h) = D(\mathrm{Im}L_h) = D(L_0) \dot{+} (E-U)\left(\left\{e^{A_1^{-1}(t-a)}f : f \in M_+\right\}\right),$$

where M_+ and M_- are subspaces of H_1 and $U : e^{A_1^{-1}(t-a)}M_+ \to e^{A_1^{-1}(b-a)}M_-$ is an isomorphism. Hence,

$$H_a = \{u(a) \in H : u(t) \in D(L_h)\},$$

$$H_b = \{u(b) \in H : u(t) \in D(L_h)\}$$

are subspaces of H. We claim that H_a or H_b must be equal to H. If it is not, because $\dim((A^{1/2}(H_a) \cap Ker(W-E)) \oplus (W-E)A^{1/2}(H_a)) = \dim A^{1/2}(H_a)$ and $\dim H < +\infty$, there exists a nonzero element f in H_1 such that for all $x \in A^{1/2}(H_a)$,

$$((W-E)x, f)_{H_1} = 0 \quad \text{and} \quad f \notin A^{1/2}(H_a) \cap Ker(W-E).$$

Thus, an extension $\tilde{L_h}$ of L_h can be constructed such that $D(\tilde{L_h}) = span\{D(L_h), A_1^{-1/2}f\}$. It is obvious that $D(\tilde{L_h}) \subset D(\tilde{L_h}^*)$ and $\tilde{L_h}$ is a hyponormal operator. This is a contradiction, so $H = H_a = H_b$ and $W : H_1 \to H_1$ is a unitary operator. Hence, $Im(L_h)$ is a self-adjoint operator on L^2 and the unitary operator W is determined uniquely by the extension of L_h.

Since L_h is a maximal hyponormal extension operator of the minimal operator L_0, for every $u(t) \in D(L_h)$ the following inequality holds:

$$\begin{aligned}(L_h u(t), L_h u(t))_{L^2} - (L_h^* u(t), L_h^* u(t))_{L^2} &= 2\left[(B(t)Au(t), u(t))'_{L_2} - (AB'(t)u(t), u(t))_{L_2}\right] \\ &= 2\left[\left(A^{1/2}\left(W^*B(b)W - B(a)\right)A^{1/2}u(a), u(a)\right)_{L_2} - (AB'(t)u(t), u(t))_{L_2}\right] \geqslant 0.\end{aligned}$$

Because $D(L_0) \subset D(L_h)$, from the last relation we obtain that

$$\left(A^{1/2}\left(W^*B(b)W - B(a)\right)A^{1/2}u(a), u(a)\right)_{L_2} - (AB'(t)(u-v)(t), (u-v)(t))_{L_2} \geqslant 0,$$

for $u \in D(L_h)$, $v \in D(L_0)$. Since $D(L_0)$ is dense, there exists a sequence $v_n \in D(L_0)$ such that $v_n \to u, n \to +\infty$. From this result and $\|B'(t)\| \in L^2(a,b)$,

$$\left(A^{1/2}\left(W^*B(b)W - B(a)\right)A^{1/2}u(a), u(a)\right)_{L_2} \geqslant 0,$$

i.e. $A^{1/2}\left(W^*B(b)W - B(a)\right)A^{1/2}$ is a positive operator on H.

In this case the adjoint operator $L_W{}^*$ is generated by the differential-operator expression $l^*(v) = -(Av(t))' + B(t)v(t)$ and for every $v(t) \in D(L_W{}^*)$ is satisfied with the boundary condition $A^{1/2}v(a) = W^*A^{1/2}v(b)$. It is easy to see that $D(L_W) \subset D(L_W{}^*)$ and the other condition of hyponormal extensions in L^2 can be easily obtained. □

3. The spectrum of maximal hyponormal extensions

In this section, the spectrum of the maximal hyponormal extension L_W of minimal operator L_0 will be investigated.

Let $U(t,s)$, $t,s\in[a,b]$ be the family of evolution operators in H_1 corresponding to the homogeneous differential equation

$$\begin{cases} U'_t(t,s)f + A_1^{-1}B(t)U(t,s)f = 0, & t,s\in[a,b] \\ U(s,s)f = f, & f\in H_1 \end{cases}$$

and A_1 and $U(t,s)$ be commutative for all $t,s\in[a,b]$ (for more detailed analysis, see [6, 16]).

Theorem 3.1 *The spectrum of the maximal hyponormal extension L_W has the form*

$$\sigma(L_W) = \{\lambda\in\mathbb{C}:\ 1\in\sigma(WU(a,b)e^{\lambda A_1^{-1}(a-b)})\} \cup \left\{\lambda\in\mathbb{C}:\ \lambda\in\sigma\left(B(t)|_{H_0}\right),\ t\in[a,b]\right\}.$$

Proof Consider the following problem for the spectrum for the maximal hyponormal extension L_W:

$$L_W u = (Au(t))' + B(t)u(t) = \lambda u(t) + f(t),\ u\in D(L_W),\ f\in L^2,\ \lambda\in\mathbb{C}.$$

It can be written as $u(t)=u_1(t)+u_0(t)$, $u_i(t)\in H_i$, $i=0,1$ for all $t\in[a,b]$, and $(Au(t))'=(Au_1(t))'$. Also, since the restriction operator A_1 on H_1 has a bounded inverse, $u_1'(t)$ exists. Therefore, the general solution of this differential equation in L^2 has the form

$$u_{\lambda,1}(t) = e^{\lambda A_1^{-1}(t-a)}U(t,a)f + \int_a^t e^{\lambda A_1^{-1}(t-s)}U(t,s)A_1^{-1}f_1(s)ds, \qquad f\in H_1,$$

$$(B(t)-\lambda E)u_{\lambda,0}(t) = f_0(t).$$

The following relation can be obtained from the boundary condition $A^{1/2}u(b) = WA^{1/2}u(a)$, where W is a unitary operator in H_1:

$$\left(WU(a,b)e^{\lambda A_1^{-1}(a-b)} - E\right)A^{1/2}f = \int_a^t e^{\lambda A_1^{-1}(a-s)}U(a,s)A_1^{-1/2}f_1(s)ds.$$

It is easy to see that $\lambda\in\mathbb{C}$ is a point of the spectrum of hyponormal extension L_W if and only if $1\in\sigma(WU(a,b)e^{\lambda A_1^{-1}(a-b)})$ or $\lambda\in\sigma\left(B(t)|_{H_0},\right)$, $t\in[a,b]$. □

Corollary 3.1 *If A is a projection operator on a nontrivial subspace, then the spectrum of the maximal hyponormal extension L_W has the following form:*

$$\sigma(L_W) = \left\{\lambda\in\mathbb{C}:\ \lambda = \frac{1}{b-a}\left(ln\mid\mu\mid + i(arg\mu + 2k\pi)\right),\ \mu\in\sigma(W^*U(b,a)),\ k\in\mathbb{Z}\right\}$$
$$\cup\left\{\lambda\in\mathbb{C}:\ \lambda\in\sigma\left(B(t)|_{H_0}\right),\ t\in[a,b]\right\}.$$

Corollary 3.2 *If the minimal operator L_0 is formally hyponormal, then $\sigma_r(L_0)=\mathbb{C}$.*

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