When do quasinilpotents lie in the Jacobson radical?

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Abstract: In this paper, we give some spectral characterizations of the Jacobson radical; that is, we will show that some conditions with λ-multiplicativity imply that the set of all quasinilpotent elements equals the Jacobson radical. We also give some conditions to make sure the quasinilpotents lie in the Jacobson radical, using the set of elements with singleton spectra.

Key words: Jacobson radical, quasinilpotent elements, elements with singleton spectra

1. Introduction

Let $\mathcal{A}$ be a unital Banach algebra. Let $[a, b] = ab - ba$ denote the commutator (or Lie product) and $\{a, b\} = ab + ba$ denote the anticommutator (or Jordan product) for every $a, b \in \mathcal{A}$. For every a in $\mathcal{A}$, $\sigma(a)$ and $\rho(a)$ denote the spectrum of a and the spectral radius of a, respectively. The set of all the quasinilpotent elements in $\mathcal{A}$ is denoted by $\mathcal{Q}(\mathcal{A})$; that is, $\mathcal{Q}(\mathcal{A}) = \{a \in \mathcal{A} : \rho(a) = 0\}$. The Jacobson radical of $\mathcal{A}$ is denoted by $\text{Rad}(\mathcal{A})$. It is well known that $\text{Rad}(\mathcal{A}) = \{a \in \mathcal{A} : ab \in \mathcal{Q}(\mathcal{A}) \text{ for all } b \in \mathcal{A}\}$. As usual, if M and N are subsets of $\mathcal{A}$, then $M + N := \{x + y : x \in M, y \in N\}$. If $M = \{x\}$, then we will denote $x + N = M + N$. We have the same definitions for MN and so on.

It is shown in [6] that $\mathcal{Q}(\mathcal{A}) = \text{Rad}(\mathcal{A})$ if and only if $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})$ if and only if $\mathcal{Q}(\mathcal{A})\mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})$. By [4], $\mathcal{Q}(\mathcal{A}) = \text{Rad}(\mathcal{A})$ if and only if $[\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{A})] \subset \mathcal{Q}(\mathcal{A})$ if and only if $\{\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{A})\} \subset \mathcal{Q}(\mathcal{A})$.

There are some similar results for the elements with finite spectra. Let $I(\mathcal{A})$ denote the set of all $a \in \mathcal{A}$ with $\#(\sigma(a)) < \infty$, where $\#(\cdot)$ means the cardinality of a set. Let $\mathcal{Q}_{\mathbb{C}}(\mathcal{A}) := \mathcal{Q}(\mathcal{A}) + \mathbb{C}1$ denote the set of all $a \in \mathcal{A}$ with singleton spectra. If $a + I(\mathcal{A}) \subset I(\mathcal{A})$ for some $a \in \mathcal{A}$, then $aI(\mathcal{A}) \subset I(\mathcal{A})$ and $[a, \mathcal{A}] \subset I(\mathcal{A})$ by [1, Corollary 5.6.4 and Lemma 5.6.5]. Moreover, if $\mathcal{A}$ is semisimple then $[a, \mathcal{A}] \subset I(\mathcal{A})$ if and only if $[a, \mathcal{A}] \subset \text{Soc}(\mathcal{A})$ if and only if every element in $[a, \mathcal{A}]$ is algebraic [2]. Here $\text{Soc}(\mathcal{A})$ is the socle of $\mathcal{A}$, i.e. the sum of the minimal left ideals of $\mathcal{A}$.

For any $\lambda \in \mathbb{C}$, let λ-product $a \circ_\lambda b$ of the elements a and b of $\mathcal{A}$ be simply

$$a \circ_\lambda b = ab + \lambda ba,$$

which is called the $-\lambda$-Lie product in [5]. It is clear that the λ-product is a normal product if $\lambda = 0$, is a Lie product if $\lambda = -1$, and is a Jordan product if $\lambda = 1$. If M, N, and L are subsets of $\mathcal{A}$, then

$M \circ_\lambda N \circ_\lambda L = (M \circ_\lambda N) \circ_\lambda L$ since the λ-product is usually nonassociative. Let $M \circ_\lambda \underbrace{M \circ_\lambda \cdots \circ_\lambda M}_{n}$ be the λ-powers of M, denoted by $M^{(\lambda, n+1)}$. Using the right-hand arrangement, we have

$$M^{(\lambda, n+1)} = M^{(\lambda, n)} \circ_\lambda M.$$

In this paper, the main technical tool we use is the Jacobson density theorem in Sinclair's form: if $\xi_1, \cdots, \xi_n$ and $\eta_1, \cdots, \eta_n$ are two linearly independent systems of vectors in the underlying space X_π of a strictly irreducible representation $\pi \in \mathrm{Irr}(\mathcal{A})$ of a Banach algebra $\mathcal{A}$ then there is an invertible element $a \in \mathcal{A}^{-1}$ such that

$$\pi(a)\xi_i = \eta_i$$

for $i = 1, \cdots, n$, where $\mathcal{A}^{-1}$ means the set of all invertible elements in $\mathcal{A}$. One can see [1, Theorem 4.2.5] and [1, Corollary 4.2.6].

Recall that a set of vectors is linearly independent if every one of its finite subsets is linearly independent. For instance, it is easy to show (see also [4, observation on page 161]) that if T is a quasinilpotent operator on a Banach space X and $x \in X$, then the set $\{x, Tx, T^2x, \cdots\} \backslash \{0\}$ is linearly independent.

The main aim of this paper is to give some necessary and sufficient conditions for $\mathcal{Q}(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$.

2. Quasinilpotent elements

Lemma 2.1 *Let $\mathcal{A}$ be a unital Banach algebra, and $p \in \mathcal{Q}(\mathcal{A})$ and $\lambda \in \mathbb{C}$. Then $p \in \mathrm{Rad}(\mathcal{A})$ if and only if $p \circ_\lambda \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})$.*

Proof   If $\lambda = 0$, the result is true by [4, Proposition 1].

We assume that $\lambda \neq 0$. If $p \in \mathcal{Q}(\mathcal{A})$, but p is not contained in $\mathrm{Rad}(\mathcal{A})$, then there exists an irreducible representation π on a Banach space X and an $x \in X$ such that $\pi(p)x \neq 0$. Thus, the set $S = \{x_1, x_2, \cdots, x_6\} \backslash \{0\}$ is linearly independent and contains x_1, x_2, where $x_k = \pi(p^{k-1})x$, $k = 1, ..., 6$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1, \ \pi(a)x_3 = x_3, \ \pi(a)x_4 = x_4, \ \pi(a)x_5 = x_5, \ \pi(a)x_6 = -\lambda x_6.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent. It can be checked that

$$\pi(p \circ_\lambda q)(x_2 + \lambda x_4) = x_2 + \lambda x_4.$$

Note that $x_2 + \lambda x_4 \neq 0$; thus,

$$\{1\} \subset \sigma(\pi(p \circ_\lambda q)) \subset \sigma(p \circ_\lambda q).$$

We have a contradiction, so $p \in \mathrm{Rad}(\mathcal{A})$. $\square$

Lemma 2.2 *Let $\mathcal{A}$ be a Banach algebra, $p^n \in \mathrm{Rad}(\mathcal{A})$, but $p^{n-1} \notin \mathrm{Rad}(\mathcal{A})$ for some $n > 1$. Then for every m, $2 \leq m \leq n$, there are a closed subalgebra $\mathcal{A}_m$, an m-dimensional linear subspace Y_m, and a bounded homomorphism τ_m from $\mathcal{A}_m$ onto $B(Y_m)$ such that $e_{i+1} = \tau_n(p)e_i$ for some basis $e_1, \cdots, e_n$ of Y_n, $g_1 = e_{n-m+1}, \cdots, g_m = e_n$ is a basis of Y_m, and $\tau_m(p)$ is the restriction $\tau_n(p)|_{Y_m}$ of $\tau_n(p)$ to Y_m for every m, $2 \leq m \leq n$.*

Proof There are $\pi \in \mathrm{Irr}(\mathcal{A})$ on X_π and $x \in X_\pi$ with $e_i = \pi(p^{i-1})x$ such that $e_1, \cdots, e_n$ are linearly independent and $e_{n+1} = 0$ since $p^n \in \mathrm{Rad}(\mathcal{A})$ but $p^{n-1} \notin \mathrm{Rad}(\mathcal{A})$. Let Y_m be a subspace generated by $g_1 = e_{n-m+1}, \cdots, g_m = e_n$, and $\mathcal{A}_m = \{a \in \mathcal{A} : Y_m \text{ is invariant for } \pi(a)\}$ for every m, $2 \leq m \leq n$. Let $b \in B(Y_m)$; then there is an element $a \in \mathcal{A}$ such that $\pi(a)g_i = bg_i$ for $i = 1, \cdots, m$ by the Jacobson density theorem. Therefore, $a \in \mathcal{A}_m$. Let $\tau_n = \pi$ and $\tau_m(p)$ be the restriction $\tau_n(p)|_{Y_m}$ of $\tau_n(p)$ to Y_m. Then τ_m is a bounded homomorphism from $\mathcal{A}_m$ onto $B(Y_m)$. $\square$

Lemma 2.3 *Let* $\lambda \in \mathbb{C}$ *and* $n > 0$. *Then* $\mathcal{Q}(\mathcal{A})^{(\lambda, n+1)} \subset \mathcal{Q}(\mathcal{A})$ *implies* $\mathcal{Q}(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$ *for every Banach algebra* $\mathcal{A}$ *if and only if* $\mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(\lambda, n+1)} \nsubseteq \mathcal{Q}(\mathbb{M}_2(\mathbb{C}))$.

Proof $\Rightarrow$ is obvious.

$\Leftarrow$ Assume that $\mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(\lambda, n+1)} \nsubseteq \mathcal{Q}(\mathbb{M}_2(\mathbb{C}))$ and $\mathcal{Q}(\mathcal{A})^{(\lambda, n+1)} \subset \mathcal{Q}(\mathcal{A})$ for some Banach algebra $\mathcal{A}$. Let $p \in \mathcal{Q}(\mathcal{A})$. We claim that $p^n \in \mathrm{Rad}(\mathcal{A})$. Indeed, if $\lambda = -1$ then $(\mathrm{ad}p)^n \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})$ and $p^n \in \mathrm{Rad}(\mathcal{A})$ by [4, Theorem 3], and if $\lambda \neq -1$ then

$$p^{(\lambda, n)} \circ_\lambda \mathcal{Q}(\mathcal{A}) = (1 + \lambda)^n p^n \circ_\lambda \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})$$

and $p^n \in \mathrm{Rad}(\mathcal{A})$ by Theorem 2.1.

Assume, to the contrary, that $p^{k-1} \notin \mathrm{Rad}(\mathcal{A})$ for some k, $1 < k \leq n$. By Lemma 2.2, there are a closed subalgebra $\mathcal{A}_m$, an m-dimensional linear subspace Y_m, and a bounded homomorphism τ_m from $\mathcal{A}_m$ onto $B(Y_m)$ such that $e_{i+1} = \tau_n(p)e_i$ for some basis $e_1, \cdots, e_n$ of Y_n, $g_1 = e_{n-m+1}, \cdots, g_m = e_n$ is a basis of Y_m, and $\tau_m(p)$ is the restriction $\tau_n(p)|_{Y_m}$ of $\tau(p)$ to Y_m for every m, $2 \leq m \leq n$.

Let $m = 2$. Since $\mathcal{Q}(\mathcal{A})^{(\lambda, n+1)} \subset \mathcal{Q}(\mathcal{A})$, we have that $\mathcal{Q}(\mathcal{A}_2)^{(\lambda, n+1)} \subset \mathcal{Q}(\mathcal{A}_2)$. By the Sinclair–Jacobson theorem, $\tau_2(\mathcal{A}_2) = \mathbb{M}_2(\mathbb{C})$ and $\tau_2(\mathcal{Q}(\mathcal{A}_2))$ coincides with the set of all nilpotents of $\mathbb{M}_2(\mathbb{C})$. Thus,

$$\tau_2(\mathcal{Q}(\mathcal{A}_2)^{(\lambda, n+1)}) = (\tau_2(\mathcal{Q}(\mathcal{A}_2)))^{(\lambda, n+1)} = \mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(\lambda, n+1)} \subset \tau_2(\mathcal{Q}(\mathcal{A}_2)) = \mathcal{Q}(\mathbb{M}_2(\mathbb{C})).$$

It is a contradiction. $\square$

Next we will introduce the generalized Kleinecke–Shirokov condition, which will be used for our main result.

Let $\mathcal{A}$ be a complex algebra and $\triangle_{(a, \lambda)}$ be the operator $x \mapsto ax - \lambda xa$ on $\mathcal{A}$ for every $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It is clear that

$$\triangle_{(a, \lambda_1 \lambda_2)}(x_1 x_2) = \triangle_{(a, \lambda_1)}(x_1)x_2 + \lambda_1 x_1 \triangle_{(a, \lambda_2)}(x_2).$$

We write $\triangle_a$ instead of $\triangle_{(a, 1)}$.

The following statement belongs to S. Rosenoer (1981, unpublished).

Lemma 2.4 *Let* $\mathcal{A}$ *be a Banach algebra,* $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, *and* $|\lambda| = 1$. *If* $\triangle_{(a, \lambda)}^2(x) = 0$ *then* $\triangle_{(a, \lambda)}(x) \in \mathcal{Q}(\mathcal{A})$.

Proof   Let

$$\widehat{a} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \lambda^{-1}a & 0 & 0 & 0 & \ddots \\ \ddots & 0 & a & 0 & 0 & \ddots \\ \ddots & 0 & 0 & \lambda a & 0 & \ddots \\ \ddots & 0 & 0 & 0 & \lambda^2 a & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \widehat{x} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & x & 0 & 0 & \ddots \\ \ddots & 0 & 0 & x & 0 & \ddots \\ \ddots & 0 & 0 & 0 & x & \ddots \\ \ddots & 0 & 0 & 0 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

be elements of $\mathbb{M}_{\mathbb{Z}}(\mathcal{A})$, the infinite matrices with entries from $\mathcal{A}$. One can consider these matrices as bounded operators on infinite direct sum $\overset{\infty}{\underset{k=-\infty}{\bigoplus}} \mathcal{A}$. As $\triangle_{\widehat{a}}^2(\widehat{x}) = 0$, we have that $\triangle_{\widehat{a}}(\widehat{x})$ is a quasinilpotent operator by the Kleinecke–Shirokov theorem. It is easy to check that $\triangle_{(a,\lambda)}(x) \in \mathcal{Q}(\mathcal{A})$.   □

Lemma 2.4 is not valid under $|\lambda| \neq 1$, but it is valid for finite dimensional algebras (V.S. Shulman, 1981, unpublished). The proof of the following lemma belongs to Yu.V. Turovskii; see [7, Theorem 14].

Lemma 2.5 *Let $\mathcal{A}$ be a finite dimensional algebra and $\triangle_{(a,\lambda)}^2(x) = 0$ for some $\lambda \neq 0$. Then $\triangle_{(a,\lambda)}(x)$ is a nilpotent element.*

Proof   Let $\dim \mathcal{A} = m$. Then there are $n \leq m$ and $\alpha_0, \cdots, \alpha_{n-1} \in \mathbb{C}$ such that

$$x^n(\triangle_{(a,\lambda)}(x))^{m-n} = \sum_{r=0}^{n-1} \alpha_r x^r(\triangle_{(a,\lambda)}(x))^{m-r}.$$

Acting by $\triangle_{(a,\lambda^m)}^n$ for both parts of this equality and using (3.1) and $\triangle_{(a,\lambda)}^2(x) = 0$ many times, we obtain (by induction) that

$$\triangle_{(a,\lambda^m)}^n(x^n(\triangle_{(a,\lambda)}(x))^{m-n}) = n!\lambda^{n(n-2)/2}(\triangle_{(a,\lambda)}(x))^m$$

while

$$\triangle_{(a,\lambda^m)}^n(x^r(\triangle_{(a,\lambda)}(x))^{m-r}) = 0$$

for every $r < n$. Thus, $(\triangle_{(a,\lambda)}(x))^m = 0$.   □

Theorem 2.6 *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{Q}(\mathcal{A})^{(\lambda,n+1)} \subset \mathcal{Q}(\mathcal{A})$ for some integer $n > 1$ and $\lambda \in \mathbb{C}$, then one of the following statements is valid:*

(1) $\mathcal{Q}(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$;

(2) $\lambda = 1$ and $a^2 \in \mathrm{Rad}(\mathcal{A})$ for every $a \in \mathcal{Q}(\mathcal{A})$.

Proof   Let $\lambda = 1$. We show that $a^2 \in \mathrm{Rad}(\mathcal{A})$ for every $a \in \mathcal{Q}(\mathcal{A})$. One may assume that $\mathrm{Rad}(\mathcal{A}) = \{0\}$. Assume, to the contrary, that $p \in \mathcal{Q}(\mathcal{A})$, but $p^2 \notin \mathrm{Rad}(\mathcal{A})$. Let $W_1 = \{p^2\}$ and $W_{m+1} = W_m \circ_1 \mathcal{Q}(\mathcal{A})$ for any $m > 0$. As $p^2 = \frac{1}{2}p \circ_1 p$ then $W_1 \subset \mathcal{Q}(\mathcal{A}) \cap \mathcal{Q}(\mathcal{A})^{(1,2)}$ and $W_m \subset \mathcal{Q}(\mathcal{A})^{(1,m)} \cap \mathcal{Q}(\mathcal{A})^{(1,m+1)}$ for any $m > 0$. As $W_n \subset \mathcal{Q}(\mathcal{A})^{(\lambda,n+1)} \subset \mathcal{Q}(\mathcal{A})$ and $W_n \circ_1 \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}(\mathcal{A})^{(1,n+1)} \subset \mathcal{Q}(\mathcal{A})$ then $W_n \subset \mathrm{Rad}(\mathcal{A}) = \{0\}$ by Lemma 2.1.

It follows that if $n > 2$ then $W_{n-1} \subset \mathcal{Q}(\mathcal{A})$. Indeed, if $s = t \circ_1 q$, where $t \in W_{n-2}$ and $q \in \mathcal{Q}(\mathcal{A})$, then $\triangle^2_{(q,-1)}(t) \in \mathrm{Rad}(\mathcal{A}) = \{0\}$ and $s = \triangle_{(q,-1)}(t) \in \mathcal{Q}(\mathcal{A})$ by Lemma 2.4.

As $W_{n-1} \circ_1 \mathcal{Q}(\mathcal{A}) = \{0\} \in \mathcal{Q}(\mathcal{A})$ then $W_{n-1} \subset \mathrm{Rad}(\mathcal{A}) = \{0\}$ by Lemma 2.1. By repeating the argument many times we obtain that

$$W_2 = p^2 \circ_1 \mathcal{Q}(\mathcal{A}) \in \mathrm{Rad}(\mathcal{A}) = \{0\} \in \mathcal{Q}(\mathcal{A})$$

, from which $p^2 \in \mathrm{Rad}(\mathcal{A})$ by Lemma 2.1.

By Lemma 2.3, for the other case and arbitrary λ, it suffices to examine $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$. In this case

$$\mathcal{Q}(\mathcal{A}) = \mathbb{C}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ -\alpha^{-1} & -1 \end{pmatrix} : 0 \neq \alpha \in \mathbb{C} \right\}.$$

It is clear that

$$\mathcal{Q}(\mathcal{A})^{(\lambda,2)} = \mathbb{C}\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda\alpha & 0 \\ 1-\lambda & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & \lambda-1 \\ 0 & \lambda\alpha \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1+\lambda-\alpha\beta^{-1}-\lambda\beta\alpha^{-1} & (1-\lambda)(\beta-\alpha) \\ (1-\lambda)(\beta^{-1}-\alpha^{-1}) & 1+\lambda-\alpha\beta^{-1}-\lambda\beta\alpha^{-1} \end{pmatrix} : 0 \neq \alpha, \beta \in \mathbb{C} \right\}.$$

If $\lambda = 1$, then $\mathcal{Q}(\mathcal{A})^{(\lambda,2)}$ is isomorphic to $\mathbb{C}$ and $\mathcal{Q}(\mathcal{A})^{(1,3)} = \mathcal{Q}(\mathcal{A}) \neq \{0\}$ while $\mathrm{Rad}(\mathcal{A}) = \{0\}$. This underlines that statement (2) is the best possible.

Assume that $\lambda \neq 1$. It is easy to check that

$$\begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & \gamma_1 \end{pmatrix} := \begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \gamma_0 \end{pmatrix} \circ_\lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ \gamma_0 + \lambda\alpha_0 & \lambda\beta_0 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \gamma_2 \end{pmatrix} := \begin{pmatrix} \alpha_1 & 0 \\ \beta_1 & \gamma_1 \end{pmatrix} \circ_\lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda\beta_1 & \alpha_1 + \lambda\gamma_1 \\ 0 & \beta_1 \end{pmatrix}.$$

For $\lambda^2 + 1 \neq 0$ we see that if $\alpha_1, \beta_1, \gamma_1$ are nonzero then $\alpha_2, \beta_2, \gamma_2$ are nonzero; indeed, for instance, $\beta_2 = 0$ implies $(\lambda^2 + 1)\beta_0 = 0$. This shows that if $\mathcal{Q}(\mathcal{A})^{(\lambda,m)}$ has a non-quasinilpotent element for $m > 1$ then so does $\mathcal{Q}(\mathcal{A})^{(\lambda,m+1)}$.

It remains to consider the case $\lambda^2 + 1 = 0$; that is, $\lambda = \pm i$. Let λ be i for definiteness. In this case, $\beta_2 = 0$, $\alpha_2 = i\beta_1$, and $\gamma_2 = \beta_1$. One may assume that $\beta_1 \neq 0$. Then

$$q := \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \circ_i \begin{pmatrix} 1 & \alpha \\ -\alpha^{-1} & -1 \end{pmatrix} = \begin{pmatrix} i-1 & 2i\alpha \\ -(1+i)\alpha^{-1} & -1-i \end{pmatrix}$$

and $\mathrm{trace}(q) = (i-1) + (-1-i) = -2$ shows that q is not a quasinilpotent element. In any case if $\mathcal{Q}(\mathcal{A})^{(\pm i,m)}$ has a non-quasinilpotent element for $m > 1$ then so does $\mathcal{Q}(\mathcal{A})^{(\pm i,m+1)}$. $\qquad \square$

3. Elements with singleton spectra

The main result of this section is the following. Recall that $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$ is the set of all elements in $\mathcal{A}$ with singleton spectra. In the proof of Theorem 3.1, we will use the Jacobson density theorem repeatedly. We will use the same symbols, such as π, a, and so on, each time, but it seems that this will not create any confusion.

Theorem 3.1 *Let $\mathcal{A}$ be a Banach algebra. The following conditions are equivalent:*
(i) $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$,
(ii) For some $\lambda \in \mathbb{C}$, $\mathcal{Q}_{\mathbb{C}}(\mathcal{A}) \circ_{\lambda} \mathcal{Q}_{\mathbb{C}}(\mathcal{A}) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$,
(iii) $\mathcal{Q}(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$.

Proof (i) $\Rightarrow$ (iii) First, we will show that $p^2 \in \mathrm{Rad}(\mathcal{A})$ for all $p \in \mathcal{Q}(\mathcal{A})$.

If there is a $p \in \mathcal{Q}(\mathcal{A})$ such that p^2 is not contained in $\mathrm{Rad}(\mathcal{A})$, then there exists an irreducible representation π on a Banach space X and an $x \in X$ such that $x_k = \pi(p^{k-1})x$ is nonzero for $k \leq 3$. Moreover, since p is quasinilpotent, then the set $S = \{x_1, x_2, x_3\}$ is linearly independent. We can find an invertible $a \in \mathcal{A}$ such that
$$\pi(a)x_1 = -x_1, \ \pi(a)x_2 = x_2, \ \pi(a)x_3 = x_2 - x_3.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $q \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. It is clear that

$$\pi(p+q)x_1 = 0, \ \pi(p+q)x_2 = x_2.$$

Thus,

$$\{0, 1\} \subset \sigma(\pi(p+q)) \subset \sigma(p+q),$$

and so $p + q$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$.

Next we will show that $p \in \mathrm{Rad}(\mathcal{A})$ for all $p \in \mathcal{Q}(\mathcal{A})$.

If not, there exists a $p \in \mathcal{Q}(\mathcal{A})$ such that p does not lie in $\mathrm{Rad}(\mathcal{A})$. Note that $p^2 \in \mathrm{Rad}(\mathcal{A})$, and there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2\}$ is linearly independent where $x_k = \pi(p^{k-1})x$, and $x_k = 0$ for every $k \geq 3$. We can find an invertible $a \in \mathcal{A}$ such that
$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $q \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$, and we have

$$\pi(p+q)(x_1+x_2) = x_1 + x_2, \ \pi(p+q)(x_1 - x_2) = x_2 - x_1.$$

Thus,

$$\{1, -1\} \subset \sigma(\pi(p+q)) \subset \sigma(p+q),$$

and so $p + q$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$.

(ii) $\Rightarrow$ (iii) There are three cases.

If $\lambda = 0$, then (ii) means that $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})\mathcal{Q}_{\mathbb{C}}(\mathcal{A}) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. We will prove that (ii) $\Rightarrow$ (i), so in this case (ii) $\Rightarrow$ (iii) holds by (i) $\Rightarrow$ (iii). In fact, for every $a \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$, $\mu \in \mathbb{C}$, and $|\mu| > \rho(a)$, note that $(a-\mu)^{-1} \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. Thus, for $b \in \mathcal{Q}(\mathcal{A})$, we have

$$a + b - \mu = (a - \mu)(1 + (a - \mu)^{-1}b) \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A}),$$

since $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})\mathcal{Q}_{\mathbb{C}}(\mathcal{A}) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. That is, $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. (i) holds.

If $\lambda \neq 0$, and there is a $p \in \mathcal{Q}(\mathcal{A})$ such that p is not contained in $\mathrm{Rad}(\mathcal{A})$, there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}\backslash\{0\}$ is

linearly independent and contains x_1, x_2, where $x_k = \pi(p^{k-1})x, 1 \le k \le 6$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1, \ \pi(a)x_3 = x_3, \ \pi(a)x_4 = x_4, \ \pi(a)x_5 = x_5, \ \pi(a)x_6 = -\lambda x_6.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $q \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$, and we have

$$\pi(p \circ_\lambda q)(x_1 + \frac{1}{\lambda}x_4) = \lambda(x_1 + \frac{1}{\lambda}x_4), \ \pi(p \circ_\lambda q)(x_2 + \lambda x_4) = x_2 + \lambda x_4.$$

Thus,

$$\{\lambda, 1\} \subset \sigma(\pi(p \circ_\lambda q)) \subset \sigma(p \circ_\lambda q),$$

and so $p \circ_\lambda q$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$ for $\lambda \ne 1$.

If $\lambda = 1$, we will show that $p^2 \in \mathrm{Rad}(\mathcal{A})$ for all $p \in \mathcal{Q}(\mathcal{A})$ in the first step.

If not, there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2, x_3, x_4, x_5\} \backslash \{0\}$ is linearly independent and contains x_1, x_2, x_3, where $x_k = \pi(p^{k-1})x, 1 \le k \le 5$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = -x_1, \ \pi(a)x_3 = -x_3, \ \pi(a)x_4 = x_4, \ \pi(a)x_5 = x_5.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $q \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$, and we have

$$\pi(pq + qp)(x_1 - x_2) = x_2 - x_1, \ \pi(pq + qp)x_3 = 0.$$

Thus,

$$\{-1, 0\} \subset \sigma(\pi(pq + qp)) \subset \sigma(pq + qp),$$

and so $pq + qp$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$.

Next we will show that $p \in \mathrm{Rad}(\mathcal{A})$ for all $p \in \mathcal{Q}(\mathcal{A})$.

If not, then the set $S = \{x_1, x_2\}$ is linearly independent and $x_k = 0$ for every $k \ge 3$. We can find an invertible $a \in \mathcal{A}$ such that
$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $1 + p, 1 + q \in \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$, and we have

$$\pi((1+p)(1+q) + (1+q)(1+p))(x_1 + x_2) = 5(x_1 + x_2), \ \pi((1+p)(1+q) + (1+q)(1+p))(x_1 - x_2) = x_1 - x_2.$$

Thus,
$$\{1, 5\} \subset \sigma(\pi((1+p)(1+q) + (1+q)(1+p))) \subset \sigma((1+p)(1+q) + (1+q)(1+p)),$$

and so $(1+p)(1+q) + (1+q)(1+p)$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$.

In a word, (ii) implies (iii).

(iii) $\Rightarrow$ (i) or (ii). As $\mathcal{Q}_{\mathbb{C}}(\mathcal{A}) = \mathbb{C}1 + \mathcal{Q}(\mathcal{A}) = \mathbb{C}1 + \mathrm{Rad}(\mathcal{A})$ by (iii), we can obtain conditions (i) or (ii) immediately. $\qquad \square$

The next theorem improves Theorem 3 in [4].

Theorem 3.2 *Let $\mathcal{A}$ be a Banach algebra, p be a quasinilpotent element in $\mathcal{A}$, and n be a positive integer. If $\Delta^n_{(p,-1)}(\mathcal{Q}(\mathcal{A})) \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$ then $p^n \in \mathrm{Rad}(\mathcal{A})$.*

Proof Suppose that $p^n \notin \text{Rad}(\mathcal{A})$. Then there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2, \cdots, x_{n+1}, \cdots\}\backslash\{0\}$ is linearly independent and contains $x_1, x_2, \cdots, x_{n+1}$, where $x_k = \pi(p^{k-1})x$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_i = x_{i+1},\ 1 \le i \le n,\ \pi(a)x_{n+1} = x_1,\ \pi(a)x_m = x_m,\ m \ge n+2.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent. We have

$$\Delta^n_{(p,-1)}(q) = \sum_{i=0}^{n}(-1)^i C_n^i p^{n-i} a^{-1} pa p^n.$$

It can be checked that

$$\pi(\Delta^n_{(p,-1)}(q))x_{n+1} = x_{n+1} - x_{2n+2},$$

$$\pi(\Delta^n_{(p,-1)}(q))x_n = -nx_n + (n-1)x_{2n+1} + x_{2n+2}.$$

However, since

$$\pi(\Delta^n_{(p,-1)}(q))x_{n+m} = 0,\ \forall m \ge 2,$$

it will be easy to calculate that

$$\pi(\Delta^n_{(p,-1)}(q))(x_{n+1} - x_{2n+2}) = x_{n+1} - x_{2n+2},$$

$$\pi(\Delta^n_{(p,-1)}(q))(x_n - \frac{n-1}{n}x_{2n+1} - \frac{1}{n}x_{2n+2}) = -n(x_n - \frac{n-1}{n}x_{2n+1} - \frac{1}{n}x_{2n+2}).$$

Thus,

$$\{1, -n\} \subset \sigma(\pi(\Delta^n_{(p,-1)}(q)) \subset \sigma(\Delta^n_{(p,-1)}(q)),$$

and so $\Delta^n_{(p,-1)}(q)$ is not contained in $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})$. □

Lemma 3.3 *Let $\lambda \in \mathbb{C}$ and $n > 0$. Then $\mathcal{Q}(\mathcal{A})^{(-1,n+1)} \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$ implies $\mathcal{Q}(\mathcal{A}) = \text{Rad}(\mathcal{A})$ for every Banach algebra $\mathcal{A}$ if and only if $\mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(-1,n+1)} \nsubseteq \mathcal{Q}_{\mathbb{C}}(\mathbb{M}_2(\mathbb{C}))$.*

Proof The proof is almost the same as the proof of Lemma 2.3. □

Theorem 3.4 *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{Q}_{\mathbb{C}}(\mathcal{A})^{(\lambda, n+1)} \subset \mathcal{Q}_{\mathbb{C}}(\mathcal{A})$ for some integer $n > 1$ and $\lambda \in \mathbb{C}$, then $\mathcal{Q}(\mathcal{A}) = \text{Rad}(\mathcal{A})$.*

Proof First, we consider the case of $\lambda = -1$. Note that for every $a \in \mathcal{A}, \mathcal{Q}_{\mathbb{C}}(\mathcal{A})^{(-1,n+1)} = \mathcal{Q}(\mathcal{A})^{(-1,n+1)}$, and so it is sufficient to prove $\mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(-1,n+1)} \nsubseteq \mathcal{Q}_{\mathbb{C}}(\mathbb{M}_2(\mathbb{C}))$ by Lemma 3.3.

In this case, we have that

$$\mathcal{Q}(\mathbb{M}_2(\mathbb{C})) = \mathbb{C}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ -\alpha^{-1} & -1 \end{pmatrix} : 0 \ne \alpha \in \mathbb{C} \right\}$$

and

$$\mathcal{Q}(\mathbb{M}_2(\mathbb{C}))^{(-1,2)} = \mathbb{C}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\alpha & 0 \\ 1 & \alpha \end{pmatrix}, \begin{pmatrix} -\alpha & 1 \\ 0 & \alpha \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -\alpha\beta^{-1} + \beta\alpha^{-1} & 2(\beta - \alpha) \\ 2(\beta^{-1} - \alpha^{-1}) & -\alpha\beta^{-1} + \beta\alpha^{-1} \end{pmatrix} : 0 \neq \alpha, \beta \in \mathbb{C} \right\}.$$

It is easy check that

$$\begin{pmatrix} -\alpha_1 & 0 \\ \beta_1 & \alpha_1 \end{pmatrix} := \begin{pmatrix} -\alpha_0 & \beta_0 \\ 0 & \alpha_0 \end{pmatrix} \circ_{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 2\alpha_0 & -\beta_0 \end{pmatrix},$$

$$\begin{pmatrix} -\alpha_2 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} := \begin{pmatrix} -\alpha_1 & 0 \\ \beta_1 & \alpha_1 \end{pmatrix} \circ_{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\beta_1 & -2\alpha_1 \\ 0 & \beta_1 \end{pmatrix}.$$

We see that if α_1, β_1 are nonzero then α_2, β_2 are nonzero. This shows that if $\mathcal{Q}((\mathbb{M}_2(\mathbb{C})))^{(-1,m)}$ has an element with 2 spectra for $m > 1$ then so does $\mathcal{Q}((\mathbb{M}_2(\mathbb{C})))^{(-1,m+1)}$.

Suppose $\lambda \neq -1$. If $\mathcal{Q}(\mathcal{A}) \neq \mathrm{Rad}(\mathcal{A})$, then there exists $p, q \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$ such that $p \circ_\lambda q \notin \mathcal{Q}_\mathbb{C}(\mathcal{A})$ by Theorem 3.1. Thus, $p \circ_\lambda q \circ_\lambda \underbrace{1 \circ_\lambda \cdots \circ_\lambda 1}_{n} = (1 + \lambda)^{n-1} p \circ_\lambda q \notin \mathcal{Q}_\mathbb{C}(\mathcal{A})$. □

The equivalence of conditions (i) and (iii) was given by Grabiner in [3, Theorem (2.1)] with another method. Now we want to give a similar result.

Remark 3.5 *Let $\mathcal{A}$ be a unital Banach algebra and $Z_1(\mathcal{A}) = \{z \in \mathcal{A} : z\mathcal{Q}_\mathbb{C}(\mathcal{A}) \subset \mathcal{Q}_\mathbb{C}(\mathcal{A})\}$. Then $Z_1(\mathcal{A}) = \mathbb{C}1 + \mathrm{Rad}(\mathcal{A})$.*

Proof It is sufficient to prove $Z_1(\mathcal{A}) \subset \mathbb{C}1 + \mathrm{Rad}(\mathcal{A})$, because $\mathbb{C}1 + \mathrm{Rad}(\mathcal{A}) \subset Z_1(\mathcal{A})$ is obvious.

Let $z \in Z_1(\mathcal{A})$. Then $z \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$. If $z \in \mathcal{Q}(\mathcal{A})$, we will show that $z^2 \in \mathrm{Rad}(\mathcal{A})$ at first.

If not, there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2, x_3\}$ is linearly independent where $x_k = \pi(z^{k-1})x$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = -x_3, \ \pi(a)x_2 = -x_2, \ \pi(a)x_3 = x_1.$$

Define $q = a^{-1}za$. Then q is quasinilpotent and so $q \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$, and we have

$$\pi(zq)x_2 = x_2, \ \pi(zq)x_3 = -x_3.$$

Thus,

$$\{1, -1\} \subset \sigma(\pi(zq)) \subset \sigma(zq),$$

and so zq is not contained in $\mathcal{Q}_\mathbb{C}(\mathcal{A})$.

Next we will show that $z \in \mathrm{Rad}(\mathcal{A})$.

If not, use the same symbol above. Then the set $S = \{x_1, x_2\}$ is linearly independent and $x_k = 0$ for every $k \geq 3$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1.$$

Define $q = a^{-1}za$. Then q is quasinilpotent and so $q \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$, and we have

$$\pi(zq)x_1 = 0, \ \pi(zq)x_2 = x_2.$$

Thus,

$$\{0, 1\} \subset \sigma(\pi(zq)) \subset \sigma(zq),$$

and so zq is not contained in $\mathcal{Q}_\mathbb{C}(\mathcal{A})$.

If $z \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$ but $z \notin \mathcal{Q}(\mathcal{A})$, then there exist a nonzero complex number $\lambda \in \mathbb{C}$ and $p \in \mathcal{Q}(\mathcal{A})$ such that $z = \lambda(1 + p)$. Next we need to prove $p \in \mathrm{Rad}(\mathcal{A})$.

First, we will show that $p^2 \in \mathrm{Rad}(\mathcal{A})$.

If not, there exist an irreducible representation π on a Banach space X and an $x \in X$ such that the set $S = \{x_1, x_2, x_3, x_4, x_5\} \backslash \{0\}$ is linearly independent and contains x_1, x_2, x_3, where $x_k = \pi(p^{k-1})x, 1 \leq k \leq 5$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_3, \ \pi(a)x_2 = x_2 - x_3, \ \pi(a)x_3 = -\frac{1}{2}x_1 - \frac{1}{2}x_2 + x_3, \ \pi(a)x_4 = x_4, \ \pi(a)x_5 = x_5.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and so $q \in \mathcal{Q}_\mathbb{C}(\mathcal{A})$, and we have

$$\pi((1 + p)q)(x_1 + x_2) = x_1 + x_2, \ \pi((1 + p)q)(x_2 + x_3) = -\frac{1}{2}(x_2 + x_3).$$

Thus,

$$\{1, -\frac{1}{2}\} \subset \sigma(\pi((1 + p)q)) \subset \sigma((1 + p)q),$$

and so $(1 + p)q$ is not contained in $\mathcal{Q}_\mathbb{C}(\mathcal{A})$.

Next we will show that $p \in \mathrm{Rad}(\mathcal{A})$.

If not, then the set $S = \{x_1, x_2\}$ is linearly independent and $x_k = 0$ for every $k \geq 3$. We can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_2, \ \pi(a)x_2 = x_1.$$

Let $q = a^{-1}pa$. Then q is quasinilpotent and we have

$$\pi((1 + p)q)x_1 = 0, \ \pi((1 + p)q)(x_1 + x_2) = x_1 + x_2.$$

Thus,

$$\{0, 1\} \subset \sigma(\pi((1 + p)q)) \subset \sigma((1 + p)q)),$$

and so $(1 + p)q$ is not contained in $\mathcal{Q}_\mathbb{C}(\mathcal{A})$.

$\square$

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