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On the starlikeness of p-valent functions

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Abstract: For an analytic function in open unit disk $\mathbb{D}$, we consider the p-valent analogue of the Noshiro–Warschawski univalence condition. We apply the Fejér–Riesz inequality to establish some sufficient conditions for functions to be p-valent or to be a Bazilevič function or to be in some other classes.

Key words: Analytic, univalent, starlike, close-to-convex, Fejér–Riesz inequality, p-valent function

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p$ be the subclass of $\mathcal{H}$ consisting of analytic functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

A function f meromorphic in a domain $D \subset \mathbb{C}$ is said to be p-valent in D if for each w the equation $f(z) = w$ has at most p roots in D, where roots are counted in accordance with their multiplicity, and there is some v such that the equation $f(z) = v$ has exactly p roots in D. Furthermore, a function $f \in \mathcal{A}_p$, $p = 1, 2, 3, \ldots$, is said to be p-valently starlike if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

The class of all such functions is usually denoted by $\mathcal{S}_p^*$. For $p = 1$ we obtain the well-known class of normalized starlike univalent functions $\mathcal{S}^* = \mathcal{S}_1^*$.

2. Main result

Theorem 2.1 *[8, Th. 2, p.93] Let $f \in \mathcal{A}_p$, $f^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, \ldots, p$ and suppose that*

$$|\arg f^{(p)}(z)| < \frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right).$$

Then f is p-valent in $\mathbb{D}$.

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Corollary 2.2 *Let* $f \in \mathcal{A}_p$, $f^{(k)}(z) \neq 0$ *in* $0 < |z| < 1$ *for* $k = 1, 2, \ldots, p$ *and suppose that*

$$p \geq e^{\pi} = 23.14\ldots.$$

If

$$|\arg f^{(p)}(z)| < \pi, \quad z \in \mathbb{D},$$

then f *is* p*-valent in* $\mathbb{D}$.

Proof Since for $p = e^{\pi}$ we have

$$\frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right) = \pi,$$

a function f is p-valent by Theorem 2.1. □

Applying Theorem 2.1, we can easily obtain the following theorem.

Theorem 2.3 *Let* $f(z) = \sum_{n=p}^{\infty} a_n z^n$, $a_p \neq 0$, $f^{(k)}(z) \neq 0$ *in* $0 < |z| < 1$ *for* $k = 1, 2, \ldots, p$ *and suppose that*

$$|\arg\{\exp(-i\alpha) f^{(p)}(z)\}| < \frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right), \quad z \in \mathbb{D},$$

where $\alpha = \arg\{a_p\}$. *Then* f *is* p*-valent in* $\mathbb{D}$.

Proof It is enough to see that $F(z) = \frac{f(z)}{a_p}$ is in $\mathcal{A}_p$ and apply Theorem 2.1. From the hypothesis, we have

$$F(z) = \frac{f(z)}{a_p} = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in \mathcal{A}_p.$$

Furthermore, $F^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, \ldots, p$ and

$$b_n = \frac{a_n}{a_p} \quad \text{for} \quad p \leq n.$$

Then, from Theorem 2.1, F is p-valent in $\mathbb{D}$. Therefore, f is p-valent in $\mathbb{D}$ too. □

Remark In [11] and [12], Ozaki proved that if f of the form (1.1) is analytic in a convex domain $D \subset \mathbb{C}$ and for some real α we have

$$\mathfrak{Re}\{\exp(i\alpha) f^{(p)}(z)\} > 0, \quad z \in D, \tag{2.1}$$

then f is at most p-valent in D. Ozaki's condition is a generalization of the well-known Noshiro–Warschawski univalence condition (see [6], [14]), where $p = 1$. An improvement of Ozaki's condition is given in [9].

Nunokawa [7] proved the following result.
If $f \in \mathcal{A}_p$ and

$$|\arg f^{(p)}(z)| < \frac{3\pi}{4}, \quad z \in \mathbb{D},$$

where $2 \leq p$, then f is p-valent in $\mathbb{D}$. To prove the main results, we also need the following integral inequality. It is due to Fejér and Riesz [1] and can be found in [3, p. 175] and in [13]. This result requires the regularity of f in the closed unit disc $\overline{\mathbb{D}}$.

Lemma 2.4 *[1] Let f be analytic in $\overline{\mathbb{D}}$, and $0 < p$. Then we have*

$$\int_{-1}^{1} |g(z)|^p \mathrm{d}z \le \frac{1}{2} \int_{|z|=1} |g(z)|^p |\mathrm{d}z|, \tag{2.2}$$

where the integral on the left is taken along the real axis.

Therefore, a change of variables in (2.2) will give

$$\int_{-r}^{r} |g(\rho e^{i\theta})|^p \mathrm{d}\rho \le \frac{r}{2} \int_{0}^{2\pi} |g(re^{i\theta})|^p \mathrm{d}\theta. \tag{2.3}$$

Applying the above lemma provides the following theorem.

Theorem 2.5 *Let $f \in \mathcal{A}_p$, $f^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, \dots p$, and suppose that*

$$\left| \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| < \frac{1}{2}\left(1 + \frac{1}{\pi}\log p\right), \quad z \in \mathbb{D}.$$

Then f is p-valent in $\mathbb{D}$.

Proof We have

$$\log \frac{f^{(p)}(z)}{p!} = \int_0^z \left(\log \frac{f^{(p)}(t)}{p!}\right)' \mathrm{d}t = \int_0^z \frac{f^{(p+1)}(t)}{f^{(p)}(t)} \mathrm{d}t,$$

and hence we obtain

$$\begin{aligned}
|\arg\{f^{(p)}(z)\}| &= \left| \mathfrak{Im} \int_0^z \frac{f^{(p+1)}(t)}{f^{(p)}(t)} \mathrm{d}t \right| \\
&= \left| \mathfrak{Im} \int_0^r \frac{f^{(p+1)}(\rho e^{i\theta})}{f^{(p)}(\rho e^{i\theta})} e^{i\theta} \mathrm{d}\rho \right| \\
&= \left| \int_0^r \mathfrak{Im} \left\{ \frac{e^{i\theta} f^{(p+1)}(\rho e^{i\theta})}{f^{(p)}(\rho e^{i\theta})} \right\} \mathrm{d}\rho \right| \\
&\le \int_0^r \left| \mathfrak{Im} \left\{ \frac{e^{i\theta} f^{(p+1)}(\rho e^{i\theta})}{f^{(p)}(\rho e^{i\theta})} \right\} \right| \mathrm{d}\rho \\
&\le \int_{-r}^r \left| \mathfrak{Im} \frac{e^{i\theta} f^{(p+1)}(\rho e^{i\theta})}{f^{(p)}(\rho e^{i\theta})} \right| \mathrm{d}\rho \\
&\le \int_{-r}^r \left| \frac{f^{(p+1)}(\rho e^{i\theta})}{f^{(p)}(\rho e^{i\theta})} \right| \mathrm{d}\rho,
\end{aligned}$$

where $z=re^{i\theta}$, $0\leq r<1$, $0\leq\rho\leq r$, and $0\leq\theta\leq 2\pi$. Now, applying (2.3) gives

$$\begin{aligned}
|\arg\{f^{(p)}(z)\}| &\leq \frac{r}{2}\int_0^{2\pi}\left|\frac{f^{(p+1)}(re^{i\theta})}{f^{(p)}(re^{i\theta})}\right|\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f^{(p+1)}(re^{i\theta})}{f^{(p)}(re^{i\theta})}\right|\mathrm{d}\theta \\
&< \frac{1}{4}\int_0^{2\pi}\left(1+\frac{1}{\pi}\log p\right)\mathrm{d}\theta \\
&= \frac{\pi}{2}\left(1+\frac{1}{\pi}\log p\right).
\end{aligned}$$

Then, by Theorem 2.1, we obtain that f is p-valent in $\mathbb{D}$. □

Applying the same method as in the proof of Theorem 2.5, we obtain the following theorem.

Theorem 2.6 *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in $\mathbb{D}$. Assume that $f'(z)\neq 0$ in $\mathbb{D}$ and*

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{1}{2}\mathfrak{Re}\left\{\frac{1+z}{1-z}\right\},\quad z\in\mathbb{D}. \tag{2.4}$$

Then f is univalent in $\mathbb{D}$.

Proof We have

$$\begin{aligned}
|\arg\{f'(z)\}| &= \left|\mathfrak{Im}\int_0^z\frac{f''(t)}{f'(t)}\mathrm{d}t\right| \\
&= \left|\mathfrak{Im}\int_0^r\frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}e^{i\theta}\mathrm{d}\rho\right| \\
&\leq \int_0^r\left|\frac{e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right|\mathrm{d}\rho \\
&\leq \int_{-r}^r\left|\frac{e^{i\theta}f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right|\mathrm{d}\rho,
\end{aligned}$$

where $z=re^{i\theta}$, $0\leq r<1$, $0\leq\rho\leq r$, and $0\leq\theta\leq 2\pi$. Now, applying (2.3) gives

$$\begin{aligned}
|\arg\{f'(z)\}| &\leq \frac{r}{2}\int_0^{2\pi}\left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right|\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right|\mathrm{d}\theta \\
&< \frac{1}{4}\int_0^{2\pi}\mathfrak{Re}\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}}\right\}\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}\mathrm{d}\theta \\
&= \frac{1}{4}\cdot 2\pi=\frac{\pi}{2}.
\end{aligned}$$

Then, by Noshiro–Warschawski univalence condition (2.1), we obtain that f is univalent in $\mathbb{D}$. □

A result related to Theorem 2.6 can be found in [10]. In [10], $zf''(z)/f'(z)$ in (2.4) is replaced by $\mathfrak{Re}\{zf''(z)/f'(z)\}$ and the right-hand side is a constant. This stronger hypothesis follows that f is univalent Janowski function.

Theorem 2.7 *Let $f \in \mathcal{A}_p$, $f^{(k)}(z) \neq 0$ in $0 < |z| < 1$ for $k = 1, 2, \ldots, p$, and suppose that*

$$\left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| < \frac{1}{2}\left(1 + \frac{1}{\pi}\log p\right)\mathfrak{Re}\left\{\frac{1+z}{1-z}\right\}, \quad z \in \mathbb{D}. \tag{2.5}$$

Then f is p-valent in $\mathbb{D}$.

Proof Using the same method as in the proof of Theorem 2.5, we obtain

$$|\arg\{f^{(p)}(z)\}| \leq \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f^{(p+1)}(re^{i\theta})}{f^{(p)}(re^{i\theta})}\right| \mathrm{d}\theta.$$

Then, by (2.5), we have

$$\begin{aligned}
|\arg\{f^{(p)}(z)\}| &\leq \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f^{(p+1)}(re^{i\theta})}{f^{(p)}(re^{i\theta})}\right| \mathrm{d}\theta \\
&\leq \frac{1}{4}\left(1 + \frac{1}{\pi}\log p\right)\int_0^{2\pi}\mathfrak{Re}\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}}\right\}\mathrm{d}\theta \\
&= \frac{1}{4}\left(1 + \frac{1}{\pi}\log p\right)\int_0^{2\pi}\frac{1-r^2}{1-2r\cos\theta + r^2}\mathrm{d}\theta \\
&= \frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right),
\end{aligned}$$

where $z = re^{i\theta}$, $0 \leq r < 1$, and $0 \leq \theta \leq 2\pi$. Then, by (2.1), we obtain that f is p-valent in $\mathbb{D}$. □

Theorem 2.8 *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ and suppose that*

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{1}{2}\mathfrak{Re}\left\{\frac{1}{1-z}\right\}, \quad z \in \mathbb{D}. \tag{2.6}$$

Then $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{D}$.

Proof It follows from (2.6) that $p(z) \neq 0$ in $\mathbb{D}$. Otherwise, we would have $p(z) = (z - z_0)^k q(z)$ for some z_0 and $q \in \mathcal{H}$ such that $|z_0| < 1$ and $q(z_0) \neq 0$ in $\mathbb{D}$. Then the left-hand side of (2.6) tends to ∞ as $z \to z_0$

while the right-hand side of (2.6) is bounded at z_0. Therefore, we have

$$\begin{aligned}
|\arg\{p(z)\}| &= |\mathfrak{Im}\log\{p(z)\}| \\
&= \left|\mathfrak{Im}\int_0^z (\log\{p(z)\})'\,\mathrm{d}z\right| \\
&= \left|\mathfrak{Im}\int_0^r \frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}e^{i\theta}\mathrm{d}\rho\right| \\
&\le \int_0^r \left|\mathfrak{Im}\left\{\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}e^{i\theta}\right\}\right|\mathrm{d}\rho \\
&\le \int_{-r}^r \left|\mathfrak{Im}\left\{\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}e^{i\theta}\right\}\right|\mathrm{d}\rho \\
&\le \int_{-r}^r \left|\frac{p'(\rho e^{i\theta})}{p(\rho e^{i\theta})}\right|\mathrm{d}\rho,
\end{aligned}$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \rho \le r$, and $0 \le \theta \le 2\pi$. Now, applying (2.3) gives

$$\begin{aligned}
|\arg\{p(z)\}| &\le \frac{r}{2}\int_0^{2\pi}\left|\frac{p'(re^{i\theta})}{p(re^{i\theta})}\right|\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right|\mathrm{d}\theta. \qquad (2.7)
\end{aligned}$$

Now, from (2.6) and (2.7), we obtain

$$\begin{aligned}
|\arg\{p(z)\}| &\le \frac{1}{4}\int_0^{2\pi}\mathfrak{Re}\left\{\frac{1}{1-re^{i\theta}}\right\}\mathrm{d}\theta \\
&= \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}.
\end{aligned}$$

This proves that $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{D}$. □

If we take $p(z)$ such that

$$p(z) = \frac{zf'(z)}{f(z)}, \quad f \in \mathcal{A}_1,$$

then Theorem 2.8 becomes the following corollary.

Corollary 2.9 *Let $f = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{1}{2}\mathfrak{Re}\left\{\frac{1}{1-z}\right\}, \quad z \in \mathbb{D}.$$

Then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad z \in \mathbb{D},$$

or f is a starlike function with respect to the origin, that is, $f \in \mathcal{S}_1^*$.

Recall that if $f \in \mathcal{A}_1$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\} > 0,\ z \in \mathbb{D}$$

for some $g \in \mathcal{S}_1^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then f is said to be close-to-convex in $\mathbb{D}$ and denoted by $f \in \mathcal{C}$. A univalent function $f \in \mathcal{A}_1$ belongs to $\mathcal{C}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

On the other hand, if $f \in \mathcal{A}_1$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0,\ z \in \mathbb{D}$$

for some $g \in \mathcal{S}_1^*$ and some $\beta \in (0, \infty)$, then f is said to be a Bazilevič function of type β and denoted by $f \in \mathcal{B}(\beta)$. If we take p such that

$$p(z) = \frac{zf'(z)}{e^{i\alpha}g(z)},\quad f \in \mathcal{A}_1,\quad g(z) \in \mathcal{S}_1^*,$$

then Theorem 2.8 becomes the following corollary.

Corollary 2.10 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathbb{D}$ *and suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \beta\frac{zg'(z)}{g(z)}\right| < \frac{1}{2}\mathfrak{Re}\left\{\frac{1}{1-z}\right\},\quad z \in \mathbb{D},$$

where $g \in \mathcal{S}_1^*$. *Then*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\} > 0,\ z \in \mathbb{D}$$

or f *is a close-to-convex function.*

If we take $p(z)$ such that

$$p(z) = \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)},\quad f \in \mathcal{A}_1,\quad g(z) \in \mathcal{S}_1^*,$$

then Theorem 2.8 becomes the following corollary.

Corollary 2.11 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathbb{D}$ *and suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)}\right| < \frac{1}{2}\mathfrak{Re}\left\{\frac{1}{1-z}\right\},\quad z \in \mathbb{D},$$

where $g(z) \in \mathcal{S}_1^*$ *and* $\beta \in (0, \infty)$. *Then*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0,\ z \in \mathbb{D}$$

or f *is a Bazilevič function of type* β, *that is,* $f \in \mathcal{B}(\beta)$.

We say that a function f is in the class $\mathcal{K}_s(\gamma)$, $0 \le \gamma < 1$, if $f \in \mathcal{A}_1$ and if there exists a function $g \in \mathcal{A}_1$, starlike of order $1/2$, such that

$$\mathfrak{Re}\,[zf'(z)/(g(z)g(-z))] > \gamma, \quad z \in \mathbb{D}.$$

The class $\mathcal{K}_s(0) = \mathcal{K}_s$ was defined by Gao and Zhou in [2], while the class $\mathcal{K}_s(\gamma)$ was introduced in [5] (see also [4]).

Corollary 2.12 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zg'(-z)}{g(-z)}\right| < \frac{1}{2}\mathfrak{Re}\left\{\frac{1}{1-z}\right\}, \quad z \in \mathbb{D},$$

where g is starlike of order $1/2$. Then

$$\mathfrak{Re}\,[zf'(z)/(g(z)g(-z))] > 0, \quad z \in \mathbb{D},$$

or f is in the class $\mathcal{K}_s(0)$.

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