

## Some classes of harmonic mappings with analytic part defined by subordination

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**Abstract:** Let  $S_H$  be the class of functions  $f = h + \bar{g}$  that are harmonic univalent and sense-preserving in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ , where  $h, g$  are analytic and  $f(0) = f'_z(0) - 1 = 0$  in  $\mathbb{U}$ . In this paper, we investigate the properties of some subclasses of  $S_H$  such that  $h(z)$  is a starlike (or convex) function defined by subordination. We provide coefficient estimates, extremal function, distortion and growth estimates of  $g$ , growth, and Jacobian estimates of  $f$ . We also obtain area estimates and covering theorems of the classes. The results presented here generalize some known results.

**Key words:** Harmonic univalent function, subordination, coefficient estimate, distortion, area estimate, covering theorem

### 1. Introduction and preliminaries

For two analytic functions  $f$  and  $g$  on  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  with  $f(0) = g(0)$ ,  $f$  is said to be subordinate to  $g$  if there exists an analytic function  $\omega$  on  $\mathbb{U}$  such that  $\omega(0) = 0, |\omega| < 1$ , and  $f(z) = g(\omega(z))$  ( $z \in \mathbb{U}$ ). We denote this subordination relation by  $f(z) \prec g(z), z \in \mathbb{U}$ . Furthermore, if the function  $g$  is univalent in  $\mathbb{U}$ , then we have the following equivalence (see [4]; also see [27]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let  $\mathcal{A}$  be the class of functions  $h(z)$  that are analytic in  $\mathbb{U}$  and let  $\mathcal{S}$  denote the subclass of functions in  $\mathcal{A}$  that are univalent in  $\mathbb{U}$ .

For  $-1 \leq B < A \leq 1$ , we let  $S^*(A, B)$  and  $K(A, B)$ , respectively, denote the subclasses of  $\mathcal{A}$  (see [12]):

$$h \in S^*(A, B) \iff \frac{zh'(z)}{h(z)} \prec \frac{1 + Az}{1 + Bz} \quad (h \in \mathcal{A}, z \in \mathbb{U})$$

and

$$h \in K(A, B) \iff \frac{(zh'(z))'}{h'(z)} \prec \frac{1 + Az}{1 + Bz} \quad (h \in \mathcal{A}, z \in \mathbb{U}).$$

It is clear that

$$h \in K(A, B) \iff zh'(z) \in S^*(A, B)$$

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and

$$K(A, B) \subset S^*(A, B), \quad K(A, B) \subset K \subset \mathcal{S}, \quad S^*(A, B) \subset S^* \subset \mathcal{S}.$$

Especially,  $S^*(1 - 2\beta, -1) = S^*_\beta$  ( $0 \leq \beta < 1$ ) and  $K(1 - 2\beta, -1) = K_\beta$  are a starlike function of order  $\beta$  and convex function of order  $\beta$ , respectively [24];  $S^*(1, -1) = S^*$  and  $K(1, -1) = K = \mathcal{CV}$  are the well-known starlike function and convex function, respectively.

In [7], a classical problem of Fekete and Szegő states that, for  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \mu \leq 0, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & 0 \leq \mu \leq 1, \\ 4\mu - 3, & \mu \geq 1. \end{cases}$$

The result is sharp.

In 1994, Ma and Minda [20] obtained the Fekete–Szegő problem for the starlike function and convex function defined by subordination. Many authors studied the problems of Fekete and Szegő and obtained some useful results (for example, see [6,17,26]).

A continuous function  $f = u + iv$  is a complex valued harmonic function in a complex domain  $\mathbb{U}$  if both  $u$  and  $v$  are real harmonic in  $\mathbb{U}$ . In any simply connected domain  $\mathbb{U}$ , we can write  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic in  $\mathbb{U}$ . We call  $h$  the analytic part and  $g$  the coanalytic part of  $f$ . The Jacobian and second complex dilatation of  $f(z)$  are given by  $J_f(z) = |h'(z)|^2 - |g'(z)|^2$  and  $\omega(z) = g'(z)/h'(z)$  ( $z \in \mathbb{U}$ ), respectively. Lewy [19] proved that a necessary and sufficient condition for  $f(z)$  to be locally univalent and sense-preserving in  $\mathbb{U}$  is  $J_f(z) > 0$ . A necessary and sufficient condition for  $f$  to be locally univalent and sense-preserving in  $\mathbb{U}$  is that  $|h'(z)| > |g'(z)|$  in  $\mathbb{U}$  (see [3]; see also [5]).

Denote by  $S_H$  the class of univalent and harmonic functions  $f$  that are sense-preserving in  $\mathbb{U}$  and has the form (see [3,5])

$$f = h + \bar{g}, \quad z \in \mathbb{U}, \tag{1.1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.2}$$

In [1,2,10,11,21,22,25,28,29,31], many authors further investigated various subclasses of  $S_H$  and obtained some important results. In [15], the authors studied the properties of a subclass  $\bar{S}_H^\alpha$  of  $S_H$ , consisting of all univalent antianalytic perturbations of the identity in the unit disk with  $|b_1| = \alpha$ , and in [16], the authors studied the class  $\bar{S}_H^\alpha$  of all  $f \in S_H$ , such that  $|b_1| = \alpha \in (0, 1)$  and  $h \in \mathcal{CV}$ , where  $\mathcal{CV}$  denotes the well-known family of normalized, univalent functions that are convex. Recently, Kans and Klimek-Smet [14] studied the properties of a subclass of  $\bar{S}_H^\alpha$  and established estimates of some functionals and bounds of the Bloch constant for the coanalytic part. Also, Hotta and Michalski [9] studied the properties of a subclass of  $L_H$  having starlike analytic part  $h$  and obtained coefficient, distortion, and growth estimates of  $g$ , and Jacobian estimates of  $f$ . Kahramaner et al. [13] investigated the class of harmonic mappings related to Janowski starlike functions. Zhu and Huang [32] studied the distortion theorems for harmonic mappings with analytic parts of convex or starlike functions of order  $\beta$ . Some sharp estimates of coefficients, distortion, and growth are obtained (also see Sun et al. [30]).

In this paper, we introduce some subclasses of  $S_H$ , such that  $h(z)$  is a subclass of starlike (or convex) functions defined by subordination. We provide coefficient estimates, distortion, and area estimates, and covering theorems of the classes. The results presented here generalize the main results in [9,13,14,32].

Now we introduce the following classes.

**Definition 1.1** Let  $A, B \in \mathbb{R}; -1 \leq B < A \leq 1$ . The function  $f(z) \in \bar{S}_H^{*,\alpha}(A, B)$  if and only if  $f \in S_H, |b_1| = \alpha \in [0, 1)$  and  $h(z) \in S^*(A, B)$ . Also, the function  $f(z) \in \bar{K}_H^\alpha(A, B)$  if and only if  $f \in S_H, |b_1| = \alpha \in [0, 1)$  and  $h(z) \in K(A, B)$ . Additionally, we define the classes

$$\bar{S}_H^*(A, B) = \bigcup_{\alpha \in [0,1)} \bar{S}_H^{*,\alpha}(A, B)$$

and

$$\bar{K}_H(A, B) = \bigcup_{\alpha \in [0,1)} \bar{K}_H^\alpha(A, B).$$

Obviously,  $\bar{K}_H^\alpha(A, B) \subset \bar{S}_H^{*,\alpha}(A, B)$  and  $\bar{K}_H(A, B) \subset \bar{S}_H^*(A, B)$ .

**Remark 1.2** Specializing the parameters, the classes  $\bar{K}_H^\alpha(A, B), \bar{S}_H^{*,\alpha}(A, B), \bar{K}_H(A, B)$ , and  $\bar{S}_H^*(A, B)$  reduce to the various subclasses of  $S_H$ :

- (i)  $\bar{K}_H^\alpha(1, -1) = \bar{K}_H^\alpha = \bar{S}_H^\alpha = \{f \in S_H : h \in \mathcal{CV}\}$  (Klimek and Michalski [16]; Kans and Klimek-Smet [14]).
- (ii)  $\bar{S}_H^{*,\alpha}(1, -1) = \bar{S}_H^{*,\alpha} = \check{L}_H^\alpha = \{f \in S_H : h \in S^*\}$  and  $\bar{S}_H^*(1, -1) = \bar{S}_H^* = \check{L}_H$  (Hotta and Michalski [9]; also [3]).
- (iii)  $\bar{S}_H^{*,\alpha}(A, B) = S_H^*(A, B)$  (Kahramaner et al. [13]).
- (iv)  $\bar{K}_H^\alpha(1 - 2\beta, -1) = S_H^\alpha(C_\beta)$  and  $\bar{S}_H^{*,\alpha}(1 - 2\beta, -1) = L_H^\alpha(S_\beta^*)$  ( $0 \leq \beta < 1$ ) (Zhu and Huang [32]).

We need the following lemmas to prove our results.

**Lemma 1.3** [18] If the function  $\omega(z) = c_0 + c_1z + \dots + c_nz^n + \dots$  is analytic and  $|\omega(z)| \leq 1$  on  $\mathbb{U}$ , then

$$|c_n| \leq 1 - |c_0|^2, \quad n = 1, 2, \dots \tag{1.3}$$

**Lemma 1.4** Let  $-1 \leq B < A \leq 1, n = 2, 3, \dots$ .

- (i) If  $h(z) = z + \sum_{n=2}^\infty a_nz^n \in S^*(A, B)$ , then  $|a_n| \leq F_n(A, B)$ , where

$$F_n(A, B) = \frac{\prod_{k=0}^{n-2} (A - B + k)}{(n - 1)!} \tag{1.4}$$

- (ii) If  $h(z) = z + \sum_{n=2}^\infty a_nz^n \in K(A, B)$ , then  $|a_n| \leq \frac{F_n(A, B)}{n}$ , where  $F_n(A, B)$  is defined by (1.4).

**Proof** If  $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(A, B)$ , there exists a positive real function  $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$  and  $|p_k| \leq A - B$ , such that  $\frac{zh'(z)}{h(z)} = p(z)$ . Comparing the coefficients of both sides of the equation, we have  $(n - 1)a_n = p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_{n-2} a_2 + p_{n-1}$ . Therefore,

$$|a_n| \leq \frac{(A - B)}{n - 1} (1 + |a_2| + \dots + |a_{n-1}|).$$

Let  $\phi(n) = 1 + |a_2| + \dots + |a_n|$ , we have

$$\phi(n) \leq \frac{\prod_{k=1}^{n-1} (A - B + k)}{(n - 1)!}, \quad n \geq 2.$$

Thus,  $|a_n| \leq \frac{\prod_{k=0}^{n-2} (A - B + k)}{(n - 1)!}$ .

If  $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K(A, B)$ , then  $zh'(z) \in S^*(A, B)$ , and by using (i), we have

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} (A - B + k)}{n!}.$$

Thus, we complete the proof of Lemma 1.4. □

In [6, Theorem 1], letting  $b = 1, \alpha = 0, \varphi(z) = \frac{1+Az}{1+Bz} (-1 \leq B < A \leq 1)$  and letting  $n = 0$  and  $n = 1$ , respectively, we have:

**Lemma 1.5** Let  $A, B \in \mathbb{R}, \mu \in \mathbb{R}$ , and  $-1 \leq B < A \leq 1$ .

(i) If  $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(A, B)$ , then

$$|a_3 - \mu a_2^2| \leq \frac{A - B}{2} \max \{1, |(1 - 2\mu)(A - B) - B|\}. \tag{1.5}$$

(ii) If  $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K(A, B)$ , then

$$|a_3 - \mu a_2^2| \leq \frac{A - B}{6} \max \left\{ 1, \left| \left(1 - \frac{3}{2}\mu\right)(A - B) - B \right| \right\}. \tag{1.6}$$

**Lemma 1.6** Let  $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ ,  $-1 \leq B < A \leq 1, |z| = r, 0 \leq r < 1$ .

(i) If  $h(z) \in K(A, B)$ , then

$$\begin{cases} (1 - Br)^{\frac{A-B}{B}} \leq |h'(z)| \leq (1 + Br)^{\frac{A-B}{B}}, & B \neq 0, \\ e^{-Ar} \leq |h'(z)| \leq e^{Ar}, & B = 0, \end{cases} \quad ([23, \text{Theorem 3.1 with } b = 1]) \quad (1.7)$$

and

$$\begin{cases} r(1 - Br)^{\frac{A-B}{2B}} \leq |h(z)| \leq r(1 + Br)^{\frac{A-B}{2B}}, & B \neq 0, \\ re^{-\frac{rA}{2}} \leq |h(z)| \leq re^{\frac{rA}{2}}, & B = 0. \end{cases} \quad ([23, \text{Theorem 3.2 with } b = 1]) \quad (1.8)$$

(ii) If  $h(z) \in S^*(A, B)$ , then

$$\begin{cases} (1 - Ar)(1 - Br)^{\frac{A-2B}{B}} \leq |h'(z)| \leq (1 + Ar)(1 + Br)^{\frac{A-2B}{B}}, & B \neq 0, \\ (1 - Ar)e^{-rA} \leq |h'(z)| \leq (1 + Ar)e^{rA}, & B = 0, \end{cases} \quad (1.9)$$

and

$$\begin{cases} r(1 - Br)^{\frac{A-B}{B}} \leq |h(z)| \leq r(1 + Br)^{\frac{A-B}{B}}, & B \neq 0, \\ re^{-Ar} \leq |h(z)| \leq re^{Ar}, & B = 0. \end{cases} \quad ([12, \text{Theorem 4}]) \quad (1.10)$$

**Proof** We need only consider the case of (1.9). Since  $h(z) \in S^*(A, B)$ , then

$$\left| \frac{zh'(z)}{h(z)} - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{(A - B)r}{1 - B^2 r^2}.$$

By simple calculation, we have

$$\frac{1 - Ar}{1 - Br} \leq \left| \frac{zh'(z)}{h(z)} \right| \leq \frac{1 + Ar}{1 + Br},$$

that is,

$$\frac{1 - Ar}{1 - Br} |h(z)| \leq |zh'(z)| \leq \frac{1 + Ar}{1 + Br} |h(z)|.$$

Applying (1.10), it is easy to obtain (1.9), so we complete the proof of Lemma 1.6.  $\square$

## 2. Main results

**Theorem 2.1** If  $f = h + \bar{g} \in \bar{S}_H^{*,\alpha}(A, B)$ , then  $F = H + \bar{G} \in \bar{K}_H^\alpha(A, B)$ , where  $H(z)$  and  $G(z)$  satisfy the conditions  $zH'(z) = h(z)$  and  $zG'(z) = g(z)$ ,  $z \in \mathbb{U}$ .

**Proof** According to the definition of  $\bar{S}_H^{*,\alpha}(A, B)$ ,  $h \in S^*(A, B)$ . By Alexander's theorem [4, p. 43], the function  $H(z) \in K(A, B)$ . Also,  $H(0) = 0$ ,  $H'(0) = \lim_{z \rightarrow 0} \frac{h(z)}{z} = h'(0) = 1$ , and  $|G'(0)| = \left| \lim_{z \rightarrow 0} \frac{g(z)}{z} \right| = |g'(0)| = \alpha$ . Let  $\Gamma := [0, h(z)] \subset h(\mathbb{U})$ ,  $z \in \mathbb{U} - \{0\}$ ; then

$$|g(z)| = \left| \int_{\Gamma} d(g \circ h^{-1}(\omega)) \right| \leq \int_{\Gamma} \left| \frac{d(g \circ h^{-1}(\omega))}{d\omega} \right| |d\omega| < \int_{\Gamma} |d\omega| = |h(z)|.$$

Hence,

$$|G'(z)| = \lim_{t \rightarrow z} \left| \frac{g(t)}{t} \right| < \lim_{t \rightarrow z} \left| \frac{h(t)}{t} \right| = |H'(z)|.$$

It shows that  $F$  is a locally univalent and sense-preserving harmonic function in  $\mathbb{U}$ . Finally, appealing to [16, Corollary 2.3], we conclude that  $F = H + \bar{G} \in \bar{K}_H^\alpha(A, B)$ .  $\square$

**Corollary 2.2** *If  $f = h + \bar{g} \in \bar{S}_H^*(A, B)$ , then  $F = H + \bar{G} \in \bar{K}_H(A, B)$ , where  $H(z)$  and  $G(z)$  satisfy the conditions  $zH'(z) = h(z)$  and  $zG'(z) = g(z)$ ,  $z \in \mathbb{U}$ .*

Next, we give coefficient estimates for functions of these classes.

**Theorem 2.3** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2) and  $F_n(A, B)$  is defined by (1.4). If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then*

$$|b_n| \leq \begin{cases} \frac{1-\alpha^2}{2} + \alpha(A-B), & n = 2, \\ \alpha F_n(A, B) + \frac{(1-\alpha^2)}{n} \left\{ 1 + \sum_{k=2}^{n-1} k F_k(A, B) \right\}, & n \geq 3. \end{cases} \quad (2.1)$$

The estimate of (2.1) is sharp and the extremal function is

$$f_0(z) = h_0(z) + \overline{g_0(z)} = \frac{z}{(1-z)^{A-B}} + \int_0^z \frac{\alpha - (\alpha^2 + \alpha - 1)t}{(1-t)} \cdot \frac{1 + (A-B-1)t}{(1-t)^{A-B+1}} dt. \quad (2.2)$$

In particular, if  $f \in \bar{S}_H^{*,\alpha}$ ,  $n = 2, 3, \dots$ , then

$$|b_n| \leq n\alpha + \frac{(n-1)(2n-1)}{6}(1-\alpha^2). \quad (2.3)$$

The estimate of (2.3) is sharp and the extremal function is

$$f_0(z) = h_0(z) + \overline{g_0(z)} = \frac{z}{(1-z)^2} + \int_0^z \frac{\alpha - (\alpha^2 + \alpha - 1)t}{(1-t)} \cdot \frac{1+t}{(1-t)^3} dt.$$

**Proof** Making use of the relation  $g' = \omega h'$  and the power series expansions (1.2), we obtain

$$nb_n = \sum_{p=1}^n p a_p c_{n-p} \quad (a_1 = 1, n = 2, 3, \dots). \quad (2.4)$$

The fact  $g' = \omega h'$ , for the case  $z = 0$ , implies that  $c_0 = b_1$ , so that by (1.3), we obtain  $|c_{n-p}| \leq 1 - |b_1|^2 = 1 - \alpha^2$ ,  $p = 1, 2, \dots, n-1$ . Therefore,

$$|b_n| \leq \frac{1}{n} \left( 1 + \sum_{k=2}^{n-1} k |a_k| \right) (1 - |c_0|^2) + |a_n| |c_0|. \quad (2.5)$$

If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ ,  $h \in S^*(A, B)$ , then from Lemma 1.4 (i) and (2.5), we have

$$|b_n| \leq \frac{(1-\alpha^2)}{n} \left\{ 1 + \sum_{k=2}^{n-1} k F_k(A, B) \right\} + \alpha F_n(A, B),$$

where  $F_k(A, B)$  is defined by (1.4). In particular,

$$|b_2| \leq \frac{|c_1|}{2} + |a_2||c_0| \leq \frac{1 - \alpha^2}{2} + \alpha(A - B).$$

Thus, the proof is completed. □

Using the same methods as in Theorem 2.3, we can obtain the following results.

**Theorem 2.4** *Let  $-1 \leq B < A \leq 1$ ,  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2), and  $F_n(A, B)$  be defined by (1.4). If  $f \in \bar{K}_H^\alpha(A, B)$ , then*

$$|b_n| \leq \begin{cases} \frac{1 - \alpha^2}{2} + \frac{\alpha(A - B)}{2}, & n = 2, \\ \frac{(1 - \alpha^2)}{n} \left\{ 1 + \sum_{k=2}^{n-1} F_k(A, B) \right\} + \frac{F_n(A, B)\alpha}{n}, & n \geq 3. \end{cases}$$

*In particular, if  $f \in \bar{K}_H^\alpha$ , then  $|b_n| \leq \alpha + \frac{(n-1)(1-\alpha^2)}{2}$  ( $n \geq 2$ ).*

From Theorem 2.3 and Theorem 2.4, we have:

**Corollary 2.5** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2) and  $F_n(A, B)$  is defined by (1.4).*

(i) *If  $f \in \bar{S}_H^*(A, B)$ , then*

$$|b_n| \leq \begin{cases} \frac{1 + (A - B)^2}{2}, & n = 2, \\ \frac{4(1 + \sum_{k=2}^{n-1} kF_k(A, B))^2 + n^2 F_n^2(A, B)}{4n(1 + \sum_{k=2}^{n-1} kF_k(A, B))}, & n \geq 3. \end{cases}$$

*Especially, if  $f \in \bar{S}_H^*$ , then*

$$|b_n| \leq \begin{cases} \frac{5}{2}, & n = 2, \\ \frac{(n-1)^2(2n-1)^2 + 9n^2}{6(n-1)(2n-1)}, & n \geq 3. \end{cases}$$

(ii) *If  $f \in \bar{K}_H(A, B)$ , then*

$$|b_n| \leq \begin{cases} \frac{1}{2} + \frac{(A - B)^2}{8}, & n = 2, \\ \frac{4(1 + \sum_{k=2}^{n-1} F_k(A, B))^2 + F_n^2(A, B)}{4n(1 + \sum_{k=2}^{n-1} F_k(A, B))}, & n \geq 3. \end{cases}$$

*Especially, if  $f \in \bar{K}_H$ , then*

$$|b_n| \leq \begin{cases} 1, & n = 2, \\ \frac{n^2 - 2n + 2}{2(n-1)}, & n \geq 3. \end{cases}$$

Also, we give the Fekete–Szegő inequalities for functions of these classes.

**Theorem 2.6** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2) and  $F_n(A, B)$  is defined by (1.4). For  $\mu \in \mathbb{R}$ , if  $f \in \bar{S}_H^{*\alpha}(A, B)$ , then*

$$|b_3 - \mu b_2^2| \leq \frac{1 - \alpha^2}{3} \left\{ 1 + \frac{3|\mu|(1 - \alpha^2)}{4} + (A - B)|2 - 3\mu b_1| \right\} + \frac{(A - B)\alpha}{2} \max \{1, |(1 - 2b_1\mu)(A - B) - B|\} \tag{2.6}$$

and

$$|b_{n+1} - b_n| \leq \begin{cases} \frac{1-\alpha^2}{2} + \alpha(A - B + 1), & n = 1, \\ \frac{(1-\alpha^2)(5+4(A-B))}{6} + \frac{\alpha(A-B)(A-B+3)}{2}, & n = 2, \\ \frac{(1-\alpha^2)}{n+1} \left\{ \frac{2n+1}{n} \left( 1 + \sum_{k=2}^{n-1} k F_k(A, B) \right) + n F_n(A, B) \right\} + \\ \alpha(F_{n+1}(A, B) + F_n(A, B)), & n \geq 3. \end{cases} \quad (2.7)$$

Especially, if  $f \in \bar{S}_H^{*,\alpha}$ , then

$$|b_3 - \mu b_2^2| \leq \frac{1-\alpha^2}{3} \left\{ 1 + \frac{3|\mu|(1-\alpha^2)}{4} + |4 - 6\mu b_1| \right\} + \alpha \max \{1, |3 - 4b_1\mu|\} \quad (2.8)$$

and

$$|b_{n+1} - b_n| \leq \frac{1-\alpha^2}{6} (4n^2 - 2n + 1) + (2n + 1)\alpha \quad (n \geq 1). \quad (2.9)$$

**Proof** From the relation (2.4), we have

$$b_1 = c_0, \quad 2b_2 = c_1 + 2a_2c_0, \quad 3b_3 = c_2 + 2a_2c_1 + 3a_3c_0$$

and

$$nb_n = \sum_{p=1}^n pa_p c_{n-p} \quad (a_1 = 1, n = 2, 3, \dots).$$

By (1.3), we have

$$|b_3 - \mu b_2^2| \leq \frac{1-\alpha^2}{3} \left\{ 1 + \frac{3|\mu|(1-\alpha^2)}{4} + |a_2||2 - 3\mu b_1| \right\} + \alpha |a_3 - \mu c_0 a_2^2|$$

and

$$|b_{n+1} - b_n| \leq \frac{(1-\alpha^2)}{n+1} \left\{ \frac{2n+1}{n} \left( 1 + \sum_{k=2}^{n-1} k |a_k| \right) + n |a_n| \right\} + \alpha (|a_{n+1}| + |a_n|).$$

If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then  $h \in S^*(A, B)$ . Applying Lemma 1.4 and Lemma 1.5, it is easy to obtain (2.6) and (2.7). Thus, we complete the proof of Theorem 2.6.

Using the same methods as in Theorem 2.6, we can get the following results.

**Theorem 2.7** Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2) and  $F_n(A, B)$  is defined by (1.4). For  $\mu \in \mathbb{R}$ , if  $f \in \bar{K}_H^\alpha(A, B)$ , then

$$|b_3 - \mu b_2^2| \leq \frac{1-\alpha^2}{3} \left\{ 1 + \frac{3|\mu|(1-\alpha^2)}{4} + \frac{(A-B)}{2} |2 - 3\mu b_1| \right\} + \frac{\alpha(A-B)}{6} \max \left\{ 1, \left| \left( 1 - \frac{3}{2}\mu b_1 \right) (A-B) - B \right| \right\}$$

and

$$|b_{n+1} - b_n| \leq \begin{cases} \frac{1-\alpha^2}{2} + \alpha \left( \frac{A-B}{2} + 1 \right), & n = 1, \\ \frac{(1-\alpha^2)(5+2(A-B))}{6} + \frac{\alpha(A-B)(A-B+4)}{6}, & n = 2, \\ \frac{(1-\alpha^2)}{n+1} \left\{ \frac{2n+1}{n} \left( 1 + \sum_{k=2}^{n-1} F_k(A, B) \right) + F_n(A, B) \right\} + \\ \alpha \left( \frac{F_{n+1}(A, B)}{n+1} + \frac{F_n(A, B)}{n} \right), & n \geq 3. \end{cases}$$



Especially, if  $f \in \bar{K}_H^\alpha$ , then

$$|b_3 - \mu b_2^2| \leq \frac{1 - \alpha^2}{3} \left\{ 1 + \frac{3|\mu|(1 - \alpha^2)}{4} + |2 - 3\mu b_1| \right\} + \frac{\alpha}{3} \max\{1, 3|1 - \mu b_1|\}$$

and

$$|b_{n+1} - b_n| \leq \frac{(2n - 1)(1 - \alpha^2)}{2} + 2\alpha.$$

From Theorem 2.6 and Theorem 2.7, it is easy to obtain the following results.

**Corollary 2.8** Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2) and  $\mu \in \mathbb{R}$ .

(i) If  $f \in \bar{S}_H^*$ , then

$$|b_{n+1} - b_n| \leq \begin{cases} 2n + 1, & 1 \leq n \leq 2, \\ \frac{(4n^2 - 2n + 1)^2 + 9(2n + 1)^2}{6(4n^2 - 2n + 1)}, & n \geq 3. \end{cases}$$

(ii) If  $f \in \bar{K}_H$ , then

$$|b_{n+1} - b_n| \leq \begin{cases} 2, & n = 1, \\ \frac{4n^2 - 4n + 5}{4n - 2}, & n \geq 2. \end{cases}$$

**Remark 2.9**

- (i) The results of Theorem 2.1 and Theorem 2.3 improve Theorem 2.1 and Theorem 2.2 in [9];
- (ii) Letting  $A = 1 - 2\beta$  ( $0 \leq \beta < 1$ ) and  $B = -1$  in Theorem 2.1, Theorem 2.4, and Theorem 2.3, we can get Lemma 4, Theorem 5, and Corollary 6 in [32].

Further, we obtain the distortion estimates of the coanalytic part  $g$  and area estimates as follows.

**Theorem 2.10** Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2),  $|z| = r$ ,  $0 \leq r < 1$ .

(i) If  $f \in \bar{K}_H^\alpha(A, B)$ , then

$$F_{\alpha,r}(A, B) \leq |g'(z)| \leq \begin{cases} \left(\frac{\alpha+r}{1+\alpha r}\right)(1+Br)^{\frac{A-B}{B}}, & B \neq 0, \\ \left(\frac{\alpha+r}{1+\alpha r}\right)e^{Ar}, & B = 0, \end{cases} \tag{2.10}$$

where  $z \in \mathbb{U}$  and

$$F_{\alpha,r}(A, B) = \begin{cases} \frac{\max\{\alpha-r, 0\}}{1-\alpha r} (1-Br)^{\frac{A-B}{B}}, & B \neq 0, \\ \frac{\max\{\alpha-r, 0\}}{1-\alpha r} e^{-Ar}, & B = 0. \end{cases} \tag{2.11}$$

(ii) If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then

$$G_{\alpha,r}(A, B) \leq |g'(z)| \leq \begin{cases} \left(\frac{\alpha+r}{1+\alpha r}\right)(1+Ar)(1+Br)^{\frac{A-2B}{B}}, & B \neq 0, \\ \left(\frac{\alpha+r}{1+\alpha r}\right)(1+Ar)e^{Ar}, & B = 0, \end{cases} \tag{2.12}$$

where  $z \in \mathbb{U}$  and

$$G_{\alpha,r}(A, B) = \begin{cases} \frac{\max\{\alpha-r, 0\}}{1-\alpha r} (1-Ar)(1-Br)^{\frac{A-2B}{B}}, & B \neq 0, \\ \frac{\max\{\alpha-r, 0\}}{1-\alpha r} (1-Ar)e^{-Ar}, & B = 0. \end{cases} \tag{2.13}$$

**Proof** Applying the relation  $g' = \omega h'$ ,  $|\omega(0)| = |g'(0)| = |b_1| = \alpha$ , we know  $\omega(z)$  satisfies the following (see [8]):

$$\left| \frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)} \right| \leq |z|. \tag{2.14}$$

Equivalently,

$$\left| \omega(z) - \frac{\omega(0)(1 - r^2)}{1 - |\omega(0)|^2 r^2} \right| \leq \frac{r(1 - |\omega(0)|^2)}{1 - |\omega(0)|^2 r^2}. \tag{2.15}$$

From (2.15), we get

$$\frac{\max\{\alpha - r, 0\}}{1 - \alpha r} \leq |\omega(z)| \leq \frac{\alpha + r}{1 + \alpha r}, \quad z \in \mathbb{U}. \tag{2.16}$$

Finally, applying (2.16) and (1.7), we obtain (2.10); also, using (2.16) and (1.9), we have (2.12). The proof is completed.  $\square$

**Theorem 2.11** Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2),  $F_{\alpha,r}(A, B)$  and  $G_{\alpha,r}(A, B)$  are given by (2.11) and (2.13), respectively, and  $|z| = r$ ,  $0 \leq r < 1$ .

(i) If  $f \in \bar{K}_H^\alpha(A, B)$ , then

$$\int_0^r F_{\alpha,t}(A, B)dt \leq |g(z)| \leq \begin{cases} \int_0^r \frac{(\alpha+t)}{(1+\alpha t)}(1+Bt)^{\frac{A-B}{B}} dt, & B \neq 0, \\ \int_0^r \frac{(\alpha+t)}{1+\alpha t} e^{At} dt, & B = 0. \end{cases} \tag{2.17}$$

(ii) If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then

$$\int_0^r G_{\alpha,t}(A, B)dt \leq |g(z)| \leq \begin{cases} \int_0^r \frac{(\alpha+t)}{(1+\alpha t)}(1+At)(1+Bt)^{\frac{A-2B}{B}} dt, & B \neq 0, \\ \int_0^r \frac{(\alpha+t)}{1+\alpha t} (1+At)e^{At} dt, & B = 0. \end{cases} \tag{2.18}$$

**Proof** By (2.10), integrating along a radial line  $\xi = te^{i\theta}$ , the right-hand side of (2.17) is obtained immediately:

$$|g(z)| \leq \begin{cases} \int_0^r \frac{(\alpha+t)}{(1+\alpha t)}(1+Bt)^{\frac{A-B}{B}} dt, & B \neq 0, \\ \int_0^r \frac{(\alpha+t)}{(1+\alpha t)} e^{At} dt, & B = 0. \end{cases} \tag{2.19}$$

In order to prove the left-hand side of (2.17), we note first that  $g$  is univalent. Let  $\Gamma = g(\{z : |z| = r\})$  and let  $\xi_1 \in \Gamma$  be the nearest point to the origin. By a rotation we may assume that  $\xi_1 > 0$ . Let  $\gamma$  be the line segment  $0 \leq \xi \leq \xi_1$  and suppose that  $z_1 = g^{-1}(\xi_1)$  and  $L = g^{-1}(\gamma)$ . With  $\varsigma$  as the variable of integration on L, we have that  $d\xi = g'(\varsigma)d\varsigma$  on L. Hence,

$$\begin{aligned} \xi_1 &= \int_0^{\xi_1} d\xi = \int_0^{z_1} g'(\varsigma)d\varsigma = \int_0^{z_1} |g'(\varsigma)||d\xi| \geq \int_0^r |g'(te^{i\theta})|dt \\ &\geq \int_0^r F_{\alpha,r}(A, B)dr = \begin{cases} \int_0^r \frac{\max\{\alpha-t, 0\}}{1-\alpha t} (1-Bt)^{\frac{A-B}{B}} dt, & B \neq 0, \\ \int_0^r \frac{\max\{\alpha-t, 0\}}{1-\alpha t} e^{-At} dt, & B = 0. \end{cases} \end{aligned} \tag{2.20}$$

By (2.12), using the same method in proof of (2.17), it is easy to prove (2.18) and so we complete the proof of Theorem 2.11.  $\square$

**Theorem 2.12** Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2),  $\mathfrak{A} = \iint_{\mathbb{U}} J_f(z) dx dy$ ,  $z = re^{i\theta} \in \mathbb{U}$ .

(i) If  $f \in \bar{K}_H^\alpha(A, B)$ , then

$$\mathfrak{A} \geq \begin{cases} 2\pi \int_0^1 (1 - Br)^{\frac{2(A-B)}{B}} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B \neq 0, \\ 2\pi \int_0^1 e^{-2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B = 0, \end{cases}$$

$$\mathfrak{A} \leq \begin{cases} 2\pi \left( \int_0^a (1 + Br)^{\frac{2(A-B)}{B}} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2} r dr + \int_a^1 (1 + Br)^{\frac{2(A-B)}{B}} r dr \right), & B \neq 0, \\ 2\pi \left( \int_0^a e^{2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2} r dr + \int_a^1 e^{2Ar} r dr \right), & B = 0. \end{cases}$$

(ii) If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then

$$\mathfrak{A} \geq \begin{cases} 2\pi \int_0^1 (1 - Ar)^2 (1 - Br)^{\frac{2(A-2B)}{B}} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B \neq 0, \\ 2\pi \int_0^1 (1 - Ar)^2 e^{-2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B = 0, \end{cases}$$

$$\mathfrak{A} \leq \begin{cases} 2\pi \left( \int_0^a \frac{(1+Ar)^2 (1-\alpha^2)(1-r^2)}{(1+Br)^{\frac{2(2B-A)}{B}} (1-\alpha r)^2} r dr + \int_a^1 \frac{(1+Ar)^2}{(1+Br)^{\frac{2(2B-A)}{B}}} r dr \right), & B \neq 0, \\ 2\pi \left( \int_0^a (1 + Ar)^2 e^{2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2} r dr + \int_a^1 (1 + Ar)^2 e^{2Ar} r dr \right), & B = 0. \end{cases}$$

**Proof** We can give the Jacobian of  $f = h + \bar{g}$  in the form

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 (1 - |\omega(z)|^2), \tag{2.21}$$

where  $\omega(z)$  is the dilatation of  $f(z)$ .

If  $f \in \bar{K}_H^\alpha(A, B)$ , applying (1.7) and (2.16) to (2.21), we obtain

$$\begin{aligned} \mathfrak{A} &= \iint_{\mathbb{U}} J_f(z) dx dy = \int_0^{2\pi} d\theta \int_0^1 J_f(re^{i\theta}) r dr \\ &= \int_0^{2\pi} d\theta \int_0^1 |h'(re^{i\theta})|^2 (1 - |\omega(re^{i\theta})|^2) r dr \\ &\geq \begin{cases} 2\pi \int_0^1 (1 - Br)^{\frac{2(A-B)}{B}} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B \neq 0, \\ 2\pi \int_0^1 e^{-2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2} r dr, & B = 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{A} &= \int_0^{2\pi} d\theta \int_0^1 |h'(re^{i\theta})|^2 (1 - |\omega(re^{i\theta})|^2) r dr \\ &\leq \begin{cases} 2\pi \int_0^1 (1 + Br)^{\frac{2(A-B)}{B}} \left( 1 - \frac{(\max\{(\alpha-r), 0\})^2}{(1-\alpha r)^2} \right) r dr, & B \neq 0, \\ 2\pi \int_0^1 e^{2Ar} \left( 1 - \frac{(\max\{(\alpha-r), 0\})^2}{(1-\alpha r)^2} \right) r dr, & B = 0, \end{cases} \\ &= \begin{cases} 2\pi \left( \int_0^a (1 + Br)^{\frac{2(A-B)}{B}} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2} r dr + \int_a^1 (1 + Br)^{\frac{2(A-B)}{B}} r dr \right), & B \neq 0, \\ 2\pi \left( \int_0^a e^{2Ar} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2} r dr + \int_a^1 e^{2Ar} r dr \right), & B = 0. \end{cases} \end{aligned}$$

Therefore, we complete the proof of (i). Applying (1.9) and (2.16) to (2.21), the proof of (ii) is similar to the above and so is omitted. □

Finally, we can deduce the growth estimate of  $f$ .

**Theorem 2.13** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2),  $z \in \mathbb{U}$ .*

(i) *If  $f \in \bar{K}_H^\alpha(A, B)$ , then*

$$|f(z)| \geq \begin{cases} \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - B\xi)^{\frac{A-B}{B}} d\xi, & B \neq 0, \\ \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} e^{-A\xi} d\xi, & B = 0. \end{cases} \quad (2.22)$$

$$|f(z)| \leq \begin{cases} \int_0^r \frac{(1+\alpha)(1+\xi)}{1+\alpha\xi} (1 + B\xi)^{\frac{A-B}{B}} d\xi, & B \neq 0, \\ \int_0^r \frac{(1+\alpha)(1+\xi)}{1+\alpha\xi} e^{A\xi} d\xi, & B = 0. \end{cases} \quad (2.23)$$

(ii) *If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then*

$$|f(z)| \geq \begin{cases} \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - A\xi)(1 - B\xi)^{\frac{A-2B}{B}} d\xi, & B \neq 0, \\ \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - A\xi) e^{-A\xi} d\xi, & B = 0. \end{cases} \quad (2.24)$$

$$|f(z)| \leq \begin{cases} \int_0^r \frac{(1+\alpha)(1+\xi)}{1+\alpha\xi} (1 + A\xi)(1 + B\xi)^{\frac{A-2B}{B}} d\xi, & B \neq 0, \\ \int_0^r \frac{(1+\alpha)(1+\xi)}{1+\alpha\xi} (1 + A\xi) e^{A\xi} d\xi, & B = 0. \end{cases} \quad (2.25)$$

**Proof** For any point  $z = re^{i\theta} \in \mathbb{U}$ , let  $\mathbb{U}_r = \mathbb{U}(0, r) = \{z \in \mathbb{U} : |z| < r\}$  and denote

$$d = \min_{z \in \mathbb{U}_r} |f(\mathbb{U}_r)|, \quad (2.26)$$

and then  $\mathbb{U}(0, d) \subseteq f(\mathbb{U}_r) \subseteq f(\mathbb{U})$ . Hence, there exists  $z_r \in \partial\mathbb{U}_r$  such that  $d = |f(z_r)|$ . Let  $L(t) = tf(z_r)$ ,  $t \in [0, 1]$ ; then  $\ell(t) = f^{-1}(L(t))$ ,  $t \in [0, 1]$  is a well-defined Jordan arc. Since  $f = h + \bar{g} \in \bar{K}_H^\alpha(A, B)$ , then using Lemma 1.6, (1.7), and (2.16), we obtain

$$\begin{aligned} d &= |f(z_r)| = \int_L |d\omega| = \int_\ell |df| = \int_\ell |h'(\eta)d\eta + \overline{g'(\eta)d\bar{\eta}}| \\ &\geq \int_\ell |h'(\eta)|(1 - |\omega(\eta)|)|d\eta| \\ &\geq \begin{cases} \int_\ell \frac{(1-\alpha)(1-|\eta|)}{1+\alpha|\eta|} (1 - B|\eta|)^{\frac{A-B}{B}} |d\eta|, & B \neq 0, \\ \int_\ell \frac{(1-\alpha)(1-|\eta|)e^{-A|\eta|}}{1+\alpha|\eta|} |d\eta|, & B = 0. \end{cases} \\ &= \begin{cases} \int_0^1 \frac{(1-\alpha)(1-|\ell(t)|)}{1+\alpha|\ell(t)|} (1 - B|\ell(t)|)^{\frac{A-B}{B}} dt, & B \neq 0, \\ \int_0^1 \frac{(1-\alpha)(1-|\ell(t)|)e^{-A|\ell(t)|}}{1+\alpha|\ell(t)|} dt, & B = 0. \end{cases} \\ &\geq \begin{cases} \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - B\xi)^{\frac{A-B}{B}} d\xi, & B \neq 0, \\ \int_0^r \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} e^{-A\xi} d\xi, & B = 0. \end{cases} \end{aligned}$$

To prove (2.23), we simply use the inequality

$$|f(z)| = |h(z) + \bar{g}(z)| \leq |h(z)| + |g(z)|.$$

By (1.7) and (2.16), with simple calculation, we have (2.23). Using the same methods, we can complete the proofs of (2.24) and (2.25).  $\square$

Using (2.22) and (2.24), we have:

**Corollary 2.14** *Let  $f = h + \bar{g}$  be such that  $h$  and  $g$  are given by (1.2),  $z \in \mathbb{U}$ .*

(i) *If  $f \in \bar{K}_H^\alpha(A, B)$ , then  $\mathbb{U}(0, R_1) \subset f(\mathbb{U})$ , where*

$$R_1 = \begin{cases} \int_0^1 \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - B\xi)^{\frac{A-B}{B}} d\xi, & B \neq 0, \\ \int_0^1 \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} e^{-A\xi} d\xi, & B = 0. \end{cases}$$

(ii) *If  $f \in \bar{S}_H^{*,\alpha}(A, B)$ , then  $\mathbb{U}(0, R_2) \subset f(\mathbb{U})$ , where*

$$R_2 = \begin{cases} \int_0^1 \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - A\xi)(1 - B\xi)^{\frac{A-2B}{B}} d\xi, & B \neq 0, \\ \int_0^1 \frac{(1-\alpha)(1-\xi)}{1+\alpha\xi} (1 - A\xi)e^{-A\xi} d\xi, & B = 0. \end{cases}$$

**Remark 2.15**

- (i) If  $A = 1 - 2\beta$  ( $0 \leq \beta < 1$ ) and  $B = -1$ , then Theorem 2.10, Theorem 2.11, Theorem 2.12, and Corollary 2.14 respectively coincide with Theorem 7, Corollary 8; Theorem 9, Corollary 10; Theorem 11, Corollary 13; Theorem 14, Corollary 15; and Theorem 16, Corollary 17 in [32];
- (ii) If  $A = 1$  and  $B = -1$ , (2.18) in Theorem 2.11 improves (2.7) of Theorem 2.4 in [14];
- (iii) If  $A = 1$  and  $B = -1$ , then Theorem 2.10, Theorem 2.12, Theorem 2.11, and Theorem 2.13 respectively coincide with Theorem 2.4, Theorem 2.6, Theorem 2.7, and Theorem 2.9 in [9].

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