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On integrability conditions, operators, and the purity conditions of the Sasakian metric with respect to lifts of $F_\lambda(7,1)$-structure on the cotangent bundle

Haşim ÇAYIR*

Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey

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Abstract: There are many structures in the cotangent bundle. These include the complete and horizontal lifts of the $F_\lambda(7,1)$-structure. The $F_\lambda(7,1)$-structure was first extended in M^n to $T^*(M^n)$ by Das, Nivas, and Pathak. Later, the horizontal and complete lift of the $F_a(K,1)$-structure in the tangent bundle was given by Prasad and Chauhan. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete lifts of the $F_\lambda(7,1)$-structure. Later, we get the results of the Tachibana operators applied to vector and covector fields according to the complete lifts of the $F_\lambda(7,1)$-structure in the cotangent bundle $T^*(M_n)$. Finally, we study the purity conditions of the Sasakian metric with respect to the complete lifts of the $F_\lambda(7,1)$-structure. In the second part, all results obtained in the first section are obtained according to the horizontal lifts of the $F_\lambda(7,1)$-structure in cotangent bundle $T^*(M_n)$.

Key words: Integrability conditions, Tachibana operators, horizontal lift, complete lift, Sasakian metric, cotangent bundle

1. Introduction

The investigation of the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, where the defining tensor field satisfies a polynomial identity, has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators; see, for example, [14]. There are many structures in the cotangent bundle. These include the complete and horizontal lifts of the $F_\lambda(7,1)$-structure. The $F_\lambda(7,1)$-structure was first extended in M^n to $T^*(M^n)$ by Das et al. [8]. Later, the horizontal and complete lift of the $F_a(K,1)$-structure in the tangent bundle was given by Prasad and Chauhan [10]. In addition, manifolds with $F(4,2)$-structure have been defined and studied by Yano et al. [13], and the complete and horizontal lifts of the $F(4,2)$-structure were extended in M^n to the cotangent bundle by Nivas and Saxena [9]. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete lifts of the $F_\lambda(7,1)$-structure. Later, we get the results of Tachibana and Wishnevskii operators applied to vector and covector fields according to the complete lifts of the $F_\lambda(7,1)$-structure in cotangent bundle $T^*(M^n)$. Finally, we study the purity conditions of the Sasakian metric with respect to the complete lifts of the $F_\lambda(7,1)$-structure. In the second part, all results obtained in the first section are obtained according to the horizontal lifts of the $F_\lambda(7,1)$-structure in cotangent bundle $T^*(M^n)$.

*Correspondence: hasim.cayir@giresun.edu.tr

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Let M^n be a differentiable manifold of class C^∞ and of dimension n and let $T^*(M^n)$ denote the cotangent bundle of M. Then $T^*(M^n)$ is also a differentiable manifold of class C^∞ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\Im_s^r(M^n)$ denotes the set of the tensor fields C^∞ and of type (r,s) on M^n. Similarly, $\Im_s^r(T^*(M^n))$ denotes the set of such tensor fields in $T^*(M^n)$.

(2) The map π is the projection of $T^*(M^n)$ onto M^n.

(3) Vector fields in M^n are denoted by X, Y, Z,... and Lie differentiation by L_X. The Lie product of vector fields X and Y is denoted by $[X,Y]$.

(4) Suffixes $a,b,c,...,h,i,j...$ take values from 1 to n and $\bar{\imath}=i+n$. Suffixes A,B,C,... take values from 1 to $2n$.

If A is a point in M^n, then $\pi^{-1}(A)$ is a fiber over A. Any point $p\in\pi^{-1}(A)$ is denoted by the ordered pair (A,p_A), where p is a 1-form in M^n and p_A is the value of p at A. Let U be a coordinate neighborhood in M^n such that $A\in U$. Then U induces a coordinate neighborhood $\pi^{-1}(U)$ in $T^*(M^n)$ and $p\in\pi^{-1}(A)$.

1.1. Complete lift of $F_\lambda(7,1)$-structure

Let $F(\neq 0)$ be a tensor field of type $(1,1)$ and class C^∞ on M^n such that [8]

$$F^7+\lambda^2F=0, \tag{1.1}$$

where λ is any complex number not equal to zero. We call the manifold M^n satisfying (1.1) an $F_\lambda(7,1)$-structure manifold. Let F_i^h be components of F at A in the coordinate neighborhood U of M^n. Then the complete lift F^C of F is also a tensor field type $(1,1)$ in $T^*(M^n)$ whose components F_B^A in $\pi^{-1}(U)$ are given by [6]

$$\tilde{F}_i^h=F_i^h;\qquad F_{\bar{\imath}}^h=0, \tag{1.2}$$

$$\tilde{F}_i^{\bar{h}}=p_a(\frac{\partial F_h^a}{\partial x^i}\frac{\partial F_i^a}{\partial x^h});\qquad \tilde{F}_{\bar{\imath}}^{\bar{h}}=F_h^i, \tag{1.3}$$

where $(x^1,x^2,...,x^n)$ are coordinates of A relative to U and p_A has a component $(p_1,p_2,...,p_n)$.

Thus, we can write

$$F^C=(\tilde{F}_B^A)=\begin{pmatrix} F_i^h & 0\\ p_a(\partial_iF_h^a-\partial_hF_i^a) & F_h^i\end{pmatrix}, \tag{1.4}$$

where $\partial_i=\partial/\partial x^i$ [8].

If we put

$$\partial_iF_h^a-\partial_hF_i^a=2\partial[iF_h^a], \tag{1.5}$$

then we can write (1.4) in the following form:

$$F^C=(F_B^A)=\begin{pmatrix} F_i^h & 0\\ 2p_a\partial[iF_h^a] & F_h^i\end{pmatrix}. \tag{1.6}$$

Thus, we have

$$(F^C)^2 = \begin{pmatrix} F_i^h & 0 \\ 2p_a\partial[iF_h^a] & f_h^i \end{pmatrix} \begin{pmatrix} F_i^h & 0 \\ 2p_a\partial[iF_h^a] & F_h^i \end{pmatrix}, \tag{1.7}$$

$$(F^C)^2 = \begin{pmatrix} F_i^h F_j^i & 0 \\ L_{hj} & F_i^j F_h^i \end{pmatrix}, \tag{1.8}$$

where $L_{hj} = 2p_a f_j^l \partial[iF_h^a] + 2p_t F_h^i \partial[jF_i^t]$ [8],
and so on. Thus,

$$(F^C)^7 = \begin{pmatrix} -\lambda^2 F_p^n & 0 \\ -\lambda^2 p_s \partial[pF_h^s] & -\lambda F_h^p \end{pmatrix}. \tag{1.9}$$

In view of (1.6) and (1.9), it follows that [8]

$$(F^C)^7 + \lambda^2(F^C) = 0. \tag{1.10}$$

Hence, the complete lift F^C of F admits an $F_\lambda(7,1)$-structure in the cotangent bundle $T^*(M^n)$.

1.2. Horizontal lift of $F_\lambda(7,1)-$structure

Let F, G be two tensor fields of type $(1,1)$ on the manifold M^n. If F^H denotes the horizontal lift of F, we have [8, 14]

$$F^H G^H + G^H F^H = (FG + GF)^H. \tag{1.11}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H, \tag{1.12}$$

and so on. Thus,

$$(F^H)^7 = (F^7)^H. \tag{1.13}$$

Since F gives on M^n, the $F_\lambda(7,1)$-structure, we have

$$F^7 + \lambda^2 F = 0. \tag{1.14}$$

Taking the horizontal lift, we obtain

$$(F^7)^H + \lambda^2(F^H) = 0. \tag{1.15}$$

In view of (1.13) and (1.15), we can write [8]

$$(F^H)^7 + \lambda^2(F^H) = 0. \tag{1.16}$$

2. Main results

2.1. The Nijenhuis tensors of $(F^7)^C$ on cotangent bundle $T^*(M^n)$

Definition 1 *Let F be a tensor field of type $(1,1)$ admitting the $F_\lambda(7,1)$-structure in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by*

$$N_F = [FX, FY] - F\,[X, FY] - F\,[FX, Y] + F^2\,[X, Y] \tag{2.1}$$

for any $X, Y \in \Im_0^1(M^n)$ [1, 11, 12]. The condition of $N_F(X,Y) = N(X,Y) = 0$ is essential to the integrability condition in these structures.

The Nijenhuis tensor N_F is defined in local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k, \tag{2.2}$$

where $X = \partial_i,\ Y = \partial_j,\ F \in \Im_1^1(M^n)$.

Proposition 1 *If X, $Y \in \Im_0^1(M^n)$, ω, $\theta \in \Im_1^0(M^n)$, and $F, G \in \Im_1^1(M^n)$, then*

$$\begin{array}{c} [\omega^V, \theta^V] = 0, \quad [\omega^V, \gamma F] = (\omega \circ F)^V, \quad [\gamma F, \gamma G] = \gamma[F, G], \\ [X^C, \omega^V] = (L_X \omega)^V, \quad [X^C, \gamma F] = \gamma(L_X F), \quad [X^C, Y^C] = [X, Y]^C, \end{array} \tag{2.3}$$

where $\omega \circ F$ is a 1-form defined by $(\omega \circ F)(Z) = \omega(FZ)$ for any $Z \in \Im_0^1(M^n)$ and L_X is the operator of Lie derivation with respect to X.

Theorem 1 *The Nijenhuis tensor $N_{(F^7)^C,(F^7)^C}(X^C, \omega^V)$ of the complete lift of F^7 vanishes if the Lie derivatives of the tensor field F with respect to X are zero and F acts as an identity operator on M^n.*

Proof

$$\begin{aligned} N_{(F^7)^C,(F^7)^C}(X^C, \omega^V) &= [(F^7)^C X^C, (F^7)^C \omega^V] - (F^7)^C[(F^7)^C X^C, \omega^V] \\ &\quad -(F^7)^C[X^C, (F^7)^C \omega^V] + (F^7)^C (F^7)^C [X^C, \omega^V] \\ &= \lambda^4\{[F^C X^C, F^C \omega^V] - (F^C)[F^C X^C, \omega^V] \\ &\quad -F^C[X^C, F^C \omega^V] + F^C F^C [X^C, \omega^V]\} \\ &= \lambda^4\{(\omega(L_{FX} F))^V - ((\omega(L_X F))F)^V \\ &\quad -((\omega \circ F) \circ (L_X F))^V + (\omega \circ (L_X F))^V \end{aligned}$$

If we suppose that $L_X F = 0$ and F acts as an identity operator on M [6], that is,

$$FX = X \qquad (\forall X \in \Im_0^1(M)),$$

then we have

$$N_{(F^7)^C(F^7)^C}(X^C, Y^C) = 0.$$

The theorem is proved. □

Theorem 2 *The Nijenhuis tensor* $N_{(F^7)^C(F^7)^C}(\omega^V,\theta^V)$ *of the complete lift* F^7 *vanishes.*

Proof

$$\begin{aligned}
N_{(F^7)^C(F^7)^C}(\omega^V,\theta^V) &= [(F^7)^C\omega^V,(F^7)^C,\theta^V]-(F^7)^C[(F^7)^C\omega^V,\theta^V] \\
&\quad -(F^7)^C[\omega^V,(F^7)^C\theta^V]+(F^7)^C(F^7)^C[\omega^V,\theta^V] \\
&= \lambda^4\{[(\omega\circ F)^V,(\theta\circ F)^V]-F^C[(\omega\circ F)^V,\theta^V] \\
&\quad -F^C[\omega^V,(\theta\circ F)^V]+(F^2)^C[\omega^V,\theta^V]\} \\
&= 0,
\end{aligned}$$

where $[\omega^V,\theta^V]=0,\ \omega,\theta\in\Im_1^0(M^n)$. □

2.2. Tachibana operators applied to vector fields according to lifts of $F_\lambda(7,1)$-structure on $T^*(M^n)$

Definition 2 *Let* $\varphi\in\Im_1^1(M^n)$, *and let* $\Im(M^n)=\sum_{r,s=0}^{\infty}\Im_s^r(M^n)$ *be a tensor algebra over* R. *A map* $\phi_\varphi\mid_{r+s\rangle 0}:\overset{*}{\Im}(M^n)\to\Im(M^n)$ *is called a Tachibana operator or* ϕ_φ *operator on* M^n *if:*

a) ϕ_φ is linear with respect to a constant coefficient,

b) $\phi_\varphi:\overset{*}{\Im}(M^n)\to\Im_{s+1}^r(M^n)$ for all r and s,

c) $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M^n)$,

d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see [2, 4, 7]),

e)

$$\begin{aligned}
(\phi_{\varphi X}\eta)Y &= (d(\imath_Y\eta))(\varphi X)-(d(\imath_Y(\eta o\varphi)))X+\eta((L_Y\varphi)X) \\
&= \phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)+\eta((L_Y\varphi)X)
\end{aligned}$$

for all $\eta\in\Im_1^0(M^n)$ and $X,Y\in\Im_0^1(M^n)$, where $\imath_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\overset{*}{\Im}{}_s^r(M^n)$, the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, and $\overset{C}{\otimes}$ is a tensor product with a contraction C [1, 3, 11] (see [12] for application to pure tensor fields).

Remark 1 *If* $r=s=0$, *then from* $c),d)$ *and* $e)$ *of Definition 2 we have* $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ *for* $\imath_Y\eta\in\Im_0^0(M^n)$, *which is not well-defined* ϕ_φ*-operator. Different choices of* Y *and* η *leading to the same function* $f=\imath_Y\eta$ *do get the same values. Consider* $M^n=R^2$ *with standard coordinates* x,y. *Let* $\varphi=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. *Consider the function* $f=1$. *This may be written in many different ways as* $\imath_Y\eta$. *Indeed, taking* $\eta=dx$, *we may choose* $Y=\frac{\partial}{\partial_x}$ *or* $Y=\frac{\partial}{\partial_x}+x\frac{\partial}{\partial_y}$. *Now the right-hand side of* $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ *is* $(\phi X)1-0=0$ *in the first case and* $(\phi X)1-Xx=-Xx$ *in the second case. For* $X=\frac{\partial}{\partial_x}$, *the latter expression is* $-1\neq 0$. *Therefore, we put* $r+s>0$ *[11].*

Remark 2 *From d) of Definition 2 we have*

$$\phi_{\varphi X}Y = [\varphi X, Y] - \varphi[X, Y]. \tag{2.4}$$

By virtue of

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \tag{2.5}$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in X, but not Y [11].

Theorem 3 *Let $(F^C)^7$ be a tensor field of type $(1,1)$ on $T^*(M^n)$ defined by (1.9). If the Tachibana operator ϕ_φ is applied to vector fields according to complete lifts of the $F_\lambda(7,1)$-structure defined by (1.10) on $T^*(M^n)$, then we get the following results:*

$$\begin{aligned}
i)\ \phi_{(F^7)^C X^C}Y^C &= \lambda^2\{((L_Y F)X)^C + \gamma(L_Y(L_X F)) - \gamma(L_{[Y,X]}F)\},\\
ii)\ \phi_{(F^7)^C X^C}\omega^V &= \lambda^2\{-(L_{(FX)}\omega)^V + (\omega \circ (L_X F))^V + ((L_X\omega)\circ F)^V\},\\
iii)\ \phi_{(F^7)^C \omega^V}X^C &= \lambda^2(\omega(L_X F))^V,\\
iv)\ \phi_{(F^7)^C \omega^V}\theta^V &= 0,
\end{aligned}$$

where complete lifts $X^C, Y^C \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof *i)*

$$\begin{aligned}
\phi_{(F^7)^C X^C}Y^C &= -(L_{Y^C}(F^7)^C)X^C\\
&= -L_{Y^C}(F^7)^C X^C + (F^7)^C L_{Y^C}X^C\\
&= \lambda^2 L_{Y^C}F^C X^C - \lambda^2 F^C[Y,X]^C\\
&= \lambda^2\{[Y^C,(FX)^C] + [Y^C, \gamma(L_X F)]\\
&\quad -(F[Y,X])^C - \gamma(L_{[Y,X]}F)\}\\
&= \lambda^2\{((L_Y F)X)^C + (F(L_Y X))^C\\
&\quad +\gamma(L_Y(L_X F)) - (F(L_Y X))^C\\
&\quad -\gamma(L_{[Y,X]}F)\}\\
&= \lambda^2\{((L_Y F)X)^C + \gamma(L_Y(L_X F))\\
&\quad -\gamma(L_{[Y,X]}F)\}
\end{aligned}$$

ii)

$$\begin{aligned}
\phi_{(F^7)^C X^C}\omega^V &= -(L_{\omega^V}(F^7)^C)X^C \\
&= -L_{\omega^V}(F^7)^C X^C + (F^7)^C L_{\omega^C} X^C \\
&= \lambda^2 L_{\omega^V} F^C X^C - \lambda^2 F^C(-(L_X\omega)^V) \\
&= \lambda^2\{[\omega^V, (FX)^C + \gamma(L_X F)] \\
&\quad +F^C(L_X\omega)^V\} \\
&= \lambda^2\{[\omega^V, (FX)^C] + [\omega^V, \gamma(L_X F)] \\
&\quad +((L_X\omega)\circ F)^V\} \\
&= \lambda^2\{-(L_{(FX)}\omega)^V + (\omega\circ(L_X F))^V \\
&\quad +((L_X\omega)\circ F)^V\}
\end{aligned}$$

iii)

$$\begin{aligned}
\phi_{(F^7)^C \omega^V} X^C &= -(L_{X^C}(F^7)^C)\omega^V \\
&= -L_{X^C}(F^7)^C\omega^V + (F^7)^C L_{X^C}\omega^V \\
&= \lambda^2 L_{X^C} F^C \omega^V - \lambda^2 F^C (L_X\omega)^V \\
&= \lambda^2\{L_{X^C}(\omega\circ F)^V - ((L_X\omega)\circ F)^V\} \\
&= \lambda^2\{(L_X(\omega\circ F))^V - ((L_X\omega)\circ F)^V\} \\
&= \lambda^2(\omega(L_X F))^V
\end{aligned}$$

iv)

$$\begin{aligned}
\phi_{(F^7)^C \omega^V}\theta^V &= -(L_{\theta^V}(F^7)^C)\omega^V \\
&= -L_{\theta^V}(F^7)^C\omega^V + (F^7)^C L_{\theta^V}\omega^V \\
&= \lambda^2 L_{\theta^V} F^C \omega^V \\
&= \lambda^2 L_{\theta^V}(\omega\circ F)^V \\
&= 0
\end{aligned}$$

$\square$

2.3. The purity conditions of Sasakian metric with respect to $(F^7)^C$

Definition 3 *A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the following three equations:*

$$ {}^S g(\omega^V, \theta^V) = (g^{-1}(\omega,\theta))^V = g^{-1}(\omega,\theta) o\pi, \tag{2.6}$$

$$ {}^S g(\omega^V, Y^H) = 0, \tag{2.7}$$

$$ {}^S g(X^H, Y^H) = (g(X,Y))^V = g(X,Y)\circ\pi. \tag{2.8}$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j,$$

where $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [14], p. 280), it follows that ${}^S g$ is completely determined by equations (2.6), (2.7), and (2.8).

Theorem 4 *Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^7)^C$ of type $(1,1)$ defined by (1.9). Sasakian metric ${}^S g$ is pure with respect to $(F^7)^C$ if $F = I$ and $\nabla F = 0$ (I = identity tensor field of type $(1,1)$).*

Proof We put

$$S(\tilde{X}, \tilde{Y}) = {}^S g((F^7)^C \tilde{X}, \tilde{Y}) - {}^S g(\tilde{X}, (F^7)^C \tilde{Y}).$$

If $S(\tilde{X}, \tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ that are of the form ω^V, θ^V or X^H, Y^H, then $S = 0$. By virtue of $(F^C)^7 + \lambda^2(F^C) = 0$ and (2.6), (2.7), and (2.8), we get:

i)

$$\begin{aligned} S(\omega^V, \theta^V) &= {}^S g((F^7)^C \omega^V, \theta^V) - {}^S g(\omega^V, (F^7)^C \theta^V) \\ &= {}^S g(-\lambda^2 F^C \omega^V, \theta^V) - {}^S g(\omega^V, -\lambda^2 F^C \theta^V) \\ &= -\lambda^2({}^S g((\omega \circ F)^V - {}^S g(\omega^V, (\theta \circ F)^V) \\ &= -\lambda^2((g^{-1}((\omega \circ F), \theta))^V - (g^{-1}(\omega, (\theta \circ F)))^V). \end{aligned}$$

ii)

$$\begin{aligned} S(X^H, \theta^V) &= {}^S g((F^7)^C X^H, \theta^V) - {}^S g(X^H, (F^7)^C \theta^V) \\ &= {}^S g(-\lambda^2 F^C X^H, \theta^V) - {}^S g(X^H, -\lambda^2 F^C \theta^V) \\ &= -\lambda^2({}^S g((FX)^H, \theta^V)) - \lambda^2({}^S g((p[\nabla F]_X)^V, \theta^V)) \\ &= -\lambda^2({}^S g((p([\nabla F]_X))^V, \theta^V)) \\ &= -\lambda^2(g^{-1}((p[\nabla F]_X), \theta)))^V. \end{aligned}$$

iii)

$$\begin{aligned} S(X^H,Y^H) &= {}^Sg((F^7)^C X^H,Y^H) -^S g(X^H,(F^7)^C Y^H) \\ &= {}^Sg(-\lambda^2 F^C X^H,Y^H) -^S g(X^H,-\lambda^2 F^C Y^H) \\ &= -\lambda^2\{{}^Sg((FX)^H+\gamma([\nabla F]_X),Y^H) \\ &\quad -{}^Sg(X^H,(FY)^H+\gamma([\nabla F]_Y))\} \\ &= -\lambda^2\{{}^Sg((FX)^H,Y^H) +^S g((p([\nabla F]_X))^V,Y^H) \\ &\quad -{}^Sg(X^H,(FY)^H) -^S g(X^H,(p([\nabla F]_Y))^V)\} \\ &= -\lambda^2({}^Sg((FX)^H,Y^H) -^S g(X^H,(FY)^H)) \\ &= -\lambda^2((g((FX),Y))^V-(g(X,(FY)))^V), \end{aligned}$$

where $F^C X^H = (FX)^H+\gamma([\nabla F]_X)$ for all $X^H \in \Im_0^1(T^*(M^n))$, $F^C \in \Im_1^1(T^*(M^n))$, and $[\nabla F]_X \in \Im_1^1(M^n)$ (see [14], p. 279). □

2.4. The Nijenhuis tensors of $(F^7)^H$ on cotangent bundle $T^*(M^n)$

Theorem 5 *The Nijenhuis tensor $N_{(F^7)^H(F^7)^H}(X^H,Y^H)$ of the horizontal lift F^7 vanishes if F is an almost complex structure, i.e. $F^2=-I$ and $R(FX,FY)=R(X,Y)$.*

Proof

$$\begin{aligned} N_{(F^7)^H(F^7)^H}(X^H,Y^H) &= [(F^7)^H X^H,(F^7)^H Y^H]-(F^7)^H[(F^7)^H X^H,Y^H] \\ &\quad -(F^7)^H[X^H,(F^7)^H Y^H]+(F^7)^H(F^7)^H[X^H,Y^H] \\ &= \lambda^4\{[F^H X^H,F^H Y^H]-F^H[F^H X^H,Y^H] \\ &\quad -F^H[X^H,F^H Y^H]+(F^H)^2[X^H,Y^H]\} \\ &= \lambda^4\{[FX,FY]^H+\gamma R(FX,FY)-F^H[(FX),Y]^H \\ &\quad -F^H\gamma R(FX,Y)-F^H[X,FY]^H-F^H\gamma R(X,FY) \\ &\quad +(F^H)^2[X,Y]^H+(F^H)^2\gamma R(X,Y) \\ &= \lambda^4\{\{[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y]\}^H \\ &\quad +\gamma\{R(FX,FY)-R(FX,Y)F-R(X,FY)F \\ &\quad +R(X,Y)F^2\}\}. \end{aligned}$$

$(F^7)^H$ is integrable if the curvature tensor R of ∇ satisfies $R(FX,FY)=R(X,Y)$ and F is an almost complex structure, and then we get $R(FX,Y)=-R(X,FY)$. Hence, using $F^2=-I$, we find $R(FX,FY)-R(FX,Y)F-R(X,FY)F+R(X,Y)F^2=0$. Therefore, it follows that $N_{(F^7)^H(F^7)^H}(X^H,Y^H)=0$. □

Theorem 6 *The Nijenhuis tensor $N_{(F^7)^H(F^7)^H}(X^H,\omega^V)$ of the horizontal lift F^7 vanishes if $\nabla F=0$.*

Proof

$$\begin{aligned}
N_{(F^7)^H(F^7)^H}(X^H,\omega^V) &= [(F^7)^H X^H,(F^7)^H\omega^V]-(F^7)^H[(F^7)^H X^H,\omega^V] \\
&\quad -(F^7)^H[X^H,(F^7)^H\omega^V]+(F^7)^H(F^7)^H[X^H,\omega^V] \\
&= \lambda^4\{[F^H X^H,F^H\omega^V]-F^H[F^H X^H,\omega^V] \\
&\quad -F^H[X^H,F^H\omega^V]+(F^H)^2[X^H,\omega^V]\} \\
&= \lambda^4\{[(FX)^H,(\omega\circ F)^V]-F^H[(FX)^H,\omega^V] \\
&\quad -F^H[X^H,(\omega\circ F)^V]+(F^H)^2(\nabla_X\omega)^V\} \\
&= \lambda^4\{\omega\circ(\nabla_{FX}F)-(\omega\circ(\nabla_X F)F\}^V,
\end{aligned}$$

where $F\in\Im_1^1(M)$, $X\in\Im_0^1(M)$, $\omega\in\Im_1^0(M)$. The theorem is proved. □

Theorem 7 *The Nijenhuis tensor $N_{(F^7)^H(F^7)^H}(\omega^V,\theta^V)$ of the horizontal lift F^7 vanishes.*

Proof

$$\begin{aligned}
N_{(F^7)^H(F^7)^H}(\omega^V,\theta^V) &= [(F^7)^H\omega^v,(F^7)^H\theta^V]-(F^7)^H[(F^7)^H\omega^V,\theta^V] \\
&\quad -(F^7)^H[\omega^V,(F^7)^H\theta^V]+(F^7)^H(F^7)^H[\omega^V,\theta^V] \\
&= \lambda^4\{[(\omega\circ F)^V,(\theta\circ F)^V]-F^H[(\omega\circ F)^H,\theta^V] \\
&\quad -F^H[\omega^V,(\theta\circ F)^V]+(F^H)^2[\omega^V,\theta^V]\} \\
&= 0
\end{aligned}$$

□

Theorem 8 *Let $(F^H)^7$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator ϕ_φ is applied to vector fields according to horizontal lifts of the $F_\lambda(7,1)$-structure defined by (1.16) on $T^*(M^n)$, then we get the following results:*

$$\begin{aligned}
i)\ \phi_{(F^7)^H X^H}Y^H &= \lambda^2\{((L_Y F)X)^H+(PR(Y,FX))^V \\
&\quad -((PR(Y,X))\circ F)^V\}, \\
ii)\ \phi_{(F^7)^H X^H}\omega^V &= \lambda^2(((\nabla_X\omega)\circ F)^V-(\nabla_{(FX)}\omega)^V), \\
iii)\ \phi_{(F^7)^H \omega^V}X^H &= \lambda^2(\omega\circ(\nabla_X F))^V, \\
iv)\ \phi_{(F^7)^H \omega^V}\theta^V &= 0,
\end{aligned}$$

where horizontal lifts $X^H,Y^H\in\Im_0^1(T^*(M^n))$ of $X,Y\in\Im_0^1(M^n)$ and the vertical lift $\omega^V,\theta^V\in\Im_0^1(T^*(M^n))$ of $\omega,\theta\in\Im_1^0(M^n)$ are given, respectively.

Proof *i)*

$$\begin{aligned}
\phi_{(F^7)^H X^H} Y^H &= -(L_{Y^H}(F^7)^H)X^H \\
&= -L_{Y^H}(F^7)^H X^H + (F^7)^H L_{Y^H} X^H \\
&= \lambda^2 L_{Y^H} F^H X^H - \lambda^2 F^H([Y,X]^H + (PR(Y,X))^V) \\
&= \lambda^2\{(L_Y FX)^H + (PR(Y,FX))^V - (FL_Y X)^H \\
&\quad -((PR(Y,X))\circ F)^V\} \\
&= \lambda^2\{((L_Y F)X)^H + (PR(Y,FX))^V - ((PR(Y,X))\circ F)^V\}
\end{aligned}$$

ii)

$$\begin{aligned}
\phi_{(F^7)^H X^H} \omega^V &= -(L_{\omega^V}(F^7)^H)X^H \\
&= -L_{\omega^V}(F^7)^H X^H + (F^7)^H L_{\omega^V} X^H \\
&= \lambda^2 L_{\omega^V}(FX)^H + \lambda^2 F^H(\nabla_X\omega)^V \\
&= -\lambda^2(\nabla_{(FX)}\omega)^V + \lambda^2((\nabla_X\omega)\circ F)^V \\
&= \lambda^2(((\nabla_X\omega)\circ F)^V - (\nabla_{(FX)}\omega)^V)
\end{aligned}$$

iii)

$$\begin{aligned}
\phi_{(F^7)^H \omega^V} X^H &= -(L_{X^H}(F^7)^H)\omega^V \\
&= -L_{X^H}(F^7)^H \omega^V + (F^7)^H L_{X^H}\omega^V \\
&= \lambda^2 L_{X^H}(\omega\circ F)^V - \lambda^2 F^H(\nabla_X\omega)^V \\
&= \lambda^2(\nabla_X(\omega\circ F))^V - \lambda^2((\nabla_X\omega)\circ F)^V \\
&= \lambda^2(\omega\circ(\nabla_X F))^V
\end{aligned}$$

iv)

$$\begin{aligned}
\phi_{(F^7)^H \omega^V} \theta^V &= -(L_{\theta^V}(F^7)^H)\omega^V \\
&= -L_{\theta^V}(F^7)^H \omega^V + (F^7)^H (L_{\theta^V}\omega^V) \\
&= \lambda^2 L_{\theta^V}(\omega\circ F)^V \\
&= 0
\end{aligned}$$

□

Theorem 9 *Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^7)^H$ of type $(1,1)$ defined by (1.16). Sasakian metric ${}^S g$ is pure with respect to $(F^7)^H$ if $F = I$ (I = identity tensor field of type $(1,1)$).*

Proof We put

$$S(\tilde{X},\tilde{Y}) = {}^S g((F^7)^H\tilde{X},\tilde{Y}) - {}^S g(\tilde{X},(F^7)^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y})=0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ that are of the form ω^V,θ^V or X^H,Y^H, then $S=0$. By virtue of $(F^H)^7+\lambda^2(F^H)=0$ and (2.6), (2.7), and (2.8), we get:

i)

$$\begin{aligned} S(\omega^V,\theta^V) &= {}^Sg((F^7)^H\omega^V,\theta^V) -{}^S g(\omega^V,(F^7)^H\theta^V) \\ &= {}^Sg(-\lambda^2F^H\omega^V,\theta^V) -{}^S g(\omega^V,-\lambda^2F^H\theta^V) \\ &= -\lambda^2({}^Sg((\omega\circ F)^V,\theta^V) -{}^S g(\omega^V,(\theta\circ F)^V)). \end{aligned}$$

ii)

$$\begin{aligned} S(X^H,\theta^V) &= {}^Sg((F^7)^HX^H,\theta^V) -{}^S g(X^H,(F^7)^H\theta^V) \\ &= {}^Sg(-\lambda^2F^HX^H,\theta^V) -{}^S g(X^H,-\lambda^2F^H\theta^V) \\ &= -\lambda^2({}^Sg((FX)^H,\theta^V) -{}^S g(X^H,(\omega\circ F)^V)) \\ &= 0. \end{aligned}$$

iii)

$$\begin{aligned} S(X^H,Y^H) &= {}^Sg((F^7)^HX^H,Y^H) -{}^S g(X^H,(F^7)^HY^H) \\ &= {}^Sg(-\lambda^2F^HX^H,Y^H) -{}^S g(X^H,-\lambda^2F^HY^H) \\ &= -\lambda^2({}^Sg((FX)^H,Y^H) -{}^S g(X^H,(FY)^H)). \end{aligned}$$

Thus, $F=I$, and then ${}^Sg$ is pure with respect to $(F^7)^H$. □

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