

## A note on polynomial expressions for sums of power of integers multiplied by exponential terms

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**Abstract:** The possible polynomial expressions for sums of powers of integers multiplied by an exponential term are investigated. We explicitly give factorization of these polynomials in terms of the roots of Apostol–Bernoulli polynomials. As a special case, alternating sums of powers of integers are also considered, and some new polynomial expressions are given.

**Key words:** Sum of powers of integers, Faulhaber’s theorem, alternating sums of power, Apostol–Bernoulli polynomials

### 1. Introduction

The sums of the form

$$\sum_{a=0}^{n-1} a^{k-1} \delta^a$$

for  $\delta = \pm 1$  and  $k \geq 2$  an integer have been studied over the centuries. The classical Faulhaber theorem states that for an even integer  $k \geq 2$  the sum

$$\sum_{a=0}^{n-1} a^{k-1}$$

is indeed a polynomial in  $n(n-1)/2$  (for notational conventions we set the upper limit of the sum to  $n-1$  and the power is fit to  $k-1$ ). The reader may refer to [6, 8, 12] for a general discussion. Faulhaber’s theorem had various generalizations in different directions. One may consider sums of fixed powers of the terms  $\{a + ib\}_{i=0}^{n-1}$ , which were studied in [4, 5, 7]. These types of sums are closely related to Bernoulli and Euler polynomials/numbers. Recently in [16], the sums of the form

$$\sum_{a=0}^{[x]-1} a^{k-1},$$

where  $x$  is a real positive number, were related to values of Bernoulli polynomials at the fractional parts of  $x$ .

Another related type of sums of interest are alternating sums of the form

$$\sum_{a=0}^{n-1} a^{k-1} (-1)^a.$$

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Different expressions for such alternating sums have been a point of interest in the last few years. In particular, expressions of various types of such sums in terms of Euler polynomials have been recently investigated (for example, see [3]). An important result on the alternating sums of powers proved by Gessel and Viennot is that

$$\sum_{a=0}^{n-1} (-1)^{n-a} a^{2m}$$

is also a polynomial in  $n(n-1)/2$  where the coefficients are the so-called Salié numbers (Determinants, paths, and plane partitions 1989, available at <http://people.brandeis.edu/~gessel/homepage/papers/index.html>). A generalization of this result, namely an explicit form for the sum

$$\sum_{a=0}^{n-1} (-1)^{n-a} (y+a)^{2m},$$

is also given in terms of both Euler and Apostol–Bernoulli polynomials (see Theorem 2.3 in [7] and Equation (1.11) in [13]).

Another direction is the study of combinatorial properties of  $q$ -analogues for sums of powers, namely the sums obtained by replacing  $a$  by  $[a] := (1 - q^a)/(1 - q)$ , where  $q$  can be seen as indeterminate. The reader is referred to [9] for similar results obtained for  $q$ -analogues of sums. The method of  $p$ -adic  $q$ -integral can also be used to obtain relations between alternating sums and families of well-known polynomials in number theory (see [11, 18, 19]).

Here we extend the matter of interest about polynomial expressions for sums of powers by considering

$$\sum_{a=0}^{n-1} a^{k-1} w^a \quad \text{or} \quad w^{-x} \sum_{a=0}^{n-1} a^{k-1} w^a \tag{1.1}$$

for an arbitrary  $w \in \mathbb{C}$  with  $w \neq 0, 1$  (the case  $w = 1$  is excluded due to poles of Apostol–Bernoulli numbers; see Section 2). In general one shall not expect that either of the sums (1.1) is equal to a polynomial in  $n$ , but here we will see that when the sums (1.1) are considered along with their counterparts obtained by replacing  $w$  by  $1/w$  then we have some nice polynomial expressions. Explicitly, we will show that for any integer  $k \geq 2$ , there exists a constant  $K$  such that the product

$$\left[ \sum_{a=0}^{n-1} a^{k-1} w^a + K \right] \cdot \left[ \sum_{a=0}^{n-1} a^{k-1} w^{-a} \pm K \right]$$

is indeed a polynomial in  $x(x-1)$  over the ring  $\mathbb{Z}[1/2]$ . Moreover, we have an explicit description of  $K_1$  and  $K_2$  (Theorem 1 and Corollary 2 below). The classical approaches for  $w \pm 1$  basically depend on elementary combinatorial identities and generating function techniques, but here we use a symmetry satisfied by Apostol–Bernoulli polynomials (see Lemma 1 below).

We proceed as follows. First we state and derive basic results about Apostol–Bernoulli polynomials, which are indeed the main tools for the rest of the paper. Then in the following section we prove the main results.

## 2. Review of Apostol–Bernoulli polynomials

Apostol–Bernoulli polynomials  $\{\beta_k(x, w)\}$  for  $k \geq 0$  are defined by the exponential generating series

$$\frac{te^{xt}}{we^t - 1} = \sum_{k=0}^{\infty} \frac{\beta_k(x, w)t^k}{k!},$$

where  $w \neq 1$  [2]. Note that  $\beta_k(x, w)$  is a polynomial in  $x$  over  $\mathbb{Z}[w, 1/(w-1)]$ . The  $k$ th Apostol–Bernoulli number at  $w$  is defined as  $\beta_k(0, w)$ . For any  $k \geq 1$ ,  $\beta_k(0, w)$  has a pole at  $w = 1$  and is analytic outside  $w = 1$ . The definition and basic properties of  $\beta_k(x, w)$ , and relations of them with the Lerch zeta function, can be found in [2].

The Apostol–Bernoulli polynomials have drawn considerable attention in the last years. They are related to Hurwitz zeta functions as explained in [2] and [14]. Also, their combinatorial properties are studied and generalized in a way similar to Bernoulli polynomials. The reader may refer to [15]. The relations among Apostol–Bernoulli polynomials and similar polynomials/numbers of combinatorial nature have been studied in many recent works, e.g., [10, 13, 20]. These polynomials also have an interpretation in terms of some specific  $p$ -adic integrals; for example, see [17].

Below we give some basic properties of  $\beta_k(x, w)$  that we will need later. These follow by direct computation. The reader may also refer to [2] for details. We may use these identities without any explanation and further reference.

First we have that  $\beta_0(x, w) = 0$ , and that  $\beta_1(x, w) = 1/(w-1)$ . The following identity is analogous to the formula well known for Bernoulli polynomials:

$$\beta_k(x, w) = \sum_{i=0}^k \binom{k}{i} \beta_i(0, w) x^{k-i}.$$

We also have that, for  $k \geq 2$ ,

$$w\beta_k(1, w) = \beta_k(0, w).$$

There are also other identities involving Apostol–Bernoulli polynomials/numbers that shall be distinguished among the other ones:

$$w\beta_k(x+1, w) = \beta_k(x, w) + kx^{k-1}, \quad (2.1)$$

$$w^x \beta_k(x, w) = \beta_k(0, w) + \sum_{a=0}^{x-1} a^{k-1} w^a, \quad x \in \mathbb{Z}, x \geq 1. \quad (2.2)$$

Note that the first one is again proved in [2]. The second one easily follows from the first one by induction. Equation (2.2) resembles the relation between the Apostol–Bernoulli polynomials and sums of powers of consecutive integers (multiplied by powers of  $w$ ). In this paper we aim to further investigate this relation and prove some certain results on expressions of such sums in terms of polynomials in  $x(x-1)$ .

We give another identity crucial for the rest of the paper. It is elementary, so the author thinks that it should be known. However, due to a lack of a suitable references, we give a complete proof here.

**Lemma 1** *For any integer  $k \geq 0$  and  $w \neq 0, 1$ , the following equality holds:*

$$(-1)^k \beta_k(x, 1/w) = w\beta_k(1-x, w). \quad (2.3)$$

**Proof** By definition we have that

$$\sum_{k=0}^{\infty} \beta_k(x, 1/w) (-1)^k \frac{t^k}{k!} = \frac{(-t)e^{x(-t)}}{(1/w)e^{-t} - 1}.$$

We may manipulate the right-hand side as

$$\frac{(-t)e^{x(-t)}}{(1/w)e^{-t} - 1} = \frac{-wte^{-xt}}{e^{-t}(1 - we^t)} = \frac{wte^{(1-x)t}}{we^t - 1} = w \sum_{k=0}^{\infty} \beta_k(1-x, w) \frac{t^k}{k!}.$$

Equating the coefficients of  $t^k$  in both expansions we obtain the desired equality. □

**Corollary 1** We have  $\beta_1(0, 1/w) = -w\beta_1(0, w)$ , and for any integer  $k \geq 2$  and  $w \neq 0, 1$ ,

$$\beta_k(0, 1/w) = (-1)^k \beta_k(0, w).$$

**Proof** The assertion for  $\beta_1$  follows since  $\beta_1(0, w) = 1/(w - 1)$ . Now for  $k \geq 2$ , we have  $w\beta_k(1, w) = \beta_k(0, w)$ , so setting  $x = 0$  in Equation (2.3) gives the desired result. □

### 3. Main results

Let  $k \geq 2$ . For any integer  $n \geq 1$ , let

$$S(n, k, w) = \sum_{a=0}^{n-1} a^{k-1} w^a$$

where  $w \in \mathbb{C}$  with  $w \neq 0, 1$ . Equation (2.2) now reads as

$$\beta_k(x, w) = w^{-x} [\beta_k(0, w) + S(n, k, w)]. \tag{3.1}$$

This equation is a natural relation between  $S(n, k, w)$  and  $\beta_k(x, w)$ , so in order to understand the possible polynomial expressions involving  $S(n, k, w)$  we shall work on  $\beta_k(x, w)$ .

We set the following convention. Whenever a result holds for a function we use the variable  $x$ , and if the result makes sense only for positive integers we use  $n$  instead of the variable  $x$ . Now we state the main result of the paper, which is about the structure of the product  $\beta_k(x, w) \cdot \beta_k(x, 1/w)$

**Theorem 1** Let  $k \geq 2$ . For any  $w \neq 0, 1$ , the product

$$\beta_k(x, w) \cdot \beta_k(x, 1/w)$$

is a polynomial in  $x(x - 1)$  over the ring  $\mathbb{Z}[w, 1/(w - 1)]$ .

**Proof** By Lemma 1 we have that

$$\beta_k(x, w) \cdot \beta_k(x, 1/w) = (-1)^k w \beta_k(x, w) \beta_k(1 - x, w).$$

Recall that

$$\beta_k(x, w) = \sum_{i=0}^k \binom{k}{i} x^{k-i} \beta_i(w).$$

As  $\beta_i(w) \in \mathbb{Z}[w, 1/(w-1)]$ , the claim for the coefficients is clear. Since  $\beta_0 = 0$  and  $\beta_1(w) = 1/(w-1)$  we have that  $(w-1)\beta_k(x, w)$  is a monic polynomial in  $x$  of degree  $k-1$ . We can factorize  $(w-1)\beta_k(x, w)$  over an algebraic closure of  $\mathbb{Q}(w)$  as

$$(w-1)\beta_k(x, w) = \prod_{i=1}^m (x - \alpha_i)^{r_i},$$

where  $\alpha_i$  denotes the distinct roots of  $\beta_k(x, w)$  with multiplicity  $r_i$ . Note that  $\alpha_i$  depends on  $w$ . Thus,

$$(w-1)\beta_k(x, 1/w) = (-1)^k w (w-1)\beta_k(1-x, w) = (-1)^k w \prod_{i=1}^m (1-x-\alpha_i)^{r_i},$$

which gives that

$$\begin{aligned} \beta_k(x, w)\beta_k(x, 1/w) &= \frac{(-1)^k w}{(w-1)^2} \prod_{i=1}^m [(x - \alpha_i)^{r_i} (1-x-\alpha_i)^{r_i}] \\ &= \frac{(-1)^k w}{(w-1)^2} \prod_{i=1}^m [(x - \alpha_i)^{r_i} (x + \alpha_i - 1)^{r_i} (-1)^{r_i}] \\ &= \frac{(-1)^k w}{(w-1)^2} (-1)^{k-1} \prod_{i=1}^m [(x(x-1) - \alpha_i(\alpha_i-1))]^{r_i} \\ &= \frac{-w}{(w-1)^2} \prod_{i=1}^m [(x(x-1) - \alpha_i(\alpha_i-1))]^{r_i}. \end{aligned}$$

□

The above theorem is also a constructive proof in the sense that it gives the factorization of the product  $\beta_k(x, w)\beta_k(x, 1/w)$  in terms of the factorization of  $\beta_k(x, w)$ . We can state this result in terms of the finite sums  $S(n, k, w)$  as follows.

**Corollary 2** *Let  $w$  be given as in Theorem 1. Then*

$$\begin{aligned} &[S(n, k, w) + \beta_k(0, w)] [S(n, k, 1/w) + (-1)^k \beta_k(0, w)] \\ &= \frac{-w}{(w-1)^2} \prod_{i=1}^m [x(x-1) - \alpha_i(\alpha_i-1)]^{r_i} \end{aligned}$$

where  $\alpha_i$  for  $i = 1, 2, \dots, m$  denotes the distinct roots of  $\beta_k(x, w)$  with multiplicity  $r_i$ .

**Proof** Let  $n \geq 1$  be an integer. First consider Equation (3.1) simultaneously for both  $w$  and  $1/w$ :

$$\begin{aligned} w^n \beta_k(n, w) &= \beta_k(0, w) + S(n, k, w), \\ w^{-n} \beta_k(n, 1/w) &= \beta_k(0, 1/w) + S(n, k, 1/w). \end{aligned}$$

Taking the product side by side gives

$$\beta_k(n, w) \cdot \beta_k(n, 1/w) = [S(n, k, w) + \beta_k(0, w)] \cdot [S(n, k, 1/w) + \beta_k(0, 1/w)].$$

Then the result follows by Corollary 1 and Theorem 1. □

Now we may specialize to the case  $w = -1$ , which has been a common interest (see the discussion in the introduction). Note that by Corollary 1,  $\beta_k(0, -1) = 0$  whenever  $k \geq 2$  is odd. Using this fact and setting  $w = -1$  in Theorem 1 and Corollary 2, we directly obtain the following result.

**Corollary 3** *Let  $w$  be given as in Theorem 1. Then for any integer  $n \geq 1$  we have*

$$[\beta_k(n, -1)]^2 = \frac{1}{4} \prod_{i=1}^m [n(n-1) - \alpha_i(\alpha_i - 1)]^{r_i} = S(n, k, w)^2, \quad \text{if } k \text{ is odd}$$

$$[\beta_k(n, -1)]^2 = \frac{1}{4} \prod_{i=1}^m [n(n-1) - \alpha_i(\alpha_i - 1)]^{r_i} = [S(n, k, w) + \beta_k(0, -1)]^2, \quad \text{if } k \text{ is even}$$

where  $\alpha_i$  for  $i = 1, 2, \dots, m$  denotes the distinct roots of  $\beta_k(x, w)$  with multiplicity  $r_i$ .

Now we obtain results for the special case  $w = -1$ . First recall that setting  $w = -1$  in Theorem 1 and its proof, we obtain

$$[\beta_k(x, -1)]^2 = \frac{1}{4} \prod_{i=1}^m [(x(x-1) - \alpha_i(\alpha_i - 1))]^{r_i}.$$

We aim to deduce the structure of  $\beta_k(x, -1)$  using this factorization of  $[\beta_k(x, -1)]^2$ . However, note that it would be wrong to directly deduce that  $\beta_k(x, -1)$  is a polynomial in  $x(x-1)$ , as we have the possibility that  $\alpha_i = 1/2$  for some  $i$ .

**Theorem 2** *Let  $w$  be given as in Theorem 1. Then there exists some polynomial  $T_k \in \mathbb{Z}[1/2][x]$  depending on  $k$  such that*

$$\beta_k(x, -1) = T_k(x(x-1)), \quad \text{if } k \text{ is odd}$$

$$\beta_k(x, -1) = \left(x - \frac{1}{2}\right) T_k(x(x-1)), \quad \text{if } k \text{ is even.}$$

**Proof** Let  $f_i(x) = x(x-1) - \alpha_i(\alpha_i - 1)$ . The roots of  $f_i$  are  $\alpha_i$  and  $1 - \alpha_i$ , so  $f_i(x)$  and  $f_j(x)$  are coprime whenever  $i \neq j$ , but also

$$[\beta_k(x, -1)]^2 = \frac{1}{4} \prod_{i=1}^m [x(x-1) - \alpha_i(\alpha_i - 1)]^{r_i}.$$

Thus, each  $[x(x-1) - \alpha_i(\alpha_i - 1)]^{r_i}$  must be a square of some polynomial. First we consider the case  $\alpha_i \neq 1/2$ . Then the roots of  $f_i$  are distinct, so  $f_i$  cannot be a square. Thus,  $r_i$  must be even if  $\alpha_i \neq 1/2$ , say  $r_i = 2s_i$ . For  $\alpha_i = 1/2$  we have that

$$x(x-1) - \alpha_i(\alpha_i - 1) = (x - 1/2)^2.$$

By reordering the terms say  $\alpha_m = 1/2$ . Thus, in any case we can write

$$4[\beta_k(x, -1)]^2 = \left( \prod_{i=1}^{m-1} [x(x-1) - \alpha_i(\alpha_i - 1)]^{2s_i} \right) (x - 1/2)^{2r_m}$$

$$\implies \pm 2\beta_k(x, -1) = \left( \prod_{i=1}^{m-1} [x(x-1) - \alpha_i(\alpha_i - 1)]^{s_i} \right) (x - 1/2)^{r_m}.$$

Now the degree of the product  $\left(\prod_{i=1}^{m-1} [x(x-1) - \alpha_i(\alpha_i - 1)]^{s_i}\right)$  is even, so the parities of the degree of  $\beta_k(x, -1)$  and of  $r_m$  must be the same (recall that the degree of  $\beta_k(x, -1)$  is  $k - 1$ ). Also, if  $r_m$  is even then  $(x - 1/2)^{r_m}$  is again a polynomial in  $x(x - 1)$ . Similarly, if  $r_m$  is odd then  $(x - 1/2)^{r_m}$  is the product of  $(x - 1/2)$  by a polynomial in  $x(x - 1)$ . Hence, we have that

$$\begin{aligned} \beta_k(x, -1) &= T_k(x(x - 1)), \quad \text{if } k \text{ is odd} \\ \beta_k(x, -1) &= \left(x - \frac{1}{2}\right) T_k(x(x - 1)), \quad \text{if } k \text{ is even and } k \neq 0 \end{aligned}$$

for some polynomial  $T_k$  depending on  $k$  as claimed.

This proves that  $\beta_k(x, -1)$  is a polynomial in  $x(x - 1)$ , but we did not say anything about the coefficients of  $T_k$ . Now we show that the coefficients of  $T_k$  belong to  $\mathbb{Z}[1/2]$ . Once we fix  $k$  we may denote  $T_k$  by  $T$ . First note that  $2\beta_k(x, -1)$  is monic. Recall that by Theorem 1

$$\beta_k(x, -1) \in \mathbb{Z}[1/2][x].$$

Let  $2T(t) = t^s + a_1 t^{s-1} + \dots + a_{s-1} t + a_s$ , so that

$$2T(x(x - 1)) = (x(x - 1))^s + a_1(x(x - 1))^{s-1} + \dots + a_{s-1}(x(x - 1)) + a_s.$$

Now consider the equalities

$$\begin{aligned} T_f(x(x - 1)) &= \beta_k(x, -1) \\ T_f(x(x - 1)) &= \frac{\beta_k(x, -1)}{\left(x - \frac{1}{2}\right)} \end{aligned}$$

respectively for odd and even  $k$ . Putting  $x = 0$  in either equality we see that  $a_s \in \mathbb{Z}[1/2]$ . Let  $y = x(x - 1)$ , so that the coefficients  $a_i$  are given by

$$2a_i = \frac{\partial T^i}{(\partial y)^i} \Big|_{y=0}.$$

However, the differential operator  $\frac{\partial}{\partial y}$  satisfies

$$\frac{\partial}{\partial y} = 2 \left(x - \frac{1}{2}\right) \frac{\partial}{\partial x},$$

so both

$$\frac{\partial \beta_k(x, -1)}{\partial y} \quad \text{and} \quad \frac{\partial(\beta_k(x, -1)/(x - 1/2))}{\partial y}$$

are equal to the product of a polynomial in  $\mathbb{Z}[1/2][x]$  and a term of the form  $(x - 1/2)^d$  where  $d \in \mathbb{Z}$ . By setting  $y = 0$  (equivalently  $x = 0$  or  $x = 1$ ) we obtain  $a_{s-1} \in \mathbb{Z}[1/2]$ .

Then in a similar way applying the operator  $\frac{\partial}{\partial y}$  and setting  $y = 0$  successively gives that  $a_i \in \mathbb{Z}[1/2][x]$  for all  $i = 1, 2, \dots, s$ . This completes the proof. □

We may state this final result in terms of the finite sums  $S(n, k, -1) = \sum_{a=0}^{n-1} a^{k-1}(-1)^a$ . It follows by Equation (3.1), Corollary 3, and the above theorem.

**Corollary 4** *We have that:*

- i) If  $k \geq 2$  is an odd integer then  $(-1)^n S(n, k, -1)$  is a polynomial of  $n(n-1)$  with coefficients in  $\mathbb{Z}[1/2]$ .*
- ii) If  $k \geq 2$  is an even integer then  $(-1)^n [S(n, k, -1) + \beta_k(0, -1)]$  is equal to the product of  $(n-1/2)$  by a polynomial of  $n(n-1)$  with coefficients in  $\mathbb{Z}[1/2]$ .*

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