Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 207 – 223

doi:10.3906/mat-1808-2

The X-coordinates of Pell equations and Padovan numbers

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Received: 01.08.2018 • **Accepted/Published Online:** 24.11.2018 • **Final Version:** 18.01.2019

Abstract: In this paper, we show that there is at most one value of the positive integer X participating in the Pell equation $X^2 - dY^2 = k$, where $k \in \{\pm 1, \pm 4\}$, which is a Padovan number, with a few exceptions that we completely characterize.

Key words: Padovan numbers, Pell equation, Linear form in logarithms, reduction method

1. Introduction

Let $\{P_l\}_{l \geq 0}$ be the Padovan sequence given by $P_l = P_{l-2} + P_{l-3}$, for $l \geq 3$, where $P_0 = 0$, P_1 and $P_2 = 1$. A few terms of this sequence are:

$$0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200 \ldots$$

Let $d > 1$ be a positive integer which is not a perfect square. Consider the Pell equations

$$X^2 - dY^2 = \pm 1, \tag{1.1}$$

and

$$X^2 - dY^2 = \pm 4. \tag{1.2}$$

It is well known that all positive solutions (X, Y) of (1.1) are given by

$$X_n + Y_n\sqrt{d} = (X_1 + Y_1\sqrt{d})^n,$$

for some positive integer n, where (X_1, Y_1) is the smallest positive solution of (1.1). Also, it is well know that all positive solutions (X, Y) of (1.2) are given by

$$\frac{X_m + Y_m\sqrt{d}}{2} = \left(\frac{X_1' + Y_1'\sqrt{d}}{2}\right)^m,$$

for some positive integer m, where (X_1', Y_1') is the smallest positive solution of (1.2).

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2010 *AMS Mathematics Subject Classification:* 11A25, 11B39, 11J86

In the literature, there are many papers investigating for which d there are members of the sequence $\{X_n\}_{n\geq 1}$ or $\{Y_m\}_{m\geq 1}$ belonging to some interesting sequences of positive integers such as the sequence of all base 10-repdigits [2], the sequence of all base b-repdigits [4], the sequence of Fibonacci numbers [5, 8], and the sequence of Tribonacci numbers [7]. For most sequences, one expects that the answer to such a question has at most one positive integer solution n for any given d except maybe for a few (finitely many) values of d. It is natural to ask what will happen if X_m is a Padovan number.

In this paper, we study when X_n and X_m can be a Padovan number. We will prove the following theorems:

Theorem 1.1 *Let $d \geq 2$ be square-free. The diophantine equation*

$$X_n = P_l, \tag{1.3}$$

has at most one solution (n, l) in positive integers with the following exceptions:

- $(n_1, l_1) = (1, 4)$, $(n_2, l_2) = (1, 5)$, $(n_3, l_3) = (1, 8)$, $(n_4, l_4) = (2, 9)$ *and* $(n_5, l_5) = (2, 16)$ *in the* $+1$ *case,*
- $(n_1, l_1) = (1, 1)$, $(n_2, l_2) = (1, 2)$, $(n_3, l_3) = (1, 3)$, $(n_4, l_4) = (1, 4)$, $(n_5, l_5) = (1, 5)$, $(n_6, l_6) = (2, 6)$, $(n_7, l_7) = (2, 10)$, *and* $(n_8, l_8) = (3, 9)$ *in the* -1 *case.*

Theorem 1.2 *Let $d \geq 2$ be square-free. The diophantine equation*

$$X_m = P_l, \tag{1.4}$$

has at most one solution (m, l) in positive integers with the following exceptions:

- $(m_1, l_1) = (1, 4)$, $(m_2, l_2) = (1, 5)$, *and* $(m_3, l_3) = (2, 11)$, *in the* -1 *case.*

We organize this paper as follows. In Section 2, we recall some results useful for the proof of two main theorems, particularly some results on the lower bounds of linear forms in logarithms and Baker–Davenport the reduction method. The proof of Theorem 1.1 is done in four steps in Section 3, and the last section is devoted to the proof of Theorem 1.2 using the same method.

2. Auxiliary results

2.1. The Padovan sequence

Here, we recall a few properties of the Padovan sequence $\{P_l\}_{l\geq 0}$ which are useful in proving our theorem.
The characteristic equation

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Further, Binet's formula is

$$P_l = a\alpha^l + b\beta^l + c\gamma^l, \quad \text{for all } l \geq 0, \tag{2.1}$$

where

$$a = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)},\ b = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)},\ c = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \bar{b}. \tag{2.2}$$

Numerically, we have

$$\begin{aligned} &1.32 < \alpha < 1.33,\\ &0.86 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.87,\\ &0.72 < a < 0.73,\\ &0.24 < |b| = |c| < 0.25. \end{aligned} \tag{2.3}$$

Using induction, we can prove that

$$\alpha^{l-2} \leq P_l \leq \alpha^{l-1}, \tag{2.4}$$

for all $l \geq 4$.

2.2. Linear forms in logarithms

The next tools are related to the transcendental approach to solve diophantine equations. For any nonzero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a_0 \prod_{i=1}^{d} \left(X - \gamma^{(i)}\right)$, we denote the usual absolute logarithmic height of γ by

$$h(\gamma) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\left(1, \left|\gamma^{(i)}\right|\right)\right).$$

We start by recalling Theorem 9.4 of [1], which is a modified version of a result of Matveev [9].

Lemma 2.1 *Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

When $s = 2$, we have the following result due Laurent (see [6]), which is better than Lemma 2.1 in this particular case.

Lemma 2.2 *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2 \log\gamma_2 - b_1 \log\gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. *Let*

$$h_j \geq \max\left\{h(\gamma_j), \frac{|\log \gamma_j|}{D}, \frac{1}{D}\right\} \quad \textit{for} \;\; j = 1, 2, \quad b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then, we have

$$\log|\Lambda| \geq -17.9 \cdot D^4 \left(\max\left\{\log b' + 0.38, \frac{30}{D}, 1\right\}\right)^2 h_1 h_2.$$

2.3. The reduction method

We recall now a slight modification of the original version of the Baker–Davenport reduction method. (See [3], Lemma 5a.)

Lemma 2.3 *Assume that* τ *and* μ *are real numbers and* M *is a positive integer. Let* p/q *be the convergent of the continued fraction of the irrational* τ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\xi = ||\mu q|| - M \cdot ||\tau q||$, *where* $||\cdot||$ *denotes the distance from the nearest integer. If* $\xi > 0$, *then there is no solution of the inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

in positive integers m, n, *and* k *with*

$$m \leq M \quad \textit{and} \quad k \geq \frac{\log(Aq/\xi)}{\log B}.$$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in four steps,

3.1. Step 1:

In this step, we will determine the relationship between n and l. So let (X_1, Y_1) the fundamental solution of the Pell equation (1.1), so

$$X_1^2 - dY_1^2 =: \varepsilon, \quad \varepsilon = \pm 1.$$

We put

$$\delta := X_1 + \sqrt{d}Y_1 \quad \text{and} \quad \eta := X_1 - \sqrt{d}Y_1 = \varepsilon\delta^{-1},$$

then

$$X_n = \frac{1}{2}(\delta^n + \eta^n). \tag{3.1}$$

Using the fact that $\delta \geq 1 + \sqrt{2}$, we get the following estimate

$$\frac{\delta^n}{\alpha^4} \leq X_n \leq \delta^n, \quad \text{for all} \quad n \geq 1. \tag{3.2}$$

We now assume that (n_1, l_1) and (n_2, l_2) are pairs of positive integers such that

$$X_{n_1} = P_{l_1} \quad \text{and} \quad X_{n_2} = P_{l_2}.$$

Without losing the generality, we can assume that $n_1 < n_2$, so $l_1 < l_2$. Putting $(n,l) = (n_i, l_i)$ for $i \in \{1,2\}$ and using inequalities (2.4) and (3.2), we obtain that

$$\alpha^{l-2} \le P_l = X_n < \delta^n \quad \text{and} \quad \frac{\delta^n}{\alpha^4} \le X_n = P_l \le \alpha^{l-1}. \tag{3.3}$$

Hence, we get

$$nc_1 - 3 \le l \le nc_1 + 2, \quad c_1 := \log\delta/\log\alpha. \tag{3.4}$$

3.2. Step 2:

In this step, we will apply Matveev's theorem of a linear form in three logarithms to get a bound to n and l in terms of $\log l$. First, we will prove the following lemma.

Lemma 3.1 *If $l > 200$, then*

$$\left|\delta^n (2a)^{-1}\alpha^{-l} - 1\right| < \frac{2}{\alpha^{3l/2}}. \tag{3.5}$$

Proof Using (2.1) and (3.1), we get

$$\frac{\delta^n}{2} - a\alpha^l = -\frac{\eta^n}{2} + b\beta^l + c\gamma^l.$$

Multiplying both sides by $a^{-1}\alpha^{-l}$, we obtain

$$\delta^n(2a)^{-1}\alpha^{-l} - 1 = -(2a)^{-1}\alpha^{-l}\eta^n + (b/a)(\beta\alpha^{-1})^l + (c/a)(\gamma\alpha^{-1})^l.$$

Thus, using (2.3), and assuming $l > 200$, we have

$$\begin{aligned}
\left|\delta^n(2a)^{-1}\alpha^{-l} - 1\right| &\le \frac{1}{2a\alpha^l\delta^n} + \frac{|b|\,|\beta|^l}{a\alpha^l} + \frac{|c|\,|\gamma|^l}{a\alpha^l},\\
&< \frac{1}{2a\alpha^l\delta^n} + \frac{2\,|b|}{a\alpha^{3l/2}},\\
&< \frac{\alpha^3}{2a\alpha^{2l}} + \frac{2\,|b|}{a\alpha^{3l/2}},\\
&< \frac{2}{\alpha^{3l/2}}.
\end{aligned}$$

Above, we used that $|b|/a < 1/2$, $|\beta| = \alpha^{-1/2}$ (see (2.3)) and that $\alpha^{l/2} > \alpha^3/(2a)$, which holds for $l > 200$. □

Now, we put

$$\Gamma := \delta^n(2a)^{-1}\alpha^{-l} - 1. \tag{3.6}$$

We will apply Lemma 2.1 to (3.6) and use Lemma 3.1 to prove the following proposition.

Proposition 3.2 *If $X_n = P_l$ and $l > 200$, then*

$$n < 3.74 \times 10^{14}(1 + \log l) \quad \textit{and} \quad l < 1.4 \times 10^{15}\log\delta(1 + \log l).$$

Proof To apply Lemma 2.1 to (3.6), we need to check that $\Gamma \neq 0$. If we assume that $\Gamma = 0$, then $\delta^n = (2a)\alpha^l$. However, the left-hand side belongs to $\mathbb{Q}(\sqrt{d})$ which is a quadratic field, while the right-hand side belongs to $\mathbb{Q}(\alpha)$ which is a field of degree 3. The intersection of these two fields is $\mathbb{Q}$. Thus, $\delta^n \in \mathbb{Q}$. Since δ is an algebraic integer and $n \geq 1$, it follows that $\delta^n \in \mathbb{Z}$. Since δ is a unit, we get that $\delta^n = 1$, so $n = 0$. We deduce a contradiction. Therefore, $\Gamma \neq 0$. To apply Lemma 2.1, we take

$$s = 3,\ \gamma_1 = \delta,\ \gamma_2 = 2a,\ \gamma_3 = \alpha,\ b_1 = n,\ b_2 = -1,\ b_3 = -l.$$

Clearly, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{d}, \alpha)$, so we can take $D = 6$. Since $\delta \geq 1 + \sqrt{2} > \alpha$, the first inequality of (3.4) implies that $n \leq l + 3$. Thus, we can take $B = l + 3$. We have

$$h(\gamma_1) = \frac{\log \delta}{2} \quad \text{and} \quad h(\gamma_3) = \frac{\log \alpha}{3} < 0.1.$$

Further, the minimal polynomial of $2a$ is $23x^3 - 46x^2 + 24x - 8$ and has roots $2a, 2b, 2c$. Since $|2b| = |2c| < 1$, then

$$h(\gamma_2) = \frac{1}{3}(\log 23 + \log 2a) < 1.2.$$

Thus, we can take

$$A_1 = 3 \log \delta, \quad A_2 = 7.2, \quad A_3 = 0.6.$$

Lemma 2.1 implies that

$$\begin{aligned} \log|\Gamma| &> -1.4 \times 30^6 \times 3^{4.5} \times 6^2(1 + \log 6)(1 + \log(l + 3))(3 \log \delta)(7.2)(0.6), \\ &> -1.87 \times 10^{14} \log \delta(1 + \log(l + 3)). \end{aligned}$$

Comparing the above inequality with (3.5), we have

$$1.5l \log \alpha - \log 2 < 1.87 \times 10^{14} \log \delta(1 + \log(l + 3)).$$

Thus,

$$l \log \alpha < 3.74 \times 10^{14} \log \delta(1 + \log l).$$

Since $\alpha^{l+3} > \delta^n$ (see the second equation in (3.3)), we get that

$$n < 3.74 \times 10^{14}(1 + \log l). \tag{3.7}$$

Furthermore, since $\alpha > 1.32$, we get

$$l < 1.4 \times 10^{15} \log \delta(1 + \log l). \tag{3.8}$$

□

3.3. Step 3:

In this step, we will use Lemma 2.2, i.e. a linear form in two logarithms to get an upper bound for n_1, n_2, l_1, and l_2. To do this, we define the following linear form in two logarithms:

$$\Lambda := (n_2 - n_1) \log 2a + (n_2 l_1 - n_1 l_2) \log \alpha. \tag{3.9}$$

Lemma 3.3 *If* $l_2 > l_1 > 200$, *then* $|\Lambda| < \frac{8n_2}{\alpha^{3l_1/2}}$.

Proof Put

$$\Lambda' := n\log\delta - \log 2a - l\log\alpha.$$

Since

$$\left|e^{\Lambda'} - 1\right| = |\Gamma| < \frac{1}{2},$$

it follows that

$$|\Lambda| < 2\left|e^{\Lambda} - 1\right| < \frac{4}{\alpha^{3l/2}}.$$

So, for $l_2 > l_1 > 200$,

$$|n_i\log\delta - \log 2a - l_i\log\alpha| < \frac{4}{\alpha^{3l_i/2}} \quad \text{holds for} \quad i = 1, 2. \tag{3.10}$$

Multiply one of the two inequalities above for $i = 1$ with n_2 and the one for $i = 2$ with n_1, subtract them and apply the triangle inequality to get that

$$\begin{aligned} |\Lambda| &= |n_2(n_1\log\delta - \log 2a - l_1\log\alpha) - n_1(n_2\log\delta - \log 2a - l_2\log\alpha)|, \\ &\le n_2|n_1\log\delta - \log 2a - l_1\log\alpha| + n_1|n_1\log\delta - \log 2a - l_1\log\alpha|, \\ &\le \frac{4n_2}{\alpha^{3l_1/2}} + \frac{4n_1}{\alpha^{3l_2/2}} < \frac{8n_2}{\alpha^{3l_1/2}}. \end{aligned}$$

□

Now, we will prove the following proposition.

Proposition 3.4 *If* $X_{n_i} = P_{l_i}$ *for* $i = 1, 2$ *with* $l_1 < l_2$ *(so* $n_1 < n_2$*), then*

$$l_1 < 6155655, \quad n_2 < 2.1 \times 10^{16}, \quad l_2 < 2.71 \times 10^{23}.$$

Proof We apply Lemma 2.2 to Λ with

$$\gamma_1 = 2a, \quad \gamma_2 = \alpha, \quad b_1 = n_2 - n_1, \quad b_2 = n_2 l_1 - l_2 n_1.$$

Since the norm of γ_1 is $8/23$ while γ_2 is a unit, γ_1 and γ_2 are multiplicatively independent. We have $\gamma_1, \gamma_2 \in \mathbb{Q}(\alpha)$, then $D = 3$. So, we have

$$\max\left\{h(\gamma_1), \frac{|\log\gamma_1|}{3}, \frac{1}{3}\right\} < 3.6$$

and

$$\max\left\{h(\gamma_2), \frac{|\log\gamma_2|}{3}, \frac{1}{3}\right\} = \frac{1}{3}.$$

Therefore, we take

$$h_1 := 3.6 \quad \text{and} \quad h_2 = \frac{1}{3}.$$

On the other hand, Lemma 3.3 implies that

$$|n_2 l_1 - n_1 l_2| \le (n_2 - n_1)\frac{|\log 2a|}{\log \alpha} + \frac{8n_2}{\alpha^{3l_1/2}\log \alpha} < 1.4n_2.$$

Hence, we get

$$b' = \frac{n_2 - n_1}{3 \times (1/3)} + \frac{|n_2 l_1 - n_1 l_2|}{3 \times 3.6} < 2n_2.$$

Thus, using Lemma 2.2 we obtain

$$\log|\Lambda| > -17.9 \times 3^4 \left(\max\{\log 2n_2 + 0.38, 10\}\right)^2 \times (1/3) \times (3.6),$$

i.e.

$$\log|\Lambda| > -1739.88 \left(\max\{\log 2n_2 + 0.38, 10\}\right)^2.$$

Combining this with Lemma 3.3, we get

$$1.5 l_1 \log \alpha - \log(8n_2) < 1739.88 \left(\max\{\log 2n_2 + 0.38, 10\}\right)^2.$$

If $\log(2n_2) + 0.38 \le 10$, then $n_2 < 7532$. The above inequality gives

$$1.5 l_1 \log \alpha < 1739.88 \times 10^2 + \log(8 \times 7532),$$

which implies that $l_1 < 116000$. Hence, $n_1 < n_2 < 7532$ in this case.

Next, assume that $n_2 > 7532$. Then

$$1.5 l_1 \log \alpha < 1739.88(\log(2n_2) + 0.38)^2 + \log(8n_2) < 1746(1 + \log n_2)^2,$$

which gives

$$l_1 < 4135(1 + \log n_2)^2. \tag{3.11}$$

Since $\alpha^{l_1+3} > \delta^{n_1} \ge \delta$ (see the second relation in (3.3)), we get

$$\log \delta < (l_1 + 3)\log \alpha < 1163(1 + \log n_2)^2.$$

Combining this with the second inequality of Proposition 3.2 with $(n, l) = (n_2, l_2)$, together with the fact that $n_2 < l_2 + 3$, we get

$$l_2 + 3 < 1.4 \times 10^{15} \times 1164 \times (1 + \log(l_2 + 3))^3,$$

giving $l_2 < 2.71 \times 10^{23}$. Inserting this into the first inequality of Proposition 3.7, we get $n_2 < 2.1 \times 10^{16}$, which together with (3.11) gives $l_1 < 6155655$. □

3.4. Step 4:

This step will conclude the proof with the final computations. Therefore, to lower the above bounds obtained, we will use continued fractions on (3.9) and Baker–Davenport reduction on (3.10).

Put $\chi := -\log 2a/\log\alpha$. Lemma 3.3 implies that

$$|(n_2 - n_1)\chi - (n_2 l_1 - n_1 l_2)| < \frac{8n_2}{\alpha^{3l_1/2}\log\alpha}. \tag{3.12}$$

Using the fact that $\log\alpha < 0.28$ and $l_1 > 200$ and Proposition 3.4, we obtain

$$\frac{16}{\log\alpha}(n_2 - n_1) < 57(n_2 - n_1) < 57n_2^2 < 3\times 10^{34} < 4\times 10^{36} < \alpha^{3l_1/2}. \tag{3.13}$$

Thus

$$\frac{8n_2}{\alpha^{3l_1/2}\log\alpha} < \frac{1}{2(n_2 - n_1)}, \tag{3.14}$$

it follows that $(n_2 l_1 - n_1 l_2)/(n_2 - n_1)$ is convergent of χ.

Obviously, $n_2 - n_1 < n_2 < 2.1\times 10^{16}$. Let $[a_0, a_1, a_2, \ldots] = [-2, 1, 2, 3, 1, 11, \ldots]$ be the continued fraction expansion of χ, and let p_k/q_k be it's k^{th} convergent. After a computer calculation, we get

$$q_{35} = 7378985365660874 < 2.1\times 10^{16} < 29361432635377315 = q_{36},$$

furthermore the maximum of a_i $(i = 0, 1, \ldots, 36)$ is $46 = a_{34}$. Hence,

$$\frac{1}{48n_2} < \frac{1}{24(n_2 - n_1)} < |(n_2 - n_1)\chi - (n_2 l_1 - n_1 l_2)| < \frac{8n_2}{\alpha^{3l_1/2}\log\alpha}.$$

Using Proposition 3.4 and comparing the leftmost and rightmost expressions, we get $l_1 \leq 195.4$. Since we assume that $l_1 > 200$, we conclude that $l_1 \leq 200$. Now (3.7) gives $n_1 \leq 64.8$.

The upper bounds on n_1 and l_1 make it possible to compute all existing n_1 and l_1. Defining

$$Q_n^+(X) := \frac{(X + \sqrt{X^2 - 1})^n + (X - \sqrt{X^2 - 1})^n}{2}$$

and

$$Q_n^-(X) := \frac{(X + \sqrt{X^2 + 1})^n + (X - \sqrt{X^2 + 1})^n}{2},$$

and using compute search on the equations

$$Q_{n_1}^+(X_1) = P_{l_1} \quad \text{and} \quad Q_{n_1}^-(X_1) = P_{l_1},$$

with $1 \leq l_1 \leq 200$ and $n_1 \leq 64$, where $n_1 < l_1 + 3$ results in only the following possibilities:

Besides the trivial case $n_1 = 1$ (for both equations), which implies $X_1 = P_{l_1}$, the only nontrivial solutions are

$$(n_1, l_1, X_1) = (2, 9, 2) \quad \text{and} \quad (n_1, l_1, X_1) = (2, 16, 5),$$

in the first case which leads to $(d, Y_1) = (3, 1)$ and $(d, Y_1) = (6, 2)$, respectively, and

$$(n_1, l_1, X_1) = (2, 6, 1) \quad \text{and} \quad (n_1, l_1, X_1) = (2, 10, 2),$$

in the second case which leads to $(d, Y_1) = (2, 1)$ and $(d, Y_1) = (5, 1)$, respectively. To determine all the solutions of equation (1.3), we apply (3.10) and Lemma 2.3. First, observe that

$$\left| n_2\frac{\log\delta}{\log\alpha} - l_2 + \chi \right| < \frac{4}{\alpha^{3l_2/2}\log\alpha} < 14.3\cdot 1.6^{-l_2}.$$

Put

$$\delta_1 = 2+\sqrt{3}, \quad \delta_2 = 5+2\sqrt{6}, \quad \delta_3 = 1+\sqrt{2}, \quad \delta_4 = 2+\sqrt{5}.$$

Taking the continued fraction expansion of $\log \delta_i / \log \alpha$ for $i = 1,2,3,4$, such that the suitable denominator of it exceeds 1.26×10^{17}, we found that

$$q_{1,42} = 657142969198152933 \approx 6.57 \times 10^{17},$$

and

$$q_{2,36} = 7249506692243760373 \approx 7.24 \times 10^{18},$$

and

$$q_{3,42} = 116521408058350539327645 \approx 1.16 \times 10^{23},$$

and

$$q_{4,36} = 194847711151769850 \approx 1.94 \times 10^{17},$$

is satisfactory for $i = 1$, $i = 2$, $i = 3$, and $i = 4$, respectively. We now apply Lemma 2.3, with $m = n_2$, $n = l_2$, $k = m_2$, $A = 14.3$, $B = 1.6$, $M = 2.1 \times 10^{16}$, $\tau = \log \delta_i / \log \alpha$, and $\mu = \chi$. Further, according to the four cases $q = q_{1,42}$, $q = q_{2,36}$, $q = q_{3,42}$ and $q = q_{4,36}$, we get $\xi_1 > 0.56$, $\xi_2 > 0.21$, $\xi_3 > 0.43$, and $\xi_4 > 0.69$. Consequently,

- In the first case: $l_2 < 104.92$ and $n_2 < 36.47$,
- In the second case: $l_2 < 112.96$ and $n_2 < 39.04$,
- In the third case: $l_2 < 134.21$ and $n_2 < 45.82$,
- In the fourth case: $l_2 < 101.56$ and $n_2 < 35.41$.

However, since we assume that $l_2 > 200$, we get a contradiction, so $l_2 \leq 200$ leading to $n_2 < 64,8$. Checking the last range we only obtained the following possibilities:

$$X_1 = 2 = P_4 = P_5, \quad X_2 = 7 = P_9, \quad \text{with} \quad d = 3,$$

and

$$X_1 = 5 = P_8, \quad X_2 = 49 = P_{16}, \quad \text{with} \quad d = 6,$$

and

$$X_1 = 1 = P_1 = P_2 = P_3, \quad X_2 = 3 = P_6 \quad X_3 = 7 = P_9, \quad \text{with} \quad d = 2,$$

and

$$X_1 = 2 = P_4 = P_5, \quad X_2 = 9 = P_{10}, \quad \text{with} \quad d = 5,$$

respectively.

Finally, in order to check the trivial cases $n_1 = 1$, $X_1 = P_{l_1}$, we used a brute force algorithm which essentially coincides with the treatment of the nontrivial cases. For any $1 \leq l_1 \leq 200$, we determine the decomposition $P_{l_1} - \varepsilon = dY_1^2$, where d is square-free. In this way we find $\delta_{l_1} = X_1 + \sqrt{d}Y_1$. Then we consider the first convergents of the continued fraction expansions of

$$\frac{\log \delta_{l_1}}{\log \alpha} \tag{3.15}$$

such that the denominator is larger than $M = 1.26 \times 10^{17}$, and the ξ value in Lemma 2.3 is positive. The upper bounds on l_2 are always less than 200, which contradicts the assertion $l_2 > 200$. Thus only cases $l_2 \leq 200$ remain to be verified. As conclusion, the trivial cases do not yield further solutions to (1.3). Theorem 1.1 is therefore proved.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 will be similar to that of Theorem 1.1 in four steps.

4.1. Step A

In this step, we will start by determining a relationship between the parameters m and l. Let (X_1', Y_1') be the fundamental solution of the Pell equation (1.2). We set

$$\sigma := \frac{X_1' + \sqrt{d}Y_1'}{2} \quad \text{and} \quad \varrho := \frac{X_1' - \sqrt{d}Y_1'}{2}.$$

One can see that $\sigma\varrho = \varepsilon$, so $\varrho = \varepsilon\sigma^{-1}$, where $\varepsilon \in \{\pm 1\}$. With

$$\sigma^m = \frac{X_m + Y_m\sqrt{d}}{2} \quad \text{and} \quad \varrho^m = \frac{X_m - Y_m\sqrt{d}}{2},$$

we get

$$X_m = \sigma^m + \varrho^m. \tag{4.1}$$

Since $\sigma \geq \dfrac{1+\sqrt{2}}{2}$, we obtain the following estimate

$$\frac{\sigma^m}{\alpha^2} \leq X_m \leq 2\sigma^m, \quad \text{for all} \quad m \geq 1. \tag{4.2}$$

We now assume (m_1, l_1) and (m_2, l_2) are pairs of positive integers such that

$$X_{m_1} = P_{l_1} \quad \text{and} \quad X_{m_2} = P_{l_2}.$$

Without loss of the generality, we can assume that $m_1 < m_2$ so $l_1 < l_2$. Put $(m, l) = (m_i, l_i)$, for $i \in \{1, 2\}$. The inequalities (2.4) and (4.2) lead to

$$\alpha^{l-2} \leq P_l = X_m < 2\sigma^m \quad \text{and} \quad \frac{\sigma^m}{\alpha^2} \leq X_m = P_l \leq \alpha^{l-1}. \tag{4.3}$$

Hence, we obtain

$$mc_2 \log\sigma - 1 \leq l \leq mc_2 \log\sigma + 3, \quad c_2 := 1/\log\alpha. \tag{4.4}$$

4.2. Step B

In this step, we apply Matveev's theorem to a form linear in three logarithms to get a bound to m and l in terms of $\log l$. First, we prove the following lemma.

Lemma 4.1 *If $l > 200$, then*

$$\left|\sigma^m a^{-1}\alpha^{-l} - 1\right| < \frac{2}{\alpha^{3l/2}}. \tag{4.5}$$

Proof The equalities (2.1) and (4.1) imply that

$$\sigma^m - a\alpha^l = -\varrho^m + b\beta^l + c\gamma^l.$$

Dividing both sides by $a\alpha^l$, we get

$$\sigma^m a^{-1}\alpha^{-l} - 1 = -a^{-1}\alpha^{-l}\varrho^m + (b/a)(\beta\alpha^{-1})^l + (c/a)(\gamma\alpha^{-1})^l.$$

Using (2.3) and assuming $l > 200$, we get

$$\begin{aligned} \left|\sigma^m a^{-1}\alpha^{-l} - 1\right| &\le \frac{1}{a\alpha^l\sigma^m} + \frac{|b|\,|\beta|^l}{a\alpha^l} + \frac{|c|\,|\gamma|^l}{a\alpha^l}, \\ &< \frac{1}{a\alpha^l\sigma^m} + \frac{2\,|b|}{a\alpha^{3l/2}}, \\ &< \frac{\alpha^3}{a\alpha^{2l}} + \frac{2\,|b|}{a\alpha^{3l/2}} < \frac{2}{\alpha^{3l/2}}. \end{aligned}$$

Above, we used that $|b|/a < 1/2$, $|\beta| = \alpha^{-1/2}$ (see (2.3)) and that $\alpha^{l/2} > \alpha^3/a$ which holds for $l > 200$. □

Now, we put

$$\Gamma_1 := \sigma^m a^{-1}\alpha^{-l} - 1. \tag{4.6}$$

We will apply Lemma 2.1 to Γ_1 given by (4.6) and use Lemma 4.1 to prove the following proposition.

Proposition 4.2 *If $X_m = P_l$ and $l > 200$, then*

$$m < 1.04 \times 10^{14}(1 + \log 2l) \quad \textit{and} \quad l < 3.7 \times 10^{14}\log\sigma(1 + \log 2l).$$

Proof Using a method similar to that of Γ, one can prove that $\Gamma_1 \neq 0$. To apply Lemma 2.1 to Γ_1, we take

$$s = 3,\ \gamma_1 = \sigma,\ \gamma_2 = a,\ \gamma_3 = \alpha,\ b_1 = m,\ b_2 = -1,\ b_3 = -l.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{d}, \alpha)$, we can take $D = 6$. Since $\sigma \ge \frac{1+\sqrt{2}}{2} > \sqrt{\alpha}$, the second inequality in (4.3) implies that $m \le 2l$, then we can take $B = 2l$. We have

$$h(\gamma_1) = \frac{\log\sigma}{2} \quad \text{and} \quad h(\gamma_3) = \frac{\log\alpha}{3}.$$

Furthermore, the minimal polynomial of a is $23x^3 - 23x^2 + 6x - 1$ and has roots a, b, c. Since $a < 1$ and $|b| = |c| < 1$, then

$$h(\gamma_2) = \frac{\log 23}{3}.$$

So, we can take

$$A_1 = 3\log\sigma, \quad A_2 = 2\log 23, \quad A_3 = 2\log\alpha.$$

Thus, Lemma 2.1 implies that

$$\begin{aligned} \log|\Gamma_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 6^2(1 + \log 6)(1 + \log 2l)(3\log\sigma)(2\log 23)(2\log\alpha), \\ &> -1.53 \times 10^{14}\log\sigma(1 + \log 2l). \end{aligned}$$

Comparing the above inequality with (4.5), we get

$$1.5l\log\alpha - \log 2 < 1.53\times 10^{14}\log\sigma(1+\log 2l).$$

Thus, we get

$$l\log\alpha < 1.03\times 10^{14}\log\sigma(1+\log 2l).$$

Since $\alpha^{l+1} > \sigma^m$ (see the second equation in (4.3)), we get that

$$m < 1.04\times 10^{14}(1+\log 2l). \tag{4.7}$$

Moreover, as $\alpha > 1.32$, we obtain

$$l < 3.71\times 10^{14}\log\sigma(1+\log 2l). \tag{4.8}$$

□

4.3. Step C

In this step, we will get an upper bound for m_1, m_2, l_1, and l_2 by applying Lemma 2.2 to a linear form in two logarithms. To do this, we define the following linear form in two logarithms:

$$\Lambda_1 := (m_2 - m_1)\log a + (m_2 l_1 - m_1 l_2)\log\alpha. \tag{4.9}$$

Lemma 4.3 *If $l_2 > l_1 > 200$, then $|\Lambda_1| < \dfrac{8m_2}{\alpha^{3l_1/2}}$.*

Proof Put

$$\Lambda_1' := m\log\sigma - \log a - l\log\alpha.$$

The fact that

$$\left|e^{\Lambda_1'} - 1\right| = |\Gamma_1| < \frac{1}{2}$$

implies that

$$|\Lambda_1| < 2\left|e^{\Lambda_1} - 1\right| < \frac{4}{\alpha^{3l/2}}.$$

So for $l_2 > l_1 > 200$,

$$|m_i\log\delta - \log a - l_i\log\alpha| < \frac{4}{\alpha^{3l_i/2}}, \quad \text{for} \quad i = 1, 2. \tag{4.10}$$

We multiply one of the two inequalities above for $i = 1$ with m_2 and the one for $i = 2$ with m_1, subtract them and apply the triangle inequality to get

$$\begin{aligned} |\Lambda_1| &= |m_2(m_1\log\sigma - \log a - l_1\log\alpha) - m_1(m_2\log\sigma - \log a - l_2\log\alpha)|, \\ &\leq m_2|m_1\log\sigma - \log a - l_1\log\alpha| + m_1|m_1\log\sigma - \log a - l_1\log\alpha|, \\ &\leq \frac{4m_2}{\alpha^{3l_1/2}} + \frac{4m_1}{\alpha^{3l_2/2}} < \frac{8m_2}{\alpha^{3l_1/2}}. \end{aligned}$$

□

Now, we will prove the following proposition.

Proposition 4.4 *If $X_{m_i} = P_{l_i}$ for $i = 1, 2$ with $l_1 < l_2$ (so $m_1 < m_2$), then*

$$l_1 < 5344168, \quad m_2 < 2 \times 10^{16}, \quad l_2 < 5.95 \times 10^{22}.$$

Proof We apply Lemma 2.2 with

$$\gamma_1 = a, \quad \gamma_2 = \alpha, \quad b_1 = m_2 - m_1, \quad b_2 = m_2 l_1 - l_2 m_1.$$

As the norm of γ_1 is $1/23$ while γ_2 is a unit, γ_1 and γ_2 are multiplicatively independent. We have $\gamma_1, \gamma_2 \in \mathbb{Q}(\alpha)$ which has $D = 3$. Similarly as above, we have

$$\max\left\{h(\gamma_1), \frac{|\log \gamma_1|}{3}, \frac{1}{3}\right\} = \frac{\log 23}{3} := h_1, \quad \max\left\{h(\gamma_2), \frac{|\log \gamma_2|}{3}, \frac{1}{3}\right\} = \frac{1}{3} := h_2,$$

$$|m_2 l_1 - m_1 l_2| \leq (m_2 - m_1)\frac{|\log a|}{\log \alpha} + \frac{8m_2}{\alpha^{3l_1/2}\log \alpha} < 2m_2,$$

and

$$b' = \frac{m_2 - m_1}{3 \times (1/3)} + \frac{|m_2 l_1 - m_1 l_2|}{3 \times (\log 23/3)} < 2m_2.$$

Thus, Lemma 2.2 leads to

$$\log |\Lambda_1| > -17.9 \times 3^4 \left(\max\left\{\log 2m_2 + 0.38, 10\right\}\right)^2 \times (1/3) \times (\log 23/3).$$

So, we obtain

$$\log |\Lambda_1| > -1516 \left(\max\left\{\log 2m_2 + 0.38, 10\right\}\right)^2.$$

Combining this with Lemma 4.3, we get

$$1.5 l_1 \log \alpha - \log(8m_2) < 1516 \left(\max\left\{\log 2m_2 + 0.38, 10\right\}\right)^2.$$

If $\log(2m_2) + 0.38 \leq 10$, then $m_2 < 7532$. The above inequality gives

$$1.5 l_1 \log \alpha < 1516 \times 10^2 + \log(8 \times 7532),$$

which implies that $l_1 < 359439$. Hence, $m_1 < m_2 < 7532$ in this case. Otherwise, we have

$$1.5 l_1 \log \alpha < 1516(\log(2m_2) + 0.38)^2 + \log(8m_2) < 1518(1 + \log m_2)^2,$$

which gives

$$l_1 < 3599(1 + \log m_2)^2. \tag{4.11}$$

The second relation in (4.3) implies that $\alpha^{l_1+1} > \sigma^{m_1} \geq \sigma$, so we obtain

$$\log \sigma < (l_1 + 1)\log \alpha < 1013(1 + \log m_2)^2.$$

Combining this with the second inequality of Proposition 4.2 with $(m, l) = (m_2, l_2)$, together with the fact that $m_2 < 2l_2$, we get

$$l_2 < 3.7 \times 10^{14} \times 1013 \times (1 + \log(2l_2 + 1))^3,$$

giving $l_2 < 5.95 \times 10^{22}$. Inserting this into the first inequality of Proposition 4.2, we get $m_2 < 2 \times 10^{16}$, which together with (4.11) gives $l_1 < 5344186$. □

4.4. Step D

For the final step of the proof, we need to lower the bounds obtained; we use continued fractions on (4.9) and Baker–Davenport reduction on (4.10).

Put $\chi' := -\log a/\log\alpha$. Lemma 4.3 implies

$$|(m_2-m_1)\chi' - (m_2l_1 - m_1l_2)| < \frac{8m_2}{\alpha^{3l_1/2}\log\alpha}. \tag{4.12}$$

Using the fact that $\log\alpha < 0.28$ and $l_1 > 200$ and Proposition 4.4, we obtain

$$\frac{16}{\log\alpha}(m_2-m_1) < 57(m_2-m_1) < 57m_2^2 < 3\times 10^{34} < 4\times 10^{36} < \alpha^{3l_1/2}. \tag{4.13}$$

Thus

$$\frac{8m_2}{\alpha^{3l_1/2}\log\alpha} < \frac{1}{2(m_2-m_1)}, \tag{4.14}$$

it follows that $(m_2l_1 - m_1l_2)/(m_2-m_1)$ is convergent of χ'.

Obviously, $m_2 - m_1 < m_2 < 2\times 10^{16}$. Let $[a_0, a_1, a_2, \ldots] = [1, 6, 2, 1, 18, 166, \ldots]$ be the continued fraction expansion of χ', and let p_k/q_k be it's k^{th} convergent. After a computer calculation, we found that

$$q_{29} = 11858488010673001 < 2\times 10^{16} < 34403713039887677 = q_{30},$$

furthermore the maximum of a_i $(i = 0, 1, \ldots, 30)$ is $166 = a_6$. Hence, we obtain

$$\frac{1}{168m_2} < \frac{1}{168(m_2-m_1)} < |(m_2-m_1)\chi' - (m_2l_1 - m_1l_2)| < \frac{8m_2}{\alpha^{3l_1/2}\log\alpha}.$$

Using Proposition 4.4 and comparing the leftmost and rightmost expressions, we get $l_1 \leq 195.4$. Since we assume that $l_1 > 198.1$, we conclude that $l_1 \leq 200$. Now (4.7) gives $m_1 \leq 64.2$.

The upper bounds on m_1 and l_1 make it possible to compute all existing m_1 and l_1. Defining

$$Q_m'^{+}(X) := \left(\frac{X+\sqrt{X^2-4}}{2}\right)^m + \left(\frac{X-\sqrt{X^2-4}}{2}\right)^m$$

and

$$Q_m'^{-}(X) := \left(\frac{X+\sqrt{X^2+4}}{2}\right)^m + \left(\frac{X-\sqrt{X^2+4}}{2}\right)^m,$$

and using compute search on the equations

$$Q'^{+}_{m_1}(X_1') = P_{l_1} \quad \text{and} \quad Q'^{-}_{m_1}(X_1) = P_{l_1},$$

with $1 \leq l_1 \leq 200$ and $1 \leq m_1 \leq 64$, where $m_1 < 2l_1$ results in only following possibilities:

Besides the trivial case $l_1 = 1$ (for both equations), which implies $X_1' = P_{l_1}$, the only nontrivial solutions are

$$(m_1, l_1, X_1') = (2, 9, 3),$$

in the first case and

$$(m_1, l_1, X_1') = (2, 6, 1), \quad (m_1, l_1, X_1') = (3, 7, 1), \quad \text{and} \quad (m_1, l_1, X_1') = (4, 9, 1),$$

in the second case which leads to $(d, Y_1') = (5, 1)$ in all cases. To determine all the solutions of equation (1.4), we apply (4.10) and Lemma 2.3. First, observe that

$$\left| m_2 \frac{\log \sigma}{\log \alpha} - l_2 + \chi' \right| < \frac{4}{\alpha^{3l_2/2} \log \alpha} < 14.3 \cdot 1.6^{-l_2}.$$

Put

$$\sigma_1 = \frac{3 + \sqrt{5}}{2}, \quad \sigma_2 = \frac{1 + \sqrt{5}}{2}.$$

Taking the continued fraction expansion of $\log \sigma_i / \log \alpha$ for $i = 1, 2$, such that the suitable denominator of it exceeds 1.2×10^{17}, we found that

$$q_{1,46} = 843503315596223623 \approx 8.43 \times 10^{17},$$

and

$$q_{2,40} = 85570068922793841671797 \approx 8.55 \times 10^{22},$$

is satisfactory for $i = 1$ and $i = 2$, respectively. We now apply Lemma 2.3, with $m = m_2$, $n = l_2$, $k = l_2$, $A = 14.3$, $B = 1.6$, $M = 2.1 \times 10^{16}$, $\tau = \log \sigma_i / \log \alpha$ and $\mu = \chi'$. Furthermore, according to the four cases $q = q_{1,46}$ and $q = q_{2,40}$, we get $\xi_1 > 0.67$ and $\xi_2 > 0.15$. Consequently,

- In the first case: $l_2 < 105.09$ and $m_2 < 34.52$,
- In the second case: $l_2 < 135.93$ and $m_2 < 44.37$.

However, since we assume that $l_2 > 200$, we get a contradiction, so $l_2 \leq 200$ leading to $m_2 < 64, 2$. Checking the last range we only obtained the possibilities:

$$X_2' = 28 = P_{14},$$

and

$$X_1' = 2 = P_4 = P_5 \quad \text{and} \quad X_2' = 12 = P_{11},$$

respectively.

Finally, in order to check the trivial cases $m_1 = 1$, $X_1' = P_{l_1}$, we used a brute force algorithm which essentially coincides with the treatment of the non-trivial cases. For any $1 \leq l_1 \leq 200$, we determine the decomposition $P_{l_1} - 4\varepsilon = d{Y'}_1^2$, where d is square-free. In this way, we find $\sigma_{l_1} = \dfrac{X_1' + \sqrt{d} Y_1'}{2}$. Then we consider the first convergents of the continued fraction expansions of

$$\frac{\log \sigma_{l_1}}{\log \alpha} \tag{4.15}$$

such that the denominator is larger than $M = 1.2 \times 10^{17}$, and the ξ value in Lemma 2.3 is positive. The upper bounds on l_2 are always less than 200, which contradicts the assertion $l_2 > 200$. Thus only cases $l_2 \leq 200$ remain to verify. As conclusion, the trivial cases do not yield further solutions to (1.4). Theorem 1.2 is therefore proved.

Acknowledgments

We thank the referees for the useful comments to improve the quality of this paper. The third author was supported in part by Purdue University Northwest.

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