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# $C$-Paracompactness and $C_2$-paracompactness

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**Abstract:** A topological space $X$ is called *$C$-paracompact* if there exist a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A topological space $X$ is called *$C_2$-paracompact* if there exist a Hausdorff paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. We investigate these two properties and produce some examples to illustrate the relationship between them and $C$-normality, minimal Hausdorff, and other properties.

**Key words:** Normal, paracompact, $C$-paracompact, $C_2$-paracompact, $C$-normal, epinormal, mildly normal, minimal Hausdorff, Fréchet, Urysohn

## 1. Introduction

We introduce two new topological properties, $C$-paracompactness and $C_2$-paracompactness. They were defined by Arhangel'skiĭ. The purpose of this paper is to investigate these two properties. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, the rational numbers by $\mathbb{Q}$, the irrational numbers by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. $T_2$ denotes the Hausdorff property. A $T_4$ space is a $T_1$ normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a $T_1$ completely regular space. We do not assume $T_2$ in the definition of compactness, countable compactness, local compactness, and paracompactness. We do not assume regularity in the definition of Lindelöfness. For a subset $A$ of a space $X$, int$A$ and $\overline{A}$ denote the interior and the closure of $A$, respectively. An ordinal $\gamma$ is the set of all ordinals $\alpha$ such that $\alpha < \gamma$. The first infinite ordinal is $\omega_0$ and the first uncountable ordinal is $\omega_1$.

## 2. $C$-paracompactness and $C_2$-paracompactness

In 2016 and in a personal communication with Kalantan, the second author, Arhangel'skiĭ introduced the following definition.

**Definition 2.1** *A topological space $X$ is called $C$-paracompact* if there exist a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A topological space $X$ is called *$C_2$-paracompact* if there exist a Hausdorff paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

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Observe that a function $f : X \longrightarrow Y$ witnessing the $C$-paracompactness ($C_2$-paracompactness) of $X$ need not be continuous: for example, the identity function from any countable complement topology on an uncountable set onto its discrete; see Theorem 2.7 below. However, it will be under some conditions. Recall that a space $X$ is *Fréchet* if for any subset $A$ of $X$ and any $x \in \overline{A}$, there exists a sequence $(x_n)$ of elements of $A$ that converges to $x$ [7].

**Theorem 2.2** *If $X$ is a $C$-paracompact ($C_2$-paracompact) Fréchet space and $f : X \longrightarrow Y$ is a witness of the $C$-paracompactness ($C_2$-paracompactness) of $X$, then $f$ is continuous.*

**Proof** Let $A$ be any nonempty subset of $X$. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that $f(x) = y$. Then $x \in \overline{A}$. Pick a sequence $(x_n) \subseteq A$ such that $x_n \longrightarrow x$. Let $B = \{x, x_n : n \in \mathbb{N}\}$; then $B$ is a compact subspace of $X$, being a convergent sequence with its limit, and hence $f|_B : B \longrightarrow f(B)$ is a homeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of $y$; then $V \cap f(B)$ is open in the subspace $f(B)$ containing $y$. Thus, $f^{-1}(V) \cap B$ is open in the subspace $B$ containing $x$. Thus, $(f^{-1}(V) \cap B) \cap \{x_n : n \in \mathbb{N}\} \neq \emptyset$, so $(f^{-1}(V) \cap B) \cap A \neq \emptyset$. Hence, $\emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A)$. Thus, $y \in \overline{f(A)}$. Therefore, $f$ is continuous. □

Since any first countable space is Fréchet, we conclude the following.

**Corollary 2.3** *If $X$ is a $C$-paracompact ($C_2$-paracompact) first countable space and $f : X \longrightarrow Y$ is a witness of the $C$-paracompactness ($C_2$-paracompactness) of $X$, then $f$ is continuous.*

**Corollary 2.4** *Any $C_2$-paracompact Fréchet space is Hausdorff.*

**Corollary 2.5** *Let $X$ be a $C_2$-paracompact Fréchet space. Then for each disjoint compact subspace $A$ and $B$, there exist two open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.*

**Proof** Let $Y$ be a $T_2$ paracompact space and $f : X \longrightarrow Y$ be a bijective function such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. By Theorem 2.2, $f$ is continuous. Let $A$ and $B$ be any disjoint compact space; then $f(A)$ and $f(B)$ are disjoint compact subspaces of $Y$. Since $Y$ is $T_2$, then $f(A)$ and $f(B)$ are disjoint closed subspaces of $Y$. Since $Y$ is $T_2$ paracompact, $Y$ is normal and thus there exist two open subsets $G$ and $H$ of $Y$ such that $f(A) \subseteq G$, $f(B) \subseteq H$, and $G \cap H = \emptyset$. By the continuity of $f$, $U = f^{-1}(G)$ and $V = f^{-1}(H)$ work. □

A $C$-paracompact Fréchet space may not be Hausdorff. Take for an example any indiscrete space containing more than one element. Another example is the space $Y$ of Example 2.25 below. Corollary 2.5 is not always true for $C$-paracompactness; see the space $X$ of Example 2.25 below. By the theorem "*A function $f$ from a $k$-space $X$ into a topological space $Y$ is continuous if and only if for every compact subspace $Z \subseteq X$ the restriction $f|_Z : Z \longrightarrow Y$ is continuous*" [7, 3.3.21], we conclude the following.

**Corollary 2.6** *If $X$ is a $C$-paracompact ($C_2$-paracompact) $k$-space and $f : X \longrightarrow Y$ is a witness of the $C$-paracompactness ($C_2$-paracompactness) of $X$, then $f$ is continuous.*

It is clear from the definitions that any $C_2$-paracompact space must be $C$-paracompact. Now, assuming that $X$ is a compact and $C_2$-paracompact space, then the witness function of $C_2$-paracompactness is a homeomorphism, which gives that $X$ is Hausdorff and $T_4$. Thus, any compact space that is not Hausdorff cannot be $C_2$-paracompact. We conclude that the following compact spaces are $C$-paracompact but not $C_2$-paracompact because they are not Hausdorff: finite complement topology on an infinite set, compact complement space [17, Example 22], modified Fort space [17, Example 27], and overlapping intervals space [17, Example 53]. In Example 2.25 below, we give a Hausdorff $C$-paracompact space that is not $C_2$-paracompact. It is clear from the definitions that any paracompact space must be $C$-paracompact. Just take $Y = X$ and use the identity function. However, in general, $C$-paracompactness does not imply paracompactness. $\omega_1$ is $C$-paracompact because it is $C_2$-paracompact, being $T_2$ locally compact (see Theorem 2.12 below), but not paracompact because it is countably compact noncompact. The following theorem can be proved in a similar way as in [3].

**Theorem 2.7** *If $X$ is a $T_1$ space such that the only compact subsets are the finite subsets, then $X$ is $C_2$-paracompact.*

We conclude that $(\mathbb{R}, \mathcal{CC})$, where $\mathcal{CC}$ is the countable complement topology [17], is $C_2$-paracompact, which is not paracompact. $(\mathbb{R}, \mathcal{CC})$ is $T_1$ but not $T_2$ and this does not contradict Corollary 2.4 because it is not Fréchet as $0 \in \overline{\mathbb{P}}$ and the only convergent sequences are the eventually constant.

Recall that a topological space $X$ is called *$C$-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$ [3]. Since any Hausdorff paracompact space is $T_4$, then it is clear that any $C_2$-paracompact space is $C$-normal. Here is an example of a $C$-normal space that is not $C_2$-paracompact.

**Example 2.8** Consider $\mathbb{R}$ with the left ray topology $\mathcal{L} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) : x \in \mathbb{R}\}$. In this space $(\mathbb{R}, \mathcal{L})$, any two nonempty closed sets must intersect; thus, $(\mathbb{R}, \mathcal{L})$ is normal and hence $C$-normal. $(\mathbb{R}, \mathcal{L})$ is not Hausdorff as any two nonempty open sets must intersect. A subset $C \subset \mathbb{R}$ is compact if and only if it has a maximum element. Suppose that $(\mathbb{R}, \mathcal{L})$ is $C_2$-paracompact. Let $Y$ be a Hausdorff paracompact space and $f : \mathbb{R} \longrightarrow Y$ be a bijection such that $f|_C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C$ of $\mathbb{R}$. Let $C = (-\infty, 0]$; then $C$ is compact in $(\mathbb{R}, \mathcal{L})$ and $C$ as a subspace is not Hausdorff because any two nonempty open sets in $C$ must intersect. However, $C$ will be homeomorphic to $f(C)$ and $f(C)$ is Hausdorff, being a subspace of a Hausdorff space, and this is a contradiction. Therefore, $(\mathbb{R}, \mathcal{L})$ cannot be $C_2$-paracompact.

There are some conditions whereby $C$-normality will imply $C_2$-paracompactness, but first we need the following lemma.

**Lemma 2.9** *If $f : X \longrightarrow Y$ is a bijection function such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$ and any finite subset of $X$ is discrete, then $Y$ is $T_1$.*

**Proof** Assume that $Y$ has more than one element and let $a$ and $b$ be any two distinct elements of $Y$. Let $c$ and $d$ be the unique elements of $X$ such that $f(c) = a$ and $f(d) = b$. Then $f|_{\{c,d\}} : \{c, d\} \longrightarrow \{a, b\}$ is a homeomorphism and $\{c, d\}$ is a discrete subspace of $X$. Thus, $f(\{c\}) = \{a\}$ and $f(\{d\}) = \{b\}$ are both open in $\{a, b\}$ as a subspace of $Y$. Thus, there exists an open neighborhood $U_a \subseteq Y$ of $a$ such that $U_a \cap \{a, b\} = \{a\}$; hence, $b \notin U_a$, and similarly there exists an open neighborhood $U_b \subseteq Y$ of $b$ such that $a \notin U_b$. Thus, $Y$ is $T_1$. □

**Theorem 2.10** *Let $X$ be a Fréchet Lindelöf space such that any finite subspace of $X$ is discrete. If $X$ is $C$-normal, then $X$ is $C_2$-paracompact.*

**Proof** Since $X$ is $C$-normal, then there exist a normal space $Y$ and a bijection function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. By Lemma 2.9, $Y$ is $T_1$ and hence $T_4$. Since $X$ is Fréchet, then $f$ is continuous [10]. Since $X$ is Lindelöf and $f$ is continuous and onto, then $Y$ is Lindelöf. Since any $T_3$ Lindelöf space is paracompact, then $Y$ is $T_2$ paracompact. Therefore, $X$ is $C_2$-paracompact. □

Any infinite particular point space [17] is not paracompact. A similar proof as in [10] shows that any infinite particular point space cannot be $C$-paracompact. Observe that any finite space that is not discrete (i.e. not $T_1$) is compact and hence paracompact, thus $C$-paracompact. Therefore, any finite space that is neither normal nor discrete will be an example of a $C$-paracompact that is neither $C_2$-paracompact nor $C$-normal. We conclude that paracompactness does not imply $C$-normality, and $C$-paracompactness does not imply $C$-normality. Here is an infinite $C$-normal space that is not $C$-paracompact.

**Example 2.11** Let $X = [0, \infty)$. Define $\mathcal{T} = \{\emptyset, X\} \cup \{[0, x) : x \in \mathbb{R}, 0 < x\}$. Note that $(X, \mathcal{T})$ is just the subspace of $(\mathbb{R}, \mathcal{L})$. That is, $\mathcal{T} = \mathcal{L}_X = \mathcal{L}_{[0,\infty)}$. Now consider $(X, \mathcal{T}_0)$, where $\mathcal{T}_0$ is the particular point topology. We have that $\mathcal{T}$ is coarser than $\mathcal{T}_0$ because any nonempty open set in $\mathcal{T}$ must contain $0$. Thus, $(X, \mathcal{T}_0)$ cannot be paracompact. Observe that $(X, \mathcal{T})$ is normal because there are no two nonempty closed disjoint subsets. Thus, $(X, \mathcal{T})$ is $C$-normal. Now, a subset $C$ of $X$ is compact if and only if $C$ has a maximal element. To see this, if $C$ has a maximal element, then any open cover for $C$ will be covered by one member of the open cover, the one that contains the maximal element. If $C$ has no maximal element, then $C$ cannot be finite. If $C$ is unbounded above, then $\{[0, n) : n \in \mathbb{N}\}$ would be an open cover for $C$ that has no finite subcover. If $C$ is bounded above, let $y = \sup C$ and pick an increasing sequence $(c_n) \subseteq C$ such that $c_n \longrightarrow y$, where the convergence is taken in the usual metric topology on $X$. Then $\{[0, c_n) : n \in \mathbb{N}\}$ would be an open cover for $C$ that has no finite subcover. Thus, $C$ would not be compact. $(X, \mathcal{T})$ is Fréchet. That is because $X$ is first countable. If $x \in X$, then $\mathcal{B}(x) = \{[0, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local base for $X$ at $x$.

Now, suppose that $X$ is $C$-paracompact. Pick a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A$ of $X$. By Corollary 2.3, $f$ is continuous. Thus, for any nonempty open subset $U$ of $Y$ we have that $f^{-1}(U)$ is open in $X$. Since $f$ is a bijective, $Y$ is infinite. For each $y \in Y$, pick an open neighborhood $U_y$ of $y$ such that the family $\{U_y : y \in Y\}$ is an infinite open cover for $Y$. Since each $U_y$ contains the element $f(0)$, then the open cover $\{U_y : y \in Y\}$ cannot have any locally finite open refinement and thus $Y$ is not paracompact, which is a contradiction. Therefore, $X$ is $C$-normal but not $C$-paracompact.

An example of a Tychonoff $C$-normal space that is not paracompact is $\omega_1 \times (\omega_1 + 1)$. It is $C$-normal because it is Hausdorff locally compact [3]. We have a great benefit from local compactness.

**Theorem 2.12** *Every Hausdorff locally compact space is $C_2$-paracompact.*

**Proof** Let $X$ be any Hausdorff locally compact topological space. By [7, 13], there exists a $T_2$ compact space $Y$ and hence $Y$ is $T_2$ paracompact, and a bijective function $f : X \longrightarrow Y$ such that $f$ is continuous. Since $f$ is continuous, then for any compact subspace $A \subseteq X$ we have that $f|_A : A \longrightarrow f(A)$ is a homeomorphism because $1-1$, onto, and continuity are inherited from $f$, and $f|_A$ is closed as $A$ is compact and $f(A)$ is Hausdorff. □

The converse of Theorem 2.12 is not true in general. Here is an example of a Tychonoff $C_2$-paracompact space that is not locally compact.

**Example 2.13** Consider the quotient space $\mathbb{R}/\mathbb{N}$. We can describe it as follows: Let $i = \sqrt{-1}$. Let $Y = (\mathbb{R} \setminus \mathbb{N}) \cup \{i\}$. Define $f : \mathbb{R} \longrightarrow Y$ as follows:

$$f(x) = \begin{cases} x & ; \text{if } x \in \mathbb{R} \setminus \mathbb{N} \\ i & ; \text{if } x \in \mathbb{N} \end{cases}$$

Now consider on $\mathbb{R}$ the usual topology $\mathcal{U}$. Define on $Y$ the topology $\mathcal{T} = \{W \subseteq Y : f^{-1}(W) \in \mathcal{U}\}$. Then $f : (\mathbb{R}, \mathcal{U}) \longrightarrow (Y, \mathcal{T})$ is a closed quotient mapping. We can describe the open neighborhoods of each element in $Y$ as follows: The open neighborhoods of $i \in Y$ are of the form $(U \setminus \mathbb{N}) \cup \{i\}$, where $U$ is an open set in $(\mathbb{R}, \mathcal{U})$ such that $\mathbb{N} \subseteq U$. The open neighborhoods of any $y \in \mathbb{R} \setminus \mathbb{N}$ are of the form $(y-\epsilon, y+\epsilon) \setminus \mathbb{N}$ where $\epsilon$ is a positive real number.

It is well known that $(Y, \mathcal{T})$ is $T_3$, which is neither locally compact nor first countable. Now, since $(Y, \mathcal{T})$ is Lindelöf, being a continuous image of $\mathbb{R}$ with its usual topology, and $T_3$, then $(Y, \mathcal{T})$ is paracompact and $T_4$. Hence, it is $C_2$-paracompact.

Recall that a topological space $(X, \mathcal{T})$ is called *submetrizable* if there exists a metric $d$ on $X$ such that the topology $\mathcal{T}_d$ on $X$ generated by $d$ is coarser than $\mathcal{T}$, i.e. $\mathcal{T}_d \subseteq \mathcal{T}$, see [8]. By a similar proof as in [3], we can get the following theorem.

**Theorem 2.14** *Every submetrizable space is $C_2$-paracompact.*

$\omega_1 + 1$ is an example of $C_2$-paracompact that is not submetrizable. Recall that a topological space $(X, \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on $X$ such that $(X, \mathcal{T}')$ is $T_4$ [2]. Epinormality implies $C$-normality [3]. We still do not know if epinormality implies $C_2$-paracompactness or not, but epinormality and Lindelöfness do. We emphasize that we do not assume $T_3$ in the definition of Lindelöfness.

**Theorem 2.15** *Every Lindelöf epinormal space is $C_2$-paracompact.*

**Proof** Let $(X, \mathcal{T})$ be any Lindelöf epinormal space. Take a coarser topology $\mathcal{T}'$ on $X$ such that $(X, \mathcal{T}')$ is $T_4$. Since $(X, \mathcal{T})$ is Lindelöf and $\mathcal{T}'$ is coarser than $\mathcal{T}$ we have that $(X, \mathcal{T}')$ is $T_3$ and Lindelöf, and hence Hausdorff paracompact. Therefore, $(X, \mathcal{T})$ is $C_2$-paracompact as the identity function $id : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}')$ works [7, 3.1.13]. □

In general, $C_2$-paracompactness does not imply epinormality. Since any epinormal space is Hausdorff, in fact $T_{2\frac{1}{2}}$ [2], any countable complement topology on an uncountable set is such an example, but $C_2$-paracompactness and the Fréchet property do.

**Theorem 2.16** *Any $C_2$-paracompact Fréchet space is epinormal.*

**Proof** Let $(X, \mathcal{T})$ be any $C_2$-paracompact Fréchet space. If $(X, \mathcal{T})$ is normal, we are done. Assume that $(X, \mathcal{T})$ is not normal. Let $(Y, \mathcal{T}')$ be a $T_2$ paracompact space and $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ be a bijective function such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Since $X$ is Fréchet, $f$ is continuous; see Theorem 2.2. Define $\mathcal{T}^\star = \{f^{-1}(U) : U \in \mathcal{T}'\}$. It clear that $\mathcal{T}^\star$ is a topology on $X$ coarser than $\mathcal{T}$ such that $f : (X, \mathcal{T}^\star) \longrightarrow (Y, \mathcal{T}')$ is continuous. If $W \in \mathcal{T}^\star$, then $W$ is of the form $W = f^{-1}(U)$ where $U \in \mathcal{T}'$. Thus, $f(W) = f(f^{-1}(U)) = U$, which gives that $f$ is open and hence homeomorphism. Thus, $(X, \mathcal{T}^\star)$ is $T_4$. Therefore, $(X, \mathcal{T})$ is epinormal. □

Recall that a topological space $X$ is called *completely Hausdorff,* $T_{2\frac{1}{2}}$ [17] (called also an *Urysohn space* [7]), if for each distinct element $a, b \in X$ there exist two open sets $U$ and $V$ such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. Since epinormality implies $T_{2\frac{1}{2}}$ [2], we have the following corollary.

**Corollary 2.17** *Any $C_2$-paracompact Fréchet space is completely Hausdorff, ($T_{2\frac{1}{2}}$).*

Any finite complement topology on an infinite set is $C$-paracompact and Fréchet but not Hausdorff; thus, Theorem 2.16 is not always true for $C$-paracompactness. The next example is an application for Theorem 2.16

**Example 2.18** Recall that two countably infinite sets are said to be *almost disjoint* [18] if their intersection is finite. Call a subfamily of $[\omega_0]^{\omega_0} = \{A \subset \omega_0 : A \text{ is infinite}\}$ a *mad family* [18] on $\omega_0$ if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let $\mathcal{A}$ be a pairwise almost disjoint subfamily of $[\omega_0]^{\omega_0}$. The *Mrówka space* $\Psi(\mathcal{A})$ is defined as follows: The underlying set is $\omega_0 \cup \mathcal{A}$, each point of $\omega_0$ is isolated, and a basic open neighborhood of $W \in \mathcal{A}$ has the form $\{W\} \cup (W \setminus F)$, with $F \in [\omega_0]^{<\omega_0} = \{B \subseteq \omega_0 : B \text{ is finite}\}$. It is well known that there exists an almost disjoint family $\mathcal{A} \subset [\omega_0]^{\omega_0}$ such that $|\mathcal{A}| > \omega_0$ and the Mrówka space $\Psi(\mathcal{A})$ is a Tychonoff, separable, first countable, and locally compact space that is neither countably compact, paracompact, nor normal. $\mathcal{A}$ is a *mad* family if and only if $\Psi(\mathcal{A})$ is pseudocompact [12]. For a *mad* family $\mathcal{A}$, the Mrówka space $\Psi(\mathcal{A})$ is $C_2$-paracompact, being $T_2$ locally compact. $\Psi(\mathcal{A})$ is also Fréchet, being first countable. We conclude that such a Mrówka space is epinormal.

We have to mention that Corollary 2.9 of [2], of the second author, is incorrect; the condition of cardinality less than continuum must be added to its hypothesis. Observe that Example 2.18 shows that $C_2$-paracompactness does not imply the Lindelöf property.

The next notion, especially in the context of compact Hausdorff spaces, has been considered many times by various topologists but the short name to label the situation was not yet introduced. Arhangel'skiĭ suggested to name it *lower compact.*

**Definition 2.19** *A topological space $(X, \mathcal{T})$ is called lower compact* if there exists a coarser topology $\mathcal{T}'$ on $X$ such that $(X, \mathcal{T}')$ is $T_2$-compact.

Observe that if we do not require the space $(X, \mathcal{T}')$ to be $T_2$ in Definition 2.19, then any space would be lower compact as the indiscrete topology will refine. If we require $T_1$, then the co-finite (finite complement) topology will refine any space to make it lower compact.

**Theorem 2.20** *Every lower compact space is $C_2$-paracompact.*

**Proof** Let $\mathcal{T}'$ be a $T_2$ compact topology on $X$ such that $\mathcal{T}' \subseteq \mathcal{T}$. Then $(X, \mathcal{T}')$ is $T_2$ paracompact and the identity function $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}')$ is a continuous bijective. If $C$ is any compact subspace of $(X, \mathcal{T})$, then the restriction of the identity function on $C$ onto $id_X(C)$ is a homeomorphism because $C$ is compact, $id_X(C)$ is Hausdorff being a subspace of the $T_2$ space $(X, \mathcal{T}')$, and "*every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism*" [7, 3.1.13]. □

The converse of Theorem 2.20 is not always true. Consider for example the countable complement topology on an uncountable set.

**Theorem 2.21** *If $(X, \mathcal{T})$ is $C_2$-paracompact countably compact Fréchet, then $(X, \mathcal{T})$ is lower compact.*

**Proof** Pick a $T_2$ paracompact space $(Y, \mathcal{T}^\star)$ and a bijection function $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}^\star)$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Since $X$ is Fréchet, then $f$ is continuous. Hence, $(Y, \mathcal{T}^\star)$ is countably compact. Since $(Y, \mathcal{T}^\star)$ is also paracompact, then $(Y, \mathcal{T}^\star)$ is $T_2$ compact. Define a topolgy $\mathcal{T}'$ on $X$ as follows: $\mathcal{T}' = \{f^{-1}(U) : U \in \mathcal{T}^\star\}$. Then $\mathcal{T}'$ is coarser than $\mathcal{T}$ and $f : (X, \mathcal{T}') \longrightarrow (Y, \mathcal{T}^\star)$ is a bijection continuous function. Let $W \in \mathcal{T}'$ be arbitrary; then $W$ is of the form $f^{-1}(U)$ for some $U \in \mathcal{T}^\star$. Thus, $f(W) = f(f^{-1}(U)) = U$. Hence, $f$ is open and so $f$ is a homeomorphism. Thus, $(X, \mathcal{T}')$ is $T_2$ compact. Therefore, $(X, \mathcal{T})$ is lower compact. □

Applying Theorem 2.21 on $\omega_1$, we get that $\omega_1$ is lower compact. Indeed, here is a coarser Hausdorff compact topology on $\omega_1$. Define a topology $\mathcal{V}$ on $\omega_1$ generated by the following neighborhood system: Each nonzero element $\beta < \omega_1$ will have the same open neighborhood as in the usual ordered topology in $\omega_1$. Each open neighborhood of 0 is of the form $U = (\beta, \omega_1) \cup \{0\}$ where $\beta < \omega_1$. Simply, the idea is to move the minimal element 0 to the top and make it the maximal element. Then $\mathcal{V}$ is coarser than the usual ordered topology on $\omega_1$ and $(\omega_1, \mathcal{V})$ is a Hausdorff compact space because it is homeomorphic to $\omega_1 + 1$.

Recall that a topology $\mathcal{T}$ on a nonempty set $X$ is said to be *minimal Hausdorff* if $(X, \mathcal{T})$ is Hausdorff and there is no Hausdorff topology on $X$ strictly coarser than $\mathcal{T}$; see [4, 5]. In the next theorem we will use the following theorem: "*A minimal Hausdorff space is compact if and only if it is completely Hausdorff* $(T_{2\frac{1}{2}})$" [14, 1.4]. Using this fact and Corollary 2.17, we get the following theorem.

**Theorem 2.22** *Any minimal Hausdorff $C_2$-paracompact Fréchet space is compact.*

**Corollary 2.23** *If $X$ is a minimal Hausdorff $C_2$-paracompact Fréchet space, then the witness ( $T_2$-paracompact) space $Y$ is unique up to homeomorphism.*

Now we give the following characterization in the class of minimal Hausdorff spaces.

**Theorem 2.24** *Let $X$ be a minimal Hausdorff second countable space. The following are equivalent.*

1. *$X$ is $C_2$-paracompact.*
2. *$X$ is locally compact.*
3. *$X$ is compact*
4. *$X$ is epinormal.*

5. *$X$ is metrizable.*

6. *$X$ is lower compact.*

7. *$X$ is minimal $T_4$.*

**Proof** (1) $\Rightarrow$ (2) Since any second countable space is first countable and any first countable space is Fréchet, then Theorem 2.22 gives that $X$ is $T_2$ compact and hence locally compact.

(2) $\Rightarrow$ (3) Since any $T_2$ locally compact space is Tychonoff, by the minimality, $X$ is compact [14, 1.4].

(3) $\Rightarrow$ (4) Any $T_2$ compact space is $T_4$.

(4) $\Rightarrow$ (5) Any epinormal space is $T_{2\frac{1}{2}}$. By minimilaity, $X$ is compact and hence $T_3$. Since any $T_3$ second countable space is metrizable, the result follows.

(5) $\Rightarrow$ (6) By minimality, $X$ is $T_{2\frac{1}{2}}$ compact and hence lower compact.

(6) $\Rightarrow$ (7) Again, by minimality, $X$ is $T_2$ compact and hence $T_4$. Since any minimal $T_4$ space is compact [4, 4.2], the result follows.

(7) $\Rightarrow$ (1) Since any minimal $T_4$ space is compact, $X$ will be $T_2$ paracompact and hence $C_2$-paracompact. □

In the next example, we give a minimal Hausdorff second countable $C$-paracompact space that is not $C_2$-paracompact. The space $X$ in the next example is due to Urysohn [14].

**Example 2.25** *Let $X = \{a, b, c_i, a_{ij}, b_{ij} : i \in \mathbb{N}, j \in \mathbb{N}\}$ where all these elements are assumed to be distinct. Define the following neighborhood system on $X$:*

*For each $i, j \in \mathbb{N}$, $a_{ij}$ is isolated and $b_{ij}$ is isolated.*

$$\text{For each } i \in \mathbb{N},\ \mathcal{B}(c_i) = \{V^n(c_i) = \{c_i, a_{ij}, b_{ij} : j \geq n\} : n \in \mathbb{N}\}.$$

$$\mathcal{B}(a) = \{V^n(a) = \{a, a_{ij} : i \geq n\} : n \in \mathbb{N}\}.$$

$$\mathcal{B}(b) = \{V^n(b) = \{b, b_{ij} : i \geq n\} : n \in \mathbb{N}\}.$$

*Let us denote the unique topology on $X$ generated by the above neighborhood system by $\mathcal{T}$. Then $\mathcal{T}$ is minimal Hausdorff and $(X, \mathcal{T})$ is not compact [4]. Since $X$ is countable and each local base is countable, then the neighborhood system is a countable base for $(X, \mathcal{T})$, so it is second countable but not $C_2$-paracompact because it is not $T_{2\frac{1}{2}}$ as the closure of any open neighborhood of $a$ must intersect the closure of any open neighborhood of $b$.*

*For each $i \in \mathbb{N}$, let $A_i = \{a_{ij} : j \in \mathbb{N}\}$ and $B_i = \{b_{ij} : j \in \mathbb{N}\}$. Let $C = \{c_i : i \in \mathbb{N}\}$.*

**Claim 1:** *A subset $E$ of $X$ is compact if and only if $E$ satisfies all of the following conditions:*

1. *$E \cap C$ is finite.*

2. *If $E \cap A_i$ or $E \cap B_i$ is infinite, then $c_i \in E$.*

3. *If $\{i \in \mathbb{N} : E \cap A_i \neq \emptyset\}$ is infinite, then $a \in E$.*

4. *If $\{i \in \mathbb{N} : E \cap B_i \neq \emptyset\}$ is infinite, then $b \in E$.*

**Proof of Claim 1:** *Let* $K_1 = \{i \in \mathbb{N} : c_i \in E\}$, $K_2 = \{i \in \mathbb{N} : E \cap A_i \neq \emptyset\}$, *and* $K_3 = \{i \in \mathbb{N} : E \cap B_i \neq \emptyset\}$. *Assume* $E$ *is compact. Suppose that* $E \cap C$ *is infinite. The family* $\{V^1(a), V^1(b), V^1(c_i) : i \in K_1\}$ *is an open cover for* $E$ *that has no finite subcover, which contradicts the compactness of* $E$. *Thus, (1) holds. Now, assume* $E$ *is compact and satisfies (1). Suppose that there exists an* $m \in \mathbb{N}$ *with* $E \cap A_m$ *infinite and* $c_m \notin E$. *The family* $\{V^1(b), V^{m+1}(a), \{a_{mj}\}, \{a_{ij}\} : j \in \mathbb{N}, i \notin K_1, i < m\} \cup \{V^1(c_i) : i \in K_1\}$ *is an open cover for* $E$ *that has no finite subcover, a contradiction. Similarly, we can show that if* $E \cap B_i$ *is infinite, then* $c_i \in E$. *Now, assume* $E$ *is compact and satisfies (1) and (2). Suppose that* $K_2$ *is infinite but* $a \notin E$. *The open cover* $\{V^1(b), V^1(c_m), \{a_{ij}\} : m \in K_1, i \in K_2, j \in \mathbb{N}\}$ *of* $E$ *has no finite subcover, a contradiction. Thus, (3) holds and in a similar way (4) does hold.*

*Now assume* $E$ *satisfies all of the four conditions. Let* $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ *be any open (open in* $X$*) cover of* $E$. *By (1), for each* $i \in K_1$ *there exists an* $\alpha_i \in \Lambda$ *such that* $c_i \in U_{\alpha_i}$. *Thus, for each* $i \in K_1$, *there exists an* $n_i \in \mathbb{N}$ *such that* $V^{n_i}(c_i) \subseteq U_{\alpha_i}$. *Observe that if there exists* $a_{n_i,j} \in E \setminus V^{n_i}(c_i)$, *then those* $a_{n_i,j}$*s are finite. Also, if there exists* $b_{n_i,j} \in E \setminus V^{n_i}(c_i)$, *then those* $b_{n_i,j}$*s are finite. Let* $\mathcal{V}_1 = \{V^{n_i}(c_i) : i \in K_1\} \cup \{\{a_{n_i,j}\}, \{b_{n_i,j}\} : i \in K_1; a_{n_i,j} \in E \setminus V^{n_i}(c_i); b_{n_i,j} \in E \setminus V^{n_i}(c_i)\}$. *Observe that* $\mathcal{V}_1$ *is finite. Now, let* $k_1 = \max K_1$. *For each* $i \in K_2 \setminus K_1$ *we have, by (2), that* $A_i \cap E$ *is finite and for each* $i \in K_3 \setminus K_1$ *we have, by (2), that* $B_i \cap E$ *is finite. Let* $\mathcal{V}_2 = \{\{a_{ij}\}, \{b_{ij}\} : a_{ij} \in E; b_{ij} \in E; i < k_1; i \notin K_1\}$. *Observe that* $\mathcal{V}_2$ *is finite. If* $K_2$ *is infinite, then, by (3),* $a \in E$. *Thus, there exists an* $\alpha_a \in \Lambda$ *such that* $a \in U_{\alpha_a}$. *Hence, there exists an* $n_a \in \mathbb{N}$ *such that* $V^{n_a}(a) \subseteq U_{\alpha_a}$. *Let* $n'_a = \max\{n_a, k_1\}$. *Then* $V^{n'_a}(a) \subseteq V^{n_a}(a) \subseteq U_{\alpha_a}$. *In this case, let* $\mathcal{V}_3 = \{V^{n'_a}(a), \{a_{ij}\} : i < n'_a; i \in K_2 \setminus K_1\}$. *Observe that* $\mathcal{V}_3$ *is finite. If* $K_2$ *is finite but* $a \in E$, *we may take the same* $\mathcal{V}_3$. *If* $K_2$ *is finite and* $a \notin E$, *we take* $\mathcal{V}_3 = \{\{a_{ij}\} : i \in K_2 \setminus K_1\}$. *Observe that* $\mathcal{V}_3$ *is also finite in this case. Similarly, if* $K_3$ *is infinite, then, by (4),* $b \in E$. *Thus, there exists an* $\alpha_b \in \Lambda$ *such that* $b \in U_{\alpha_b}$. *Hence, there exists an* $n_b \in \mathbb{N}$ *such that* $V^{n_b}(b) \subseteq U_{\alpha_b}$. *Let* $n'_b = \max\{n_b, k_1\}$. *Then* $V^{n'_b}(b) \subseteq V^{n_b}(b) \subseteq U_{\alpha_b}$. *In this case, let* $\mathcal{V}_4 = \{V^{n'_b}(b), \{b_{ij}\} : i < n'_b; i \in K_3 \setminus K_1\}$. *Observe that* $\mathcal{V}_4$ *is finite. If* $K_3$ *is finite but* $b \in E$, *we may take the same* $\mathcal{V}_3$. *If* $K_3$ *is finite and* $b \notin E$, *we take* $\mathcal{V}_4 = \{\{b_{ij}\} : i \in K_3 \setminus K_1\}$. *Observe that* $\mathcal{V}_4$ *is also finite in this case. Now* $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$ *is a finite refinement of* $\mathcal{U}$. *Thus,* $E$ *is compact.*

**Claim 2:** $(X, \mathcal{T})$ *is* $C$*-paracompact.*

**Proof of Claim 2:** *Let* $Y = X$ *and let* $Y$ *have the following neighborhood system: For each* $y \in Y \setminus C$, *let* $y$ *have the same neighborhoods as in* $X$. *For each* $i \in \mathbb{N}$, *let* $H_i = A_i \cup B_i = \{a_{ij}, b_{ij} : j \in \mathbb{N}\}$. *For each* $i \in \mathbb{N}$ *and each* $n \in \mathbb{N}$, *let* $V^n(c_i)$ *be the same as in* $X$. *For* $i \in \mathbb{N}$, *an open neighborhood of* $c_i$ *is of the form* $V^n(c_i) \cup (\cup_{k \geq l} (V^n(c_k) \setminus F_k))$,*where* $l > i$ *and* $F_k$ *is a finite subset of* $H_k$. *That is, we add an "*$l$*-tail"* $D^n_l = \cup_{k \geq l} V^n(c_k)$, *where* $l > i$, *to* $V^n(c_i)$, *but we delete from each* $V^n(c_k)$, $k \geq l$, *a finite subset* $F_k \subset H_k$. *The open neighborhoods of the* $c_i$*s are the only difference between the neighborhood system of* $X$ *and of* $Y$. *Note that if* $c_m \in V^n(c_i) \cup (\cup_{k \geq l} (V^n(c_k) \setminus F_k))$, *where* $i, l, n \in \mathbb{N}$ *with* $l > i$ *and* $m \neq i$, *then* $m \geq l > i$. *Consider now* $V^n(c_m) \setminus F_m$. *Since* $F_m$ *is finite, we can find an* $n' \in \mathbb{N}$ *such that* $V^{n'}(c_m) \subseteq V^n(c_m) \setminus F_m$. *Thus,* $V^{n'}(c_m) \cup (\cup_{k \geq m+1} (V^{n'}(c_k) \setminus F_k)) \subseteq V^n(c_i) \cup (\cup_{k \geq l} (V^n(c_k) \setminus F_k))$. *Thus, this neighborhood system on* $Y$ *will generate a unique topology* $\mathcal{T}'$*; see [7, 1.2.3]. Since any* $V^n(c_i) \cup (\cup_{k \geq l} (V^n(c_k) \setminus F_k))$, *where* $l > i$ *and* $F_k$ *is a finite subset of* $H_k$, *is open in* $X$, *then* $\mathcal{T}'$ *is coarser than* $\mathcal{T}$. *If* $i_1 \neq i_2$, *then any open neighborhood of* $c_{i_1}$ *will intersect any open neighborhood of* $c_{i_2}$ *and thus* $Y$ *is not Hausdorff. Let* $\{U_\alpha : \alpha \in \Lambda\}$ *be any open* *cover of* $Y$*. There exists an* $\alpha_a \in \Lambda$ *such that* $a \in U_{\alpha_a}$*. There exists an* $n_a \in \mathbb{N}$ *such that* $V^{n_a}(a) \subseteq U_{\alpha_a}$*. There exists an* $\alpha_b \in \Lambda$ *such that* $b \in U_{\alpha_b}$*. There exists an* $n_b \in \mathbb{N}$ *such that* $V^{n_b}(b) \subseteq U_{\alpha_b}$*. There exists an* $\alpha_1 \in \Lambda$ *such that* $c_1 \in U_{\alpha_1}$*. There exists an* $n_1, l_1 \in \mathbb{N}$ *such that* $V^{n_1}(c_1) \cup (\cup_{k \geq l_1} (V^{n_1}(c_k) \setminus F_k)) \subseteq U_{\alpha_1}$*. Let* $l = \max\{l_1, n_a, n_b\}$*. For each* $1 < i < l$*, there exists an* $\alpha_i \in \Lambda$ *such that* $c_i \in U_{\alpha_i}$*. The set* $L_i = H_i \setminus U_{\alpha_i}$ *is finite for each* $i < l$*. Thus,* $\{V^l(a), V^l(b), U_{\alpha_i} : 1 \leq i < l\} \cup \{\{x\} : x \in L_i ; i < l\}$ *is a finite refinement of* $\{U_\alpha : \alpha \in \Lambda\}$*. Therefore,* $Y$ *is compact and hence paracompact. Let* $E$ *be any compact subspace of* $X$*. We show that the topology on* $E$ *inherited from* $X$ *coincides with the topology on* $E$ *inherited from* $Y$*. Since* $\mathcal{T}'$ *is coarser than* $\mathcal{T}$*, we just need to show the other containment. Since the only differences are the neighborhoods of the elements of* $C$*, let* $c_i \in E$ *be arbitrary and let* $V^n(c_i) \cap E$ *be any open neighborhood of* $c_i$ *in* $E$ *as a subspace of* $X$*. Since* $E$ *is compact in* $X$*, then, by part 1 of Claim 1,* $E \cap C$ *is finite. Let* $l = \max\{i \in \mathbb{N} : c_i \in E\}$*. Thus, by part 2 of Claim 1, for each* $k \geq l+1$ *we have that* $H_k \cap E$ *is finite. For each* $k \geq l+1$*, let* $H_k \cap E = F_k$*. Then* $G = V^n(c_i) \cup (\cup_{k \geq l+1} (V^n(c_k) \setminus F_k))$ *is an open neighborhood of* $c_i$ *in* $Y$ *such that* $G \cap E = V^n(c_i) \cap E$*. Thus,* $V^n(c_i) \cap E$ *is an open neighborhood of* $c_i$ *in* $E$ *as a subspace of* $Y$*. Hence, the two topologies on* $E$ *coincide. Thus,* $Y$ *and the identity function from* $X$ *onto* $Y$ *will give the* $C$*-paracompactness of* $X$*.*

**Theorem 2.26** *$C$-paracompactness ($C_2$-paracompactness) is a topological property.*

**Proof** Let $X$ be a $C$-paracompact ($C_2$-paracompact) space and let $X \cong Z$. Let $Y$ be a paracompact (Hausdorff paracompact) space and $f : X \longrightarrow Y$ be a bijective function such that the restriction $f|_C : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. Let $g : Z \longrightarrow X$ be a homeomorphism. Then $Y$ and $f \circ g : Z \longrightarrow Y$ satisfy the requirements. □

**Theorem 2.27** *$C$-paracompactness ($C_2$-paracompactness) is an additive property.*

**Proof** Let $X_\alpha$ be a $C$-paracompact ($C_2$-paracompact) space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is $C$-paracompact ($C_2$-paracompact). For each $\alpha \in \Lambda$, pick a paracompact (a Hausdorff paracompact) space $Y_\alpha$ and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha_{|C_\alpha}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each compact subspace $C_\alpha$ of $X_\alpha$. Since $Y_\alpha$ is paracompact (Hausdorff paracompact) for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is paracompact (Hausdorff paracompact), [7, 2.2.7, 5.1.30]. Consider the function sum [7, 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is compact in $X_\alpha$ for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)|_C$ is a homeomorphism because $f_{\alpha_{|C \cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. □

Let $X$ be any $T_1$ topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in $X'$ by $x'$ and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ be open in $X$ with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$, which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff duplicate of* $X$ [6]. It is well known that if $X$ is paracompact (Hausdorff), then so is its Alexandroff duplicate $A(X)$ [1]. By a similar argument as in [3] we have the following theorem.

**Theorem 2.28** *If* $X$ *is* $C$*-paracompact (*$C_2$*-paracompact), then so is its Alexandroff duplicate* $A(X)$*.*

Recall that a subset $A$ of a space $X$ is called a *closed domain* [7], and called also *regularly closed, $\kappa$-closed,* if $A = \overline{\text{int}A}$. A space $X$ is called *mildly normal* [16], called also *$\kappa$-normal* [15], if for any two disjoint closed domains $A$ and $B$ of $X$ there exist two disjoint open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$; see also [9, 11]. The space $X$ in Example 2.11 is mildly normal, being normal, but not $C$-paracompact. Here is an example of a $C_2$-paracompact space that is not mildly normal.

**Example 2.29** Recall that the Dieudonné Plank [17] is defined as follows: Let

$$X = ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_1, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_1, n \rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0 \rangle : \alpha < \omega_1\}$, and $N = \{\langle \alpha, n \rangle : \alpha < \omega_1$ and $n < \omega_0\}$. The topology $\mathcal{T}$ on $X$ is generated by the following neighborhood system: For each $\langle \alpha, n \rangle \in N$, let $\mathcal{B}(\langle \alpha, n \rangle) = \{\{\langle \alpha, n \rangle\}\}$. For each $\langle \omega_1, n \rangle \in A$, let $\mathcal{B}(\langle \omega_1, n \rangle) = \{V_\alpha(n) = (\alpha, \omega_1] \times \{n\} : \alpha < \omega_1\}$. For each $\langle \alpha, \omega_0 \rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0 \rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. It is well known that the Dieudonné plank is a Tychonoff space that is neither locally compact, normal, nor paracompact [17]. Now, a subset $C \subseteq X$ is compact if and only if $C$ satisfies all of the following conditions:

(i) $C \cap A$ and $C \cap B$ are both finite.

(ii) If $\langle \omega_1, n \rangle \in C$, then the set $(\omega_1 \times \{n\}) \cap C$ is finite.

(iii) The set $\{\langle \alpha, n \rangle \in C : \langle \alpha, \omega_0 \rangle \notin C\}$ is finite.

Now, define $Y = X = A \cup B \cup N$. Generate a topology $\mathcal{T}\,'$ on $Y$ by the following neighborhood system: Elements of $B \cup N$ have the same local base as in $X$. For each $\langle \omega_1, n \rangle \in A$, let $\mathcal{B}(\langle \omega_1, n \rangle) = \{\{\langle \omega_1, n \rangle\}\}$. Then $Y$ is Hausdorff paracompact. Now, $Y$ and the identity function $id : X \longrightarrow Y$ will witness the $C_2$-paracompactness of the Dieudonné plank $X$, in a similar way as in [3, Example 1.10].

$X$ is not normal because $A$ and $B$ are closed disjoint subsets, which cannot be separated by two disjoint open sets. Let $E = \{n < \omega_0 : n \text{ is even}\}$ and $O = \{n < \omega_0 : n \text{ is odd}\}$. Let $K$ and $L$ be subsets of $\omega_1$ such that $K \cap L = \emptyset$, $K \cup L = \omega_1$, and the cofinality of $K$ and $L$ is $\omega_1$; for instance, let $K$ be the set of limit ordinals in $\omega_1$ and $L$ be the set of successor ordinals in $\omega_1$. Then $K \times E$ and $L \times O$ are both open, being subsets of $N$. Define $C = \overline{K \times E}$ and $D = \overline{L \times O}$; then $C$ and $D$ are closed domains in $X$, being closures of the open set, and they are disjoint. Note that $C = \overline{K \times E} = (K \times E) \cup (K \times \{\omega_0\}) \cup (\{\omega_1\} \times E)$ and $D = \overline{L \times O} = (L \times O) \cup (L \times \{\omega_0\}) \cup (\{\omega_1\} \times O)$. Let $U \subseteq X$ be any open set such that $C \subseteq U$. For each $n \in E$ there exists an $\alpha_n < \omega_1$ such that $V_{\alpha_n}(n) \subseteq U$. Let $\beta = \sup\{\alpha_n : n \in E\}$; then $\beta < \omega_1$. Since $L$ is cofinal in $\omega_1$, then there exists $\gamma \in L$ such that $\beta < \gamma$ and then any basic open set of $\langle \gamma, \omega_0 \rangle \in D$ will meet $U$. Thus, $C$ and $D$ cannot be separated. Therefor, the Dieudonné plank $X$ is $C_2$-paracompact but not mildly normal.

**Open Problem:** (Arhangel'skiĭ, 2016)

Is there a $T_4$ space that is not $C_2$-paracompact?

The class of all $C_2$-paracompact spaces is very wide, but, intuitively, we think the answer is positive even though we have not found such a space yet. Observe that such a space is not in the class of minimal Hausdorff spaces (see Theorem 2.24 and [14, 1.4]), or in the class of minimal $T_4$ spaces as any minimal $T_4$ space is compact

[4, 4.2], and hence $C_2$-paracompact. Also, such a space cannot be an ordinal because any ordinal space is $T_2$ locally compact, and hence $C_2$-paracompact; see Theorem 2.12. It cannot be submetrizable; see Theorem 2.14. It cannot be Lindelöf; see Theorem 2.15. It cannot be lower compact; see Theorem 2.20. It could be the case that such a space is a LOTS, a linearly ordered topological space, or any other space, but this LOTS must be neither Lindelöf, locally compact, nor paracompact. Observe also that the existence of such a space, $T_4$ but not $C_2$-paracompact, will show that $C_2$-paracompactness is not hereditary just by taking a compactification of it.

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