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On a theorem of Terzioğlu*

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Abstract: The theory of compact linear operators acting on a Banach space has a classical core and is familiar to many. Perhaps less known is the characterization theorem of Terzioğlu for compact maps. This theorem has a number of important connections that deserves illumination. In this paper we survey Terzioğlu's characterization theorem for compact maps and some of its consequences. We also prove a similar characterization theorem for Q-compact maps.

Key words: Compact operators, approximation schemes, Q-compact operators

1. Introduction

Let X and Y be Banach spaces and $T : X \to Y$ be an operator. We say that T is compact if and only if it maps closed unit ball B_X of X into a precompact subset of Y. In other words, T is compact if and only if for every norm bounded sequence $\{x_n\}$ of X, the sequence $\{Tx_n\}$ has a norm convergent subsequence in Y. Equivalently, T is compact if and only if for every $\epsilon > 0$, there exists elements $y_1, y_2, \dots, y_n \in Y$ such that

$$T(B_X) \subseteq \bigcup_{k=1}^{n} \{y_k + \epsilon B_Y\},$$

where by B_X and B_Y we mean the closed unit balls of X and Y, respectively. Every compact linear operator is bounded, hence continuous, but clearly not every bounded linear map is compact since one can take the identity operator on an infinite dimensional space X. Compact operators are natural generalizations of finite rank operators and thus dealing with compact operators provides us with the closest analogy to the usual theorems of finite dimensional spaces. Recall that $\mathcal{L}(X, Y)$ denotes the normed vector space of all continuous operators from X to Y and $\mathcal{L}(X)$ stands for $\mathcal{L}(X, X)$ and $\mathcal{K}(X, Y)$ is the collection of all compact operators from X to Y. It is well known that if Y is a Hilbert space then any compact $T : X \to Y$ is a limit of finite rank operators; in other words, if $\mathcal{F}(X, Y)$ denotes the class of finite rank maps, then

$$\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)},$$

where the closure is taken in the operator norm. However, the situation is quite different for Banach spaces; not every compact operator between Banach spaces is a uniform limit of finite rank maps. For further information we refer the reader to a well-known example due to Enflo [8], in which Enflo constructs a Banach space without

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2010 *AMS Mathematics Subject Classification:* Primary 41A25; Secondary 41A50, 46B20 the approximation property. The following classical results on compact operators will be used for our discussion later.

Theorem 1.1 *For Banach spaces X, Y, and Z, we have the following:*

1. *$\mathcal{K}(X,Y)$ is a norm closed vector subspace of $\mathcal{L}(X,Y)$.*

2. *If $X \xrightarrow{S} Y \xrightarrow{T} Z$ are continuous operators and either S or T is compact, then TS is likewise compact.*

If one consider the continuous operators on a Banach space X, the above theorem asserts the fact that compact operators on X form a two-sided ideal in $\mathcal{L}(X)$. The following theorem of Schauder simply states that an operator is compact if and only if its adjoint is compact.

Theorem 1.2 (Schauder) *A norm bounded operator $T : X \to Y$ between Banach spaces is compact if and only if its adjoint $T^* : Y^* \to X^*$ is compact.*

The main idea in proving Schauder's theorem lies in the fact that

$$||P_n T - T|| \to 0 \quad \text{implies} \quad ||T^* P_n - T^*|| \to 0,$$

where $P_n : X \to \text{span}\{e_1, \ldots, e_n\}$. A well-known proof of Schauder's theorem may be found in Yosida's work [[21], p. 282].

For our discussion below we also need the following characterization of compact sets in a Banach space; in some sense, it is a comment on the smallness of compact sets.

Theorem 1.3 (Grothendieck) *A subset of a Banach space is compact if and only if it is included in the closed convex hull of a sequence that converges in norm to zero.*

In other words, if we have K, a compact subset of a Banach space X, then we can find a sequence $\{x_n\}$ in X such that

$$||x_n|| \to 0 \quad \text{and} \quad K \subseteq \overline{\text{co}}\{x_n\}.$$

For a proof we refer the reader to [[7], p. 3].

2. Terzioğlu's theorem

Theorem 2.1 (Terzioğlu [19]) *An operator $T : X \to Y$ between two Banach spaces is compact if and only if there exists a sequence $\{u_n\}$ of linear functionals in X^* with $||u_n|| \to 0$ such that the inequality*

$$||Tx|| \leq \sup_n |< u_n, x >|$$

holds for every $x \in X$.

Proof Suppose $T : X \to Y$ is compact. Then by Schauder's theorem $T^* : Y^* \to X^*$ is compact, and thus, by definition, if V denotes the closed unit ball of Y^*, $T^*(V)$ is a norm totally bounded subset of X^*. Now applying

Grothendieck's result, we have a sequence $\{u_n\}$ of elements of X^* with $||u_n|| \to 0$ and $T^*(V) \subseteq \overline{\text{co}\{u_n\}}$, or in other words, each element of $T^*(V)$ can be written the form

$$\sum_{n=1}^{\infty} \alpha_n u_n \quad \text{with} \quad \sum_{n=1}^{\infty} |\alpha_n| \leq 1.$$

Thus, for each $x \in X$, we have

$$||Tx|| = \sup_{||v|| \leq 1} |< T^*v, x >| \leq \left(\sum_{n=1}^{\infty} |\alpha_n| \right) \sup_n | < u_n, x > |.$$

Suppose that T satisfies the inequality $||Tx|| \leq \sup_n |< u_n, x >|$ for some sequence $\{u_n\} \in X^*$. For $\epsilon > 0$ choose N such that $||u_n|| < \epsilon$ for $n > N$ and set

$$M_\epsilon = \{x \in X : < u_i, x >= 0 \quad \text{for } i = 1, 2, \dots N\},$$

and then one can have

$$T^*(\mathring{V}) \subset \epsilon \mathring{U} + M_\epsilon^\perp,$$

where U denotes the unit ball of X and for each linear subspace M of X, the polar of M, denoted by $\mathring{M}$, is a linear subspace of X^* defined as:

$$\mathring{M} := \{a \in X^* : \quad | < x, a > | \leq 1 \quad \text{for} \quad x \in M\}.$$

This shows that T^* is compact and hence T is compact.

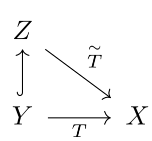
□

An application of the above characterization theorem (Theorem 2.1) implies the fact that every compact mapping of a Banach space into a $\mathcal{P}_\lambda$-space is ∞-nuclear. To understand this result, we need the following definitions.

Definition 2.2 *We say that X is a $\mathcal{P}_\lambda$ space, ($\lambda \geq 1$), if for every bounded linear operator T from a Banach space Y to X and every $Z \supset Y$ there is a linear extension $\tilde{T}$ of Z to X with*

$$||\tilde{T}|| \leq \lambda ||T||.$$

This is illustrated in the following diagram:



If $||\tilde{T}|| = ||T||$ in the above definition, we call X extendible. This property is related to the existence of a Hahn–Banach type extension. Lindenstrauss in [12] examined the problem of when the extension $\tilde{T}$ is compact if T itself is compact and his results are diverse and numerous, touching upon many related topics. For the case $\lambda = 1$, it is known that a Banach space is a P_1 space if and only if it is isometric to the space $C(K)$ of all continuous functions on extremally disconnected compact Housdorff space K with the sup norm. This result is due to Nachbin [14].

Next, we define infinite nuclear mappings. This concept was first introduced in [15].

Definition 2.3 *Let X and Y be Banach spaces and $T : X \to Y$ a linear operator. Then T is said to be infinite-nuclear if there are sequences $\{u_n\} \subset X^*$ and $\{y_n\} \subset Y$ such that $\lim_n ||u_n|| = 0$,*

$$\sup_{||v|| \leq 1} \left\{ \sum_{n=1}^{\infty} | < v, y_n > | : \ v \in Y^* \right\} < +\infty,$$

and

$$Tx = \sum_{n=1}^{\infty} < u_n, x > y_n$$

for $x \in X$.

As an application of Theorem 2.1, under the condition that $T : X \to Y$ where Y is a $\mathcal{P}_\lambda$-space, Terzioğlu also obtains a precise expression for Tx, which we state in the following theorem.

Theorem 2.4 ([19]) *Let T be a compact mapping of a Banach space X into a $\mathcal{P}_\lambda$ space Y. Then for every $\epsilon > 0$ there exists a sequence $\{u_n\}$ in X^* with*

$$\lim_{n \to \infty} ||u_n|| = 0 \quad \textit{and} \quad \sup_n ||u_n|| \leq ||T|| + \epsilon$$

and a sequence $\{y_n\} \in Y$ with $\sup_{||v|| \leq 1} \sum_{n=1}^{\infty} | < v, y_n > | \leq \lambda$ such that T has the form

$$Tx = \sum_{n=1}^{\infty} < u_n, x > y_n.$$

The complete details of the proof can be found in [19]. However, it is worth pointing out that the main idea of the proof relies on a factorization of a compact map through the space c_0 as follows:
Using Theorem 2.1, choose the sequence $\{u_n\}$ in X^* satisfying

$$\lim_n ||u_n|| = 0 \text{ and } \sup_n ||u_n|| \leq ||T|| + \epsilon \text{ and } ||Tx|| \leq \sup | < u_n, x > |.$$

Define the linear mapping

$$S : X \to c_0 \quad \text{by} \quad Sx = \{< u_n, x >\},$$

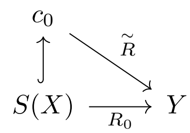
and observe that S is compact. Then define a linear mapping R_0:

$$R_0 : S(X) \to Y \text{ by } R_0(Sx) = Tx.$$

The inequality

$$||R_0(Sx)|| = ||Tx|| \leq \sup | < u_n, x > | = ||Sx||$$

implies that $||R_0|| \leq 1$.



Since Y is a $\mathcal{P}_\lambda$ space, there exists an extension $\tilde{R}$ of R_0 such that $\tilde{R}: c_0 \to Y$ with $||\tilde{R}|| \leq \lambda ||R_0|| = \lambda$ and

$$E \xrightarrow{S} c_0 \xrightarrow{\widetilde{R}} Y.$$

Evidently $T = \widetilde{R}S$.

By considering $\{e_n\} \in c_0$ and setting $y_n = \widetilde{R}(e_n)$ we obtain

$$\sup_{||v|| \leq 1} \sum_{n=1}^{\infty} |< v, y_n >| \leq \lambda, \text{ and } Tx = \sum_{n=1}^{\infty} < u_n, x > y_n.$$

Another related representation theorem for compact maps emphasizing the factorization through c_0 can be found in [17]. Using all of the above results of Terzioğlu, the following conclusions are shown in [20].

Corollary 2.5 ([20]) *1. Every $\mathcal{P}_\lambda$ space has the approximation property.*

2. Every compact linear operator of an L^∞ space into a Banach space is infinite-nuclear.

3. Let T be a compact linear map of an infinite-dimensional space X into a Banach space Y. Then there exists an infinite-dimensional closed subspace M of X such that $T_M : M \to T(M)$ is infinite nuclear.

Terzioğlu's characterization for compact maps found its use in more current research on compact maps as well. See [9, 10, 18].

3. Compactness with approximation scheme

Approximation schemes were introduced by Butzer and Scherer for Banach spaces in 1968 [6] and later by Brudnyi and Krugljak [5]. These concepts find their best application in a paper by Pietsch [16], where he defined approximation spaces; proved embedding, reiteration, and representation results; and established connection to interpolation spaces.

Let X be a Banach space and $\{A_n\}$ be a sequence of subsets of X satisfying:

1. $A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq X$.
2. $\lambda A_n \subseteq A_n$ for all scalars λ and $n = 1, 2, \ldots$.
3. $A_m + A_n \subseteq A_{m+n}$ for $m, n = 1, 2, \ldots$.

For example, if we consider the space $X = L_p[0,1]$, for $1 \le p < \infty$, then the collection of sets $\{A_n\} = \{L_{p+\frac{1}{n}}\}$ forms an approximation scheme like above. Pietsch's approximation spaces X^{ρ}_{μ} $(0 < \rho < \infty,\ 0 < \mu \le \infty)$ are defined by considering the nth approximation number $\alpha_n(f,X)$, where

$$\alpha_n(f,X) := \inf\{||f-a|| :\ a \in A_{n-1}\}$$

and

$$X^{\rho}_{\mu} = \{f \in X :\ \{n^{\rho-\frac{1}{\mu}}\,\alpha_n(f,X)\} \in \ell^{\mu}\}.$$

In the same paper, [16], embeddings, composition, and representation interpolation of such spaces are studied and applications to the distribution of Fourier coefficients and eigenvalues of integral operators are given.

In the following we consider for each $n \in \mathbb{N}$ a family of subsets Q_n of X satisfying the same three conditions stated above. An example for Q_n could be the set of all at most n-dimensional subspaces of any Banach space X, or if our Banach space $X = \mathcal{L}(E)$, namely the set of all bounded linear operators on another Banach space E, then we can take $Q_n = N_n(E)$ the set of all n-nuclear maps on E.

Compactness relative to an approximation scheme for bounded sets and linear operators can be defined by using Kolmogorov diameters as follows.

Let $D \subset X$ be a bounded subset and U_X denote the closed unit ball of X. Suppose $Q = (Q_n(X)_{n\in\mathbb{N}})$ be an approximation scheme on X; then the nth Kolmogorov diameter of D with respect to this scheme Q is denoted by $\delta_n(D,Q)$ and defined as

$$\delta_n(D,Q) = \inf\{r > 0 :\ D \subset rU_X + A \quad \text{for some} \quad A \in Q_n(X)\}.$$

Letting Y be another Banach space and $T \in \mathcal{L}(Y,X)$, then the nth Kolmogorov diameter of T with respect to this scheme Q is denoted by $\delta_n(T,Q)$ and defined as

$$\delta_n(T,Q) = \delta_n(T(U_X),Q).$$

Definition 3.1 *We say D is Q-compact set if*

$$\lim_{n\to\infty} \delta_n(D,Q) = 0,$$

and similarly $T \in \mathcal{L}(Y,X)$ is a Q-compact map if

$$\lim_{n\to\infty} \delta_n(T,Q) = 0.$$

The following example illustrates that not every Q-compact operator is compact.

Example 3.2 *Let $\{r_n(t)\}$ be the space spanned by the Rademacher functions. It can be seen from the Khinchin inequality [13] that*

$$\ell^2 \approx \{r_n(t)\} \subset L_p[0,1] \text{ for all } 1 \le p \le \infty. \tag{3.1}$$

We define an approximation scheme A_n on $L_p[0,1]$ as follows:

$$A_n = L_{p+\frac{1}{n}}. \tag{3.2}$$

$L_{p+\frac{1}{n}} \subset L_{p+\frac{1}{n+1}}$ *gives us* $A_n \subset A_{n+1}$ *for* $n = 1, 2, \ldots,$ *and it is easily seen that* $A_n + A_m \subset A_{n+m}$ *for* $n, m = 1, 2, \ldots,$ *and that* $\lambda A_n \subset A_n$ *for all* λ. *Thus,* $\{A_n\}$ *is an approximation scheme.*

Next, we claim that for $p \geq 2$ *the projection* $P : L_p[0,1] \to R_p$ *is a* Q*-compact map, but not compact, where* R_p *denotes the closure of the span of* $\{r_n(t)\}$ *in* $L_p[0,1]$.

$$\begin{array}{ccc} L_p & \xrightarrow{i} & L_2 \\ P\downarrow & & \downarrow P_2 \\ R_p & \xleftarrow{j} & R_2 \end{array}$$

We know that for $p \geq 2$, $L_p[0,1] \subset L_2[0,1]$, *and* R_2 *is a closed subspace of* $L_2[0,1]$ *and*

$$P = j \circ P_2 \circ i,$$

where i, j *are isomorphisms shown in the above figure.* P *is not a compact operator, because* $dim R_p = \infty$. *On the other hand, it is a* Q*-compact operator because, if we let* U_{R_p}, U_{L_p} *denote the closed unit balls of* R_p *and* L_p, *respectively, it is easily seen that* $P(U_{L_p}) \subset \|P\| U_{R_p}$, *but* $U_{R_p} \subset C U_{R_{P+\frac{1}{n}}}$, *where* C *being a constant follows from the Khinchin inequality. Therefore,*

$$P(U_{L_p}) \subset L_{p+\frac{1}{n}}, \quad \text{which gives} \quad \delta_n(P, Q) \to 0.$$

The above example shows that Q-compact maps are different from compact maps, and for further properties of Q-compact maps we refer to [2]. Next we give a characterization of Q-compact sets as subsets of the closed convex hull of certain uniform null-sequences.

Definition 3.3 *Suppose* X *is a Banach space with an approximation scheme* Q_n. *A sequence* $\{x_{n,k}\}_{n,k}$ *in* X *is called an order* c_0*-sequence if*

1. $\forall n = 1, 2, \ldots,$ *there exists* $A_n \in Q_n$ *and a sequence* $\{x_{n,k}\}_k \subset A_n$;
2. $||x_{n,k}|| \to 0$ *as* $n \to \infty$ *uniformly in* k.

Theorem 3.4 *Let* X *be a Banach space with an approximation scheme with sets* $A_n \in Q_n$ *satisfying the condition* $|\lambda| A_n \subset A_n$ *for* $|\lambda| \leq 1$. *A bounded subset* D *of* X *is* Q*-compact if and only if there is an order* c_0*-sequence* $\{x_{n,k}\}_k \subset A_n$ *such that*

$$D \subset \left\{ \sum_{n=1}^{\infty} \lambda_n x_{n,k(n)} : \quad x_{n,k(n)} \in (x_{n,k}), \quad \sum_{n=1}^{\infty} |\lambda_n| \leq 1 \right\}.$$

Proof of the above theorem can be obtained from the one given for p-Banach spaces in [3]. Clearly this is an analog of Grothendieck's theorem for Q-compact sets (compare with Theorem 1.3 above).

4. Terzioğlu's theorem for Q-compact maps

Terzioğlu's characterization of compact maps relies on both the Grothendieck and Schauder theorems. To obtain the "Terzioğlu type" of a theorem for Q-compact maps one needs to check if these two theorems for Q-compact maps are valid or not. Theorem 3.4 above is Grothendieck's theorem for Q-compact sets; therefore, we turn our attention to seeing whether or not Schauder's theorem is true for Q-compact maps. In other words, we need to understand the relationship between T being Q-compact and its transpose T^* being Q^*-compact. Recall that a map $T \in \mathcal{L}(X,Y)$ is a Q-compact map if

$$\lim_{n\to\infty} \delta_n(T,Q) = 0.$$

As usual, by $\delta_n(T,Q)$ we mean

$$\delta_n(T,Q) = \delta_n(T(U_X),Q) = \inf\{r>0 : \ T(U_X) \subset rU_Y + A \quad \text{for some } A \in Q_n(Y)\},$$

and similarly $\delta_n(T^*,Q^*)$ defined as $\delta_n(T^*,Q^*) = \delta_n(T^*(U_{Y^*}),Q^*)$.

To obtain a "Schauder's type theorem" for Q-compact maps, one seeks a relationship between $\delta_n(T,Q)$ and $\delta_n(T^*,Q^*)$, which is not known. However, Astala in [4] proved that under the assumption that the Banach space X has the lifting property and the Banach space Y has the extension property, for a map $T \in \mathcal{L}(X,Y)$, one has $\gamma(T) = \gamma(T^*)$, where $\gamma(T)$ denotes the measure of noncompactness of T. The relationship between Kolmogorov diameters and the measure of noncompactness can be found in [1], which is expressed as

$$\lim_{n\to\infty} \delta_n(T,Q) = \gamma(T,Q).$$

In the following we present a result analogous to Terzioğlu's characterization theorem for Q-compact maps under the assumption that both T and T^* are Q-compact.

Theorem 4.1 *Let X and Y be Banach spaces, $T \in \mathcal{L}(X,Y)$, and assume that both T and T^* are Q-compact maps. Then there exists a sequence $\{u_{n,k}\} \in Q_n$ with $||u_{n,k}|| \to 0$ for $n \to \infty$ uniformly in k, such that the inequality*

$$||Tx|| \le sup| < u_{n,k(n)}, x > |$$

holds for every $x \in X$. Here Q_n is a "special" class of subsets of X^* with the property that $u_{n,k(n)} \in \{u_{n,k}\}$.

Proof Since $T^* : Y^* \to X^*$ is Q-compact, thus by the Theorem 3.4, $T^*(U_{Y^*})$ is a Q-compact set. Thus, there exists a sequence $\{u_{n,k}\}_k \subset A_n \in Q_n$ such that $||u_{n,k}|| \to 0$ as $n \to \infty$ uniformly in k and

$$T^*(U_{Y^*}) \subset \left\{ \sum_{n=1}^{\infty} \lambda_n u_{n,k(n)} : \quad u_{n,k(n)} \in (u_{n,k}), \quad \sum_{n=1}^{\infty} |\lambda_n| \le 1 \right\}.$$

Then for each $x \in X$, we have

$$||Tx|| = \sup_{v \in U_{Y^*}} | < v, Tx > | = \sup_{v \in U_{Y^*}} | < T^*v, x > | = \sup_n | < \sum_{n=1}^{\infty} \lambda_n u_{n,k(n)}, x > |,$$

and thus

$$||Tx|| \le \sum_{n=1}^{\infty} |\lambda_n| \sup_n | < u_{n,k(n)}, x > | \le \sup_n | < u_{n,k(n)}, x > |.$$

□

Remark 4.2 *Even though the relationship between* $\delta_n(T,Q)$ *and* $\delta_n(T^*,Q^*)$ *is not known, for the closely defined concept of approximation numbers, we know more. The relationship between the approximation numbers of* T *and* T^* *was studied by several authors. It is shown in [11] that for* $T \in \mathcal{L}(X)$, *we have*

$$dist(T,\mathcal{F}) \leq 3\, dist(T^*,\mathcal{F}^*),$$

where $\mathcal{F}$ *and* $\mathcal{F}^*$ *denote the class of all finite rank operators on* X *and* X^*, *respectively. Central to the proof of such a result is the assumption of local reflexivity possessed by all Banach spaces (see [13]). It is not hard to show that if we assume that our space* X, *with approximation scheme* Q_n, *satisfies the slight modification of this property, called the extended local reflexivity principle, then we have*

$$\alpha_n(T,Q) \leq 3\,\alpha_n(T^*,Q^*).$$

By $\alpha_n(T,Q)$ *we mean the* $n\,th$ *approximation number defined with respect to the given approximation scheme as*

$$\alpha_n(T,Q) = \inf\{||T-B||:\ B\in\mathcal{L}(X),\ B(X)\in Q_n(X)\}.$$

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