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A Neutral relation between metallic structure and almost quadratic ϕ-structure

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Abstract: In this paper, we give a neutral relation between metallic structure and almost quadratic metric ϕ-structure. Considering N as a metallic Riemannian manifold, we show that the warped product manifold $\mathbb{R}\times_f N$ has an almost quadratic metric ϕ-structure. We define Kenmotsu quadratic metric manifolds, which include cosymplectic quadratic manifolds when $\beta = 0$. Then we give nice almost quadratic metric ϕ-structure examples. In the last section, we construct a quadratic ϕ-structure on the hypersurface M^n of a locally metallic Riemannian manifold $\tilde{M}^{n+1}$.

Key words: Polynomial structure, golden structure, metallic structure, almost quadratic ϕ-structure

1. Introduction

In [10] and [9], Goldberg and Yano and Goldberg and Petridis respectively defined a new type of structure called a polynomial structure on an n-dimensional differentiable manifold M. The polynomial structure of degree 2 can be given by

$$J^2 = pJ + qI, \tag{1.1}$$

where J is a $(1,1)$ tensor field on M, I is the identity operator on the Lie algebra $\Gamma(TM)$ of vector fields on M, and p,q are real numbers. This structure can be also viewed as a generalization of the following well known structures:

· If $p = 0$, $q = 1$, then J is called an almost product or almost para complex structure and denoted by F [12, 16],;

· If $p = 0$, $q = -1$, then J is called an almost complex structure [18];

· If $p = 1$, $q = 1$, then J is called a golden structure [6, 7];

· If $p \in \mathbb{R} - (-2, 2)$ and $q = -1$, then J is called a poly-Norden structure [17];

· If $p = -1$, $q = \frac{3}{2}$, then J is called an almost complex golden structure [1];

· If p and q are positive integers, then J is called a metallic structure [11].

If a differentiable manifold is endowed with a metallic structure J then the pair (M, J) is called a metallic

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manifold. Any metallic structure J on M induces two almost product structures on M:

$$F_{\pm} = \pm\left(\frac{2}{2\sigma_{p,q}-p}J - \frac{p}{2\sigma_{p,q}-p}I\right),$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number, which is the positive solution of the equation $x^2 - px - q = 0$ for p and q nonzero natural numbers. Conversely, any almost product structure F on M induces two metallic structures on M:

$$J_{\pm} = \pm\frac{2\sigma_{p,q}-p}{2}F + \frac{p}{2}I.$$

If M is Riemannian, the metric g is said to be compatible with the polynomial structure J if

$$g(JX, Y) = g(X, JY) \tag{1.2}$$

for $X, Y \in \Gamma(TM)$. In this case, (g, J) is called a metallic Riemannian structure and (M, g, J) a metallic Riemannian manifold [8]. By (1.1) and (1.2), one can get

$$g(JX, JY) = pg(JX, Y) + qg(X, Y),$$

for $X, Y \in \Gamma(TM)$. The Nijenhuis torsion N_K for arbitrary tensor field K of type $(1,1)$ on M is a tensor field of type $(1,2)$ defined by

$$N_K(X, Y) = K^2[X, Y] + [KX, KY] - K[KX, Y] - K[X, KY], \tag{1.3}$$

where $[X, Y]$ is the commutator for arbitrary differentiable vector fields $X, Y \in \Gamma(TM)$. The polynomial structure J is said to be integrable if $N_J \equiv 0$. A metallic Riemannian structure J is said to be locally metallic if $\nabla J = 0$, where ∇ is the Levi-Civita connection with respect to g. Thus, one can deduce that a locally metallic Riemannian manifold is always integrable.

On the other hand, Debnath and Konar [8] recently introduced a new type of structure named the almost quadratic ϕ-structure (ϕ, η, ξ) on an n-dimensional differentiable manifold M, determined by a $(1,1)$-tensor field ϕ, a unit vector field ξ, and a 1-form η, which satisfy the following relations:

$$\phi\xi = 0,$$

$$\phi^2 = a\phi + b(I - \eta\otimes\xi); \quad a^2 + 4b \neq 0, \tag{1.4}$$

where a is an arbitrary constant and b is a nonzero constant. If M is a Riemannian manifold the Riemannian metric g is said to be compatible with the polynomial structure ϕ if

$$g(\phi X, Y) = g(X, \phi Y),$$

which is equivalent to

$$g(\phi X, \phi Y) = ag(\phi X, Y) + b(g(X, Y) - \eta(X)\eta(Y)). \tag{1.5}$$

In this case, (g, ϕ, η, ξ) is called an almost quadratic metric ϕ-structure. The manifold M is said to be an almost quadratic metric ϕ-manifold if it is endowed with an almost quadratic metric ϕ-structure [8]. They proved the necessary and sufficient conditions for an almost quadratic ϕ-manifold to induce an almost contact or almost paracontact manifold.

Recently, Blaga and Hretcanu [3] characterized the metallic structure on the product of two metallic manifolds in terms of metallic maps and provided a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic. Moreover, Özkan and F. Yılmaz [15] investigated integrability and parallelism conditions for the metallic structure on a differentiable manifold.

This paper is organized in the following way.

Section 2 is the preliminaries section, where we recall some properties of an almost quadratic metric ϕ-structure and warped product manifolds. In Section 3, we define the (β,ϕ)-Kenmotsu quadratic metric manifold and cosymplectic quadratic metric manifold. We mainly prove that if (N,g,∇,J) is a locally metallic Riemannian manifold, then $\mathbb{R}\times_f N$ is a $(-\frac{f'}{f},\phi)$-Kenmotsu quadratic metric manifold, and we show that every differentiable manifold M endowed with an almost quadratic ϕ-structure (ϕ,η,ξ) admits an associated Riemannian metric. We prove that on a (β,ϕ)-Kenmotsu quadratic metric manifold the Nijenhuis tensor $N_\phi\equiv 0$. We also give examples of (β,ϕ)-Kenmotsu quadratic metric manifolds. Section 4 is devoted to quadratic ϕ-hypersurfaces of metallic Riemannian manifolds. We show that there are almost quadratic ϕ-structures on hypersurfaces of metallic Riemannian manifolds. Then we give the necessary and sufficient condition for the characteristic vector field ξ to be Killing in a quadratic metric ϕ-hypersurface. Furthermore, we obtain the Riemannian curvature tensor of a quadratic metric ϕ-hypersurface.

2. Preliminaries

Let M^n be an almost quadratic ϕ-manifold. As in almost contact manifolds, Debmath and Konar [8] proved that $\eta\circ\phi=0, \eta(\xi)=1$, and $rank\ \ \phi=n-1$. They also showed that the eigenvalues of the structure tensor ϕ are $\frac{a+\sqrt{a^2+4b}}{2}$, $\frac{a-\sqrt{a^2+4b}}{2}$, and 0. If λ_i, σ_j, and ξ are eigenvectors corresponding to the eigenvalues $\frac{a+\sqrt{a^2+4b}}{2}$, $\frac{a-\sqrt{a^2+4b}}{2}$, and 0 of ϕ, respectively, then λ_i, σ_j, and ξ are linearly independent. Denote the following distributions:

$\cdot\Pi_r=\{X\in\Gamma(TM):\alpha LX=-\phi^2X-(\frac{\sqrt{a^2+4b}-a}{2})\phi, \alpha=-2b-\frac{a^2+a\sqrt{a^2+4b}}{2}\}; \dim\Pi_r=r,$

$\cdot\Pi_s=\{X\in\Gamma(TM):\beta QX=-\phi^2X+(\frac{\sqrt{a^2+4b}+a}{2})\phi X, \beta=-2b-\frac{a^2-a\sqrt{a^2+4b}}{2}\}; \dim\Pi_s=s,$

$\cdot\Pi_1=\{X\in\Gamma(TM):bRX=\phi^2X-a\phi X-bX=-b\eta(X)\xi\}; \dim\Pi_1=1.$

By the above notations, Debmath and Konar proved following theorem.

Theorem 2.1 ([8]) *The necessary and sufficient condition that a manifold M^n will be an almost quadratic ϕ-manifold is that at each point of the manifold M^n it contains distributions Π_r,Π_s, and Π_1 such that $\Pi_r\cap\Pi_s=\{\varnothing\},\Pi_r\cap\Pi_1=\{\varnothing\},\Pi_s\cap\Pi_1=\{\varnothing\}$, and $\Pi_r\cup\Pi_s\cup\Pi_1=TM$.*

Let (M^m,g_M) and (N^n,g_N) be two Riemannian manifolds and $\tilde{M}=M\times N$. The warped product metric $<,>$ on $\tilde{M}$ is given by

$$<\tilde{X},\tilde{Y}>=g_M(\pi_*\tilde{X},\pi_*\tilde{Y})+(f\circ\pi)^2g_N(\sigma_*\tilde{X},\sigma_*\tilde{Y})$$

for every $\tilde{X}$ and $\tilde{Y}\in\Gamma(T\tilde{M})$ where $f:M\overset{C^\infty}{\to}\mathbb{R}^+$ and $\pi:M\times N\to M$, $\sigma:M\times N\to N$ the canonical projections (see [2]). The warped product manifolds are denoted by $\tilde{M}=(M\times_f N,<,>)$. The function f is called the warping function of the warped product. If the warping function f is 1, then $\tilde{M} = (M \times_f N, <,>)$ reduces the Riemannian product manifold. The manifolds M and N are called the base and the fiber of $\tilde{M}$, respectively. For a point $(p,q) \in M \times N$, the tangent space $T_{(p,q)}(M \times N)$ is isomorphic to the direct sum $T_{(p,q)}(M \times q) \oplus T_{(p,q)}(p \times N) \equiv T_pM \oplus T_qN$. Let $\mathcal{L}_{\mathcal{H}}(M)$ (resp. $\mathcal{L}_{\mathcal{V}}(N)$) be the set of all vector fields on $M \times N$, which is the horizontal lift (resp. the vertical lift) of a vector field on M (a vector field on N). Thus, a vector field on $M \times N$ can be written as $\bar{E} = \bar{X} + \bar{U}$, with $\bar{X} \in \mathcal{L}_{\mathcal{H}}(M)$ and $\bar{U} \in \mathcal{L}_{\mathcal{V}}(N)$. One can see that

$$\pi_*(\mathcal{L}_{\mathcal{H}}(M)) = \Gamma(TM) \ , \ \sigma_*(\mathcal{L}_{\mathcal{V}}(N)) = \Gamma(TN)$$

and so $\pi_*(\bar{X}) = X \in \Gamma(TM)$ and $\sigma_*(\bar{U}) = U \in \Gamma(TN)$. If $\bar{X}, \bar{Y} \in \mathcal{L}_{\mathcal{H}}(M)$, then $[\bar{X}, \bar{Y}] = \overline{[X,Y]} \in \mathcal{L}_{\mathcal{H}}(M)$ and similarly for $\mathcal{L}_{\mathcal{V}}(N)$, and also if $\bar{X} \in \mathcal{L}_{\mathcal{H}}(M), \bar{U} \in \mathcal{L}_{\mathcal{V}}(N)$ then $[\bar{X}, \bar{U}] = 0$ [13].

The Levi-Civita connection $\bar{\nabla}$ of $M \times_f N$ is related to the Levi-Civita connections of M and N as follows:

Proposition 2.2 ([13]) *For $\bar{X}, \bar{Y} \in \mathcal{L}_{\mathcal{H}}(M)$ and $\bar{U}, \bar{V} \in \mathcal{L}_{\mathcal{V}}(N)$,*

(a) $\bar{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{L}_{\mathcal{H}}(M)$ *is the lift of* ${}^M\nabla_X Y$*, that is,* $\pi_*(\bar{\nabla}_{\bar{X}}\bar{Y}) = {}^M\nabla_X Y$*;*

(b) $\bar{\nabla}_{\bar{X}}\bar{U} = \bar{\nabla}_{\bar{U}}\bar{X} = \frac{X(f)}{f}U$*;*

(c) $\bar{\nabla}_{\bar{U}}\bar{V} = {}^N\nabla_U V - \frac{<U,V>}{f} gradf$*, where* $\sigma_*(\bar{\nabla}_{\bar{U}}\bar{V}) = {}^N\nabla_U V$.

Here the notation is simplified by writing f for $f \circ \pi$ and $gradf$ for $grad(f \circ \pi)$.

Now we consider the special warped product manifold

$$\tilde{M} = I \times_f N, \quad <,>= dt^2 + f^2(t) g_N.$$

In practice, $(-)$ is omitted from lifts. In this case,

$$\tilde{\nabla}_{\partial_t}\partial_t = 0, \tilde{\nabla}_{\partial_t} X = \tilde{\nabla}_X \partial_t = \frac{f'(t)}{f(t)} X \text{ and } \tilde{\nabla}_X Y = {}^N\nabla_X Y - \frac{< X, Y >}{f(t)} f'(t)\partial_t. \tag{2.1}$$

3. Almost quadratic metric ϕ-structure

Let (N, g, J) be a metallic Riemannian manifold with metallic structure J. By (1.1) and (1.2) we have

$$g(JX, JY) = pg(X, JY) + qg(X, Y).$$

Let us consider the warped product $\tilde{M} = \mathbb{R} \times_f N$, with warping function $f > 0$, endowed with the Riemannian metric

$$<,>= dt^2 + f^2 g.$$

Now we will define an almost quadratic metric ϕ-structure on $(\tilde{M}, \tilde{g})$ by using a method similar to that in [5]. Denote arbitrarily any vector field on $\tilde{M}$ by $\tilde{X} = \eta(\tilde{X})\xi + X$, where X is any vector field on N and $dt = \eta$. By the help of tensor field J, a new tensor field ϕ of type $(1,1)$ on $\tilde{M}$ can be given by

$$\phi\tilde{X} = JX, \quad X \in \Gamma(TN), \tag{3.1}$$

for $\tilde{X} \in \Gamma(T\tilde{M})$. Thus, we get $\phi\xi = \phi(\xi + 0) = J0 = 0$ and $\eta(\phi\tilde{X}) = 0$, for any vector field $\tilde{X}$ on $\tilde{M}$. Hence, we obtain

$$\phi^2\tilde{X} = p\phi\tilde{X} + q(\tilde{X} - \eta(\tilde{X})\xi) \tag{3.2}$$

and arrive at

$$\begin{aligned} <\ \phi\tilde{X}\ ,\tilde{Y}> &= f^2 g(JX,Y) \\ &= f^2 g(X,JY) \\ &= <\tilde{X}\ ,\phi\tilde{Y}>, \end{aligned}$$

for $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$. Moreover, we get

$$\begin{aligned} <\ \phi\tilde{X},\phi\tilde{Y}> &= f^2 g(JX,JY) \\ &= f^2(pg(X,JY) + qg(X,Y)) \\ &= p<\tilde{X} - \eta(\tilde{X})\xi,\phi\tilde{Y}> + q(<\tilde{X},\tilde{Y}> - \eta(\tilde{X})\eta(\tilde{Y})) \\ &= p<\tilde{X},\phi\tilde{Y}> + q(<\tilde{X},\tilde{Y}> - \eta(\tilde{X})\eta(\tilde{Y})). \end{aligned}$$

Thus, we have proved the following proposition.

Proposition 3.1 *If (N, g, J) is a metallic Riemannian manifold, then there is an almost quadratic metric ϕ-structure on warped product manifold $(\tilde{M} = \mathbb{R} \times_f N, <,>= dt^2 + f^2 g)$.*

An almost quadratic metric ϕ-manifold $(M, g, \nabla, \phi, \xi, \eta)$ is called a (β, ϕ)-Kenmotsu quadratic metric manifold if

$$(\nabla_X\phi)Y = \beta\{g(X,\phi Y)\xi + \eta(Y)\phi X\}, \beta \in C^\infty(M). \tag{3.3}$$

Taking $Y = \xi$ in (3.3) and using (1.4), we obtain

$$\nabla_X\xi = -\beta(X - \eta(X)\xi). \tag{3.4}$$

Moreover, by (3.4) we get $d\eta = 0$. If $\beta = 0$, then this kind of manifold is called a cosymplectic quadratic manifold.

Theorem 3.2 *If (N, g, ∇, J) is a locally metallic Riemannian manifold, then $\mathbb{R} \times_f N$ is a $(-\frac{f'}{f}, \phi)$-Kenmotsu quadratic metric manifold.*

Proof We consider $\tilde{X} = \eta(\tilde{X})\xi + X$ and $\tilde{Y} = \eta(\tilde{Y})\xi + Y$ vector fields on $\mathbb{R} \times_f N$, where $X, Y \in \Gamma(TN)$ and $\xi = \frac{\partial}{\partial t} \in \Gamma(\mathbb{R})$. By help of (3.1), we have

$$\begin{aligned} (\tilde{\nabla}_{\tilde{X}}\phi)\tilde{Y} &= \tilde{\nabla}_{\tilde{X}}\phi\tilde{Y} - \phi\tilde{\nabla}_{\tilde{X}}\tilde{Y} \\ &= \tilde{\nabla}_X JY + \eta(\tilde{X})\tilde{\nabla}_\xi JY - \phi(\tilde{\nabla}_X\tilde{Y} + \eta(\tilde{X})\tilde{\nabla}_\xi\tilde{Y}) \\ &= \tilde{\nabla}_X JY + \eta(\tilde{X})\tilde{\nabla}_\xi JY - \phi(\tilde{\nabla}_X Y + X(\eta(\tilde{Y}))\xi + \eta(\tilde{Y})\tilde{\nabla}_X\xi \\ &\quad + \eta(\tilde{X})\tilde{\nabla}_\xi Y + \xi(\eta(\tilde{Y}))\eta(\tilde{X})\xi). \end{aligned} \tag{3.5}$$

Using (2.1) in (3.5), we get

$$(\tilde{\nabla}_{\tilde{X}}\phi)\tilde{Y} = (\nabla_X J)Y - \frac{f'}{f} < X, JY > \xi + \eta(\tilde{X})\frac{f'}{f}JY - \phi(\eta(\tilde{Y})\frac{f'}{f}X + \eta(\tilde{X})\frac{f'}{f}Y)$$
$$= (\nabla_X J)Y - \frac{f'}{f}(< \tilde{X}, \phi\tilde{Y} > \xi + \eta(\tilde{Y})\phi\tilde{X}).$$

Since $\nabla J = 0$, the last equation is reduced to

$$(\tilde{\nabla}_{\tilde{X}}\phi)\tilde{Y} = -\frac{f'}{f}(< \tilde{X}, \phi\tilde{Y} > \xi + \eta(\tilde{Y})\phi\tilde{X}). \tag{3.6}$$

Using $\tilde{\nabla}_X\xi = \frac{f'}{f}X$, we have

$$\tilde{\nabla}_{\tilde{X}}\xi = \frac{f'}{f}(\tilde{X} - \eta(\tilde{X})\xi).$$

Thus, $\mathbb{R} \times_f N$ is a $(-\frac{f'}{f}, \phi)$-Kenmotsu quadratic metric manifold. □

Corollary 3.3 *Let (N, g, ∇, J) be a locally metallic Riemannian manifold. Then product manifold $\mathbb{R} \times N$ is a cosymplectic quadratic metric manifold.*

Example 3.4 *Blaga and Hretcanu [3] constructed a metallic structure on $\mathbb{R}^{n+m}$ in the following manner:*

$$J(x_1, ..., x_n, y_1, ..., y_m) = (\sigma x_1, ..., \sigma x_n, \bar{\sigma} y_1, ..., \bar{\sigma} y_m),$$

where $\sigma = \sigma_{p,q} = \frac{p+\sqrt{p^2+4pq}}{2}$ and $\bar{\sigma} = \bar{\sigma}_{p,q} = \frac{p-\sqrt{p^2+4pq}}{2}$ for p, q positive integers. By Theorem 3.2 $H^{n+m+1} = \mathbb{R} \times_{e^t} \mathbb{R}^{n+m}$ is a $(-1, \phi)$-Kenmotsu quadratic metric manifold.

M is said to be metallic shaped hypersurface in a space form $N^{n+1}(c)$ if the shape operator A of M is a metallic structure (see [14]).

Example 3.5 *In [14], Özgür and Yılmaz Özgür proved that an $S^n(\frac{2}{p+\sqrt{p^2+4pq}})$ sphere is a locally metallic shaped hypersurfaces in $\mathbb{R}^{n+1}$. Using Theorem 3.2, we have*

$$H^{n+1} = \mathbb{R} \times_{\cosh(t)} S^n(\frac{2}{p + \sqrt{p^2 + 4q)}}),$$

a $(-\tanh t, \phi)$-Kenmotsu quadratic metric manifold.

Example 3.6 *Debnath and Konar [8] gave an example of an almost quadratic ϕ-structure on $\mathbb{R}^4$ as follows:*

If the $(1,1)$ tensor field ϕ, 1-form η, and vector field ξ are defined as

$$\phi = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 9 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \eta = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$\phi^2 = 4\phi + 5(I_4 - \eta \otimes \xi).$$

Thus, $\mathbb{R}^4$ *has an almost quadratic* ϕ*-structure.*

Theorem 3.7 *Every differentiable manifold* M *endowed with an almost quadratic* ϕ*-structure* (ϕ, η, ξ) *admits an associated Riemannian metric.*

Proof Let $\tilde{h}$ be any Riemannian metric. Putting

$$h(X,Y) = \tilde{h}(\phi^2 X, \phi^2 Y) + \eta(X)\eta(Y),$$

we have $\eta(X) = h(X, \xi)$. We now define g by

$$g(X,Y) = \frac{1}{\alpha+\delta}[\alpha h(X,Y) + \beta h(\phi X, \phi Y) + \frac{\gamma}{2}(h(\phi X, Y) + h(X, \phi Y)) + \delta\eta(X)\eta(Y)],$$

where $\alpha, \beta, \gamma, \delta, q$ are nonzero constants satisfying $\beta q = p\frac{\gamma}{2} + \alpha,\ \alpha + \delta \neq 0$. It is clearly seen that

$$g(\phi X, \phi Y) = pg(\phi X, Y) + q(g(X,Y) - \eta(X)\eta(Y))$$

for any $X, Y \in \Gamma(TM)$. □

Remark 3.8 *If we choose* $\alpha = \delta = q, \beta = \gamma = 1$*, then we have* $p = 0$*. In this case, we obtain Theorem 4.1 of [8].*

Proposition 3.9 *Let* $(M, g, \nabla, \phi, \xi, \eta)$ *be a* (β, ϕ)*-Kenmotsu quadratic metric manifold. Then quadratic structure* ϕ *is integrable; that is, the Nijenhuis tensor* $N_\phi \equiv 0$.

Proof Using (3.2) in (1.3), we have

$$\begin{aligned} N_\phi(X,Y) &= \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ &= p\phi[X,Y] + q([X,Y] - \eta([X,Y])\xi) + \tilde{\nabla}_{\phi X}\phi Y \\ &\quad -\nabla_{\phi Y}\phi X - \phi(\nabla_{\phi X}Y - \nabla_Y \phi X) - \phi(\nabla_X \phi Y - \nabla_{\phi Y} X) \\ &= p\phi\nabla_X Y - p\phi\nabla_Y X + q\nabla_X Y - q\nabla_Y X - q\eta([X,Y])\xi) \\ &\quad +(\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + \phi\nabla_Y\phi X - \phi\nabla_X\phi Y \end{aligned} \tag{3.7}$$

for $X, Y \in \Gamma(TM)$. By using (3.2), we have

$$\begin{aligned} p\phi\nabla_X Y - \phi\nabla_X\phi Y &= p\phi\nabla_X Y + (\nabla_X\phi)\phi Y - \nabla_X\phi^2 Y \\ &= -p(\nabla_X\phi)Y + (\nabla_X\phi)\phi Y - q\nabla_X Y; \\ &\quad +qX(\eta(Y))\xi + q(\eta(Y))\nabla_X\xi. \end{aligned}$$

If we write the last equation in (3.7), we get

$$\begin{aligned} N_\phi(X,Y) &= -p(\nabla_X\phi)Y + p(\nabla_Y\phi)X + (\nabla_X\phi)\phi Y - (\nabla_Y\phi)\phi X \\ &\quad +(\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + q(X\eta(Y)\xi - Y\eta(X)\xi - \eta([X,Y])\xi) \\ &\quad +q(\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi). \end{aligned} \tag{3.8}$$

Employing (3.6) and (3.2) in (3.8), we deduce that

$$\begin{aligned} N_\phi(X,Y) &= q(X\eta(Y)\xi - Y\eta(X)\xi - \eta([X,Y])\xi) \\ &= 0. \end{aligned}$$

This completes the proof of the theorem. □

4. Quadratic metric ϕ-hypersurfaces of metallic Riemannian manifolds

Theorem 4.1 *Let $\tilde{M}^{n+1}$ be a differentiable manifold with metallic structure J and M^n be a hypersurface of $\tilde{M}^{n+1}$. Then there is an almost quadratic ϕ-structure (ϕ, η, ξ) on M^n.*

Proof Denote by ν a unit normal vector field of M^n. For any vector field X tangent to M^n, we put

$$JX = \phi X + \eta(X)\nu, \tag{4.1}$$

$$J\nu = q\xi + p\nu, \tag{4.2}$$

$$J\xi = \nu, \tag{4.3}$$

where ϕ is a $(1,1)$ tensor field on M^n, $\xi \in \Gamma(TM)$ and η is a 1-form such that $\eta(\xi) = 1$ and $\eta \circ \phi = 0$. On applying operator J to the above equality (4.1) and using (4.2), we have

$$\begin{aligned} J^2X &= J(\phi X) + \eta(X)J\nu \\ &= \phi^2 X + \eta(X)(q\xi + p\nu). \end{aligned} \tag{4.4}$$

Using (1.1) in (4.4),

$$p\phi X + p\eta(X)\nu + qX = \phi^2 X + \eta(X)(q\xi + p\nu).$$

Hence, we are led to the conclusion:

$$\phi^2 X = p\phi X + q(X - \eta(X)\xi). \tag{4.5}$$

□

Let M^n be a hypersurface of an $n+1$-dimensional metallic Riemannian manifold $\tilde{M}^{n+1}$ and let ν be a globally unit normal vector field on M^n. Denote $\tilde{\nabla}$ the Levi-Civita connection with respect to the Riemannian metric $\tilde{g}$ of $\tilde{M}^{n+1}$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\nu,$$

$$\tilde{\nabla}_X \nu = -AX$$

for any $X, Y \in \Gamma(TM)$, where g denotes the Riemannian metric of M^n induced from $\tilde{g}$ and A is the shape operator of M^n.

Proposition 4.2 *Let $(\tilde{M}^{n+1}, <,>, \tilde{\nabla}, J)$ be a locally metallic Riemannian manifold. If (M^n, g, ∇, ϕ) is a quadratic metric ϕ-hypersurface of $\tilde{M}^{n+1}$, then*

$$(\nabla_X \phi)Y = \eta(Y)AX + g(AX, Y)\xi, \tag{4.6}$$

$$\nabla_X \xi = pAX - \phi AX, \;\; A\xi = 0, \tag{4.7}$$

and

$$(\nabla_X \eta)Y = pg(AX, Y) - g(AX, \phi Y). \tag{4.8}$$

Proof If we take the covariant derivatives of the metallic structure tensor J according to X by (4.1)–(4.3), the Gauss and Weingarten formulas, we get

$$\begin{aligned} 0 &= (\nabla_X \phi)Y - \eta(Y)AX - qg(AX, Y)\xi \\ &\quad +(g(AX, \phi Y) + X(\eta(Y)) - \eta(\nabla_X Y) - pg(AX, Y))\nu. \end{aligned} \tag{4.9}$$

If we identify the tangential components and the normal components of the equation (4.9), respectively, we have

$$(\nabla_X \phi)Y - \eta(Y)AX - qg(AX, Y)\xi = 0. \tag{4.10}$$

$$g(AX, \phi Y) + X(\eta(Y)) - \eta(\nabla_X Y) - pg(AX, Y) = 0.$$

Using the compatible condition of J and (4.1), we get

$$\begin{aligned} g(JX, JY) &= pg(X, JY) + qg(X, Y) \\ &= pg(X, \phi Y) + qg(X, Y). \end{aligned} \tag{4.11}$$

Expressed in another way, by help of (1.5) and (4.1), we obtain

$$\begin{aligned} g(JX, JY) &= g(\phi X, \phi Y) + \eta(X)\eta(Y) \\ &= pg(X, \phi Y) + q(g(X, Y) - \eta(X)\eta(Y)) + \eta(X)\eta(Y) \\ &= pg(X, \phi Y) + qg(X, Y) + (1 - q)\eta(X)\eta(Y)). \end{aligned} \tag{4.12}$$

Considering (4.11) and (4.12), we get $q = 1$. By (4.10) we arrive at (4.6). If we put $Y = \xi$ in (4.10) we get

$$\phi \nabla_X \xi = -AX - g(AX, \xi)\xi. \tag{4.13}$$

If we apply ξ on both sides of (4.13), we have $A\xi = 0$.

Applying ϕ on both sides of the equation (4.13) and using $A\xi = 0$,

$$\begin{aligned} -\phi AX &= p\phi \nabla_X \xi + (\nabla_X \xi - \eta(\nabla_X \xi)\xi) \\ &= -pAX + \nabla_X \xi. \end{aligned}$$

Hence, we arrive at the first equation of (4.7). By help of (4.7), we readily obtain (4.8). This completes the proof. □

Proposition 4.3 ([4]) *Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on M induced by g. For every vector field X on M, the following conditions are equivalent:*

(1) X is a Killing vector field; that is, $L_X g = 0$.

(2) $g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$ for all $Y, Z \in \chi(M)$.

Proposition 4.4 *Let $(M^n, g, \nabla, \phi, \eta, \xi)$ be a quadratic metric ϕ-hypersurface of a locally metallic Riemannian manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\nabla}, J)$. The characteristic vector field ξ is a Killing vector field if and only if $\phi A + A\phi = 2pA$.*

Proof From Proposition 4.3, we have

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0.$$

Making use of (4.7) in the last equation, we get

$$pg(AX, Y) - g(\phi AX, Y) + pg(AY, X) - g(\phi AY, X) = 0.$$

Using the symmetric property of A and ϕ, we obtain

$$2pg(AX, Y) = g(\phi AX, Y) + g(A\phi X, Y). \tag{4.14}$$

We arrive at the desired equation from (4.14). □

Proposition 4.5 *If $(M^n, g, \nabla, \phi, \xi)$ is a (β, ϕ)-Kenmotsu quadratic hypersurface of a locally metallic Riemannian manifold on $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\nabla}, J)$, then $\phi A = A\phi$ and $A^2 = \beta pA + \beta^2(I - \eta \otimes \xi)$.*

Proof Since $d\eta = 0$, using (4.7), we have

$$\begin{aligned} 0 &= g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi) \\ &= pg(Y, AX) - g(Y, \phi AX) - pg(X, AY) + g(X, \phi AY) \\ &= g(A\phi X - \phi AX, Y). \end{aligned}$$

Thus, we get $\phi A = A\phi$. By (3.3) and (4.6), we get

$$\beta(g(X, \phi Y)\xi + \eta(Y)\phi X) = \eta(Y)AX + g(AX, Y)\xi.$$

If we apply ξ on both sides of the last equation, we obtain

$$\beta g(X, \phi Y) = g(AX, Y).$$

Namely,

$$\beta\phi X = AX. \tag{4.15}$$

Putting AX instead of X and using (4.5) in (4.15), we get $A^2X = \beta pAX + \beta^2(X - \eta(X)\xi)$. This completes the proof. □

By help of (4.15) we obtain the following:

Corollary 4.6 *Let $(M^n, g, \nabla, \phi, \xi)$ be a cosymplectic quadratic metric ϕ-hypersurface of a locally metallic Riemannian manifold. Then M is totally geodesic.*

Remark 4.7 *Hretcanu and Crasmareanu [11] investigated some properties of the induced structure on a hypersurface in a metallic Riemannian manifold, but the argument in Proposition 4.2 is to get the quadratic ϕ-hypersurface of a metallic Riemannian manifold. In the same paper, they proved that the induced structure on M is parallel to the induced Levi-Civita connection if and only if M is totally geodesic.*

By Proposition 4.2, we have the following.

Proposition 4.8 *Let $(M^n, g, \nabla, \phi, \xi)$ be a quadratic metric ϕ-hypersurface of a locally metallic Riemannian manifold. Then*

$$R(X,Y)\xi = p((\nabla_X A)Y - (\nabla_Y A)X) - \phi((\nabla_X A)Y - (\nabla_Y A)X),$$

for any $X, Y \in \Gamma(TM)$.

Corollary 4.9 *Let $(M^n, g, \nabla, \phi, \xi)$ be a quadratic metric ϕ-hypersurface of a locally metallic Riemannian manifold. If the second fundamental form is parallel, then $R(X,Y)\xi = 0$.*

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