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Notes on certain analytic functions

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Abstract: Let $\mathcal{A}(n)$ be the class of functions

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots (n \in \mathbb{N}),$$

which are analytic in the open unit disk $\mathbb{U}$, where $a_n \neq 0$. For $f(z) \in \mathcal{A}(n)$, Miller and Mocanu in 1978 showed a very interesting result for $f(z)$. Applying the result due to Miller and Mocanu, we would like to consider some new results for such functions. Our results in this paper are generalizations for results by Nunokawa in 1992.

Key words: Analytic function, the class $\mathcal{S}(n,\alpha)$, the class $\mathcal{K}(n,\alpha)$

1. Introduction

Let $\mathcal{A}_0(n)$ be the class of functions $f(z)$ of the form

$$f(z) = f(0) + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where $a_n \neq 0$ and $f(0) > 0$. Also, let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.2}$$

which are analytic in $\mathbb{U}$, where $a_n \neq 0$. If $f(z) \in \mathcal{A}(n)$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some real α $(0 \leq \alpha < n)$, then $f(z)$ is said to be in the class $\mathcal{S}(n,\alpha)$. Furthermore, if $f(z) \in \mathcal{A}(n)$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.4}$$

for some real α $(0 \leq \alpha < n)$, then $f(z)$ is said to be in the class $\mathcal{K}(n,\alpha)$. Then $f(z) \in \mathcal{K}(n,\alpha)$ if and only if $\frac{1}{n} zf'(z) \in \mathcal{S}(n,\alpha)$. Also, we say that $\mathcal{S}(n,0) = \mathcal{S}(n)$ and $\mathcal{K}(n,0) = \mathcal{K}(n)$ for $\alpha = 0$.

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For $f(z) \in \mathcal{A}(n)$, we have to recall here the following result due to Miller and Mocanu [2].

Lemma 1.1 *Let $f(z)$ be in the class $\mathcal{A}(n)$. If $|f(z)|$ attains its maximum value on the circle $|z| = r_0 < 1$ at a point z_0, then*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k \tag{1.5}$$

and

$$\mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right) \geq k, \tag{1.6}$$

where $k \geq n$.

If $n = 1$ in Lemma 1.1, we have the result due to Jack [1]. Therefore, Lemma 1.1 is a generalization of the result by Jack [1].

Applying the result by Jack [1] for $n = 1$, Nunokawa [3] showed the following result.

Theorem A *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\mathrm{Re}p(z) > 0 \quad (|z| < |z_0| < 1) \tag{1.7}$$

and

$$\mathrm{Re}p(z_0) = 0, \quad p(z_0) \neq 0, \tag{1.8}$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik, \tag{1.9}$$

where k is real and $|k| \geq 1$.

In the present paper, we would like to consider a generalization for Theorem A applying Lemma 1.1.

2. Generalization results

Applying Lemma 1.1 by Miller and Mocanu [2], we show the following.

Theorem 2.1 *For $f(z) \in \mathcal{A}_0(n)$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\mathrm{Re}f(z) > 0 \quad (|z| < |z_0| < 1) \tag{2.1}$$

and

$$\mathrm{Re}f(z_0) = 0, \quad f(z_0) \neq 0. \tag{2.2}$$

Then we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik, \tag{2.3}$$

where k is real and $|k| \geq n$.

Proof Let us define a function $w(z)$ by

$$w(z) = \frac{f(0) - f(z)}{f(0) + f(z)} \tag{2.4}$$

for $f(z) \in \mathcal{A}_0(n)$. Since $w(0) = 0$, we have that $w(z) \in \mathcal{A}(n)$ and

$$\frac{zw'(z)}{w(z)} = \frac{-2zf'(z)f(0)}{f^2(0) - f^2(z)} \quad (z \in \mathbb{U}). \tag{2.5}$$

For such $w(z)$, we see that $w(0) = 0$, $|w(z)| < 1$ $(|z| < |z_0| < 1)$, and $|w(z_0)| = 1$. Since $\mathrm{Re} f(z_0) = 0$, we write that $f(z_0) = i\alpha$ $(\alpha \in \mathbb{R})$. Applying Lemma 1.1, we have that

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{-2z_0 f'(z_0) f(0)}{f^2(0) + \alpha^2} = k \quad (k \geq n); \tag{2.6}$$

that is,

$$z_0 f'(z_0) = -\frac{k}{2f(0)} \left(f^2(0) + \alpha^2\right). \tag{2.7}$$

Therefore, if $\alpha > 0$, we obtain that

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{k\left(f^2(0) + \alpha^2\right)}{i2\alpha f(0)} = \frac{k\left(f^2(0) + \alpha^2\right)}{2\alpha f(0)} i. \tag{2.8}$$

If $\alpha < 0$, then

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{k\left(f^2(0) + \alpha^2\right)}{i2\alpha f(0)} = -\frac{k\left(f^2(0) + \alpha^2\right)}{2|\alpha| f(0)} i. \tag{2.9}$$

If follows from the above that

$$\mathrm{Im}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = \frac{k\left(f^2(0) + \alpha^2\right)}{2\alpha f(0)} \geq k \quad (\alpha > 0)$$

and

$$\mathrm{Im}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = -\frac{k\left(f^2(0) + \alpha^2\right)}{2|\alpha| f(0)} \leq -k \quad (\alpha < 0).$$

Since $\mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = 0$, we have that

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik \quad (|k| \geq n). \tag{2.10}$$

□

Remark 2.2 *If $f(0) = 1$ and $n = 1$ in Theorem 2.1, then we have Theorem A by Nunokawa [3].*

Theorem 2.3 *Let $f(z) \in \mathcal{A}_0(n)$ satisfy the same conditions in Theorem 2.1. If $f(0) > 0$, we have*

$$\mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right) \geq 0. \tag{2.11}$$

Proof Defining the function $w(z)$ by (2.4) for $f(z) \in \mathcal{A}_0(n)$, we have that $w(0) = 0$, $|w(z)| < 1$ $(|z| < |z_0| < 1)$, and $|w(z_0)| = 1$. Note that

$$\frac{z_0 w''(z_0)}{w'(z_0)} = \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{2z_0 f'(z_0)}{f(0) + f(z_0)}. \tag{2.12}$$

Using Lemma 1.1 for $w(z)$, we see that

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)} + 1\right) &= \operatorname{Re}\left\{\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right) - \frac{2z_0 f'(z_0)}{f(0) + f(z_0)}\right\} \\ &\geq k \quad (k \geq n). \end{aligned} \tag{2.13}$$

This gives us that

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right) &\geq k + \operatorname{Re}\left(\frac{2z_0 f'(z_0)}{f(0) + f(z_0)}\right) \\ &= k + \operatorname{Re}\left(\frac{-k\left(f^2(0) + \alpha^2\right)}{(f(0) + i\alpha) f(0)}\right) \\ &= k - k = 0 \end{aligned} \tag{2.14}$$

where $k \geq n$, $f(z_0) = i\alpha$ $(\alpha \in \mathbb{R})$, and $f(0) > 0$. □

Furthermore, we have:

Theorem 2.4 *If* $f(z) \in \mathcal{A}_0(n)$ *satisfies*

$$\left|\operatorname{Im}\left(\frac{z f'(z)}{f(z)}\right)\right| < n \quad (z \in \mathbb{U}), \tag{2.15}$$

then $\operatorname{Re} f(z) > 0$ $(z \in \mathbb{U})$.

Proof It follows from (2.15) that $f(z) \neq 0$ $(z \in \mathbb{U})$. Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re} f(z_0) = 0. \tag{2.16}$$

Since $f(z_0) \neq 0$, $f(z)$ satisfies the conditions of Theorem 2.1, we say that

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik \quad (k \geq n); \tag{2.17}$$

that is,

$$\left|\operatorname{Im}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)\right| = k \geq n. \tag{2.18}$$

This contradicts our condition (2.15). Thus, there is no $z_0 \in \mathbb{U}$ such that $\operatorname{Re} f(z_0) = 0$ $(z_0 \in \mathbb{U})$. This means that $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{U}$.

Remark 2.5 *Letting* $n = 1$ *in Theorem 2.4, we have Theorem 2 by Nunokawa [3].*

□

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