Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 284 – 292
© TÜBİTAK
doi:10.3906/mat-1808-30

Univalent harmonic mappings and Hardy spaces

Ali EBADIAN[1], **Saman AZIZI**[2], **Sibel YALÇIN**[3*]
[1]Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, Iran
[2]Department of Mathematics, Payame Noor University, Tehran, Iran
[3]Department of Mathematics, Faculty of Arts and Sciences, Bursa Uludağ University, Bursa, Turkey

Received: 07.08.2018 • **Accepted/Published Online:** 29.11.2018 • **Final Version:** 18.01.2019

Abstract: The main purpose of this paper is to establish a relationship between univalent harmonic mappings and Hardy spaces. The main result obtained in this paper improves previously published results. Moreover, we generalize some nice results in the analytic case to the harmonic case.

Key words: Univalent harmonic mapping, Hardy space

1. Introduction

Let Ω be a domain in the complex plane $\mathbb{C}$ and f be a complex-valued function of class C^1 in Ω. The Jacobian of f is given by

$$J_f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2.$$

It is well known that f is locally univalent if $J_f(z) \neq 0$ in Ω and the converse is also true if f is analytic. A theorem of Lewy [9] asserts that the converse remains true for harmonic mappings in the plane. Thus, a locally univalent harmonic mapping is either sense-preserving (if $J_f(z) > 0$ in Ω) or sense-reversing (if $J_f(z) < 0$). A harmonic mapping of the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ has the unique representation $f = h + \overline{g}$, where h and g are analytic in U and $g(0) = 0$. This is called the canonical representation of f. Note that f is sense-preserving if and only if its dilatation $w_f(z) = g'(z)/h'(z)$ satisfies the inequality $|w_f(z)| < 1$ for all $z \in U$. This implies that $h'(z) \neq 0$ in U, so that h is locally univalent.

Let $\mathcal{H}$ be the class of harmonic mappings $f = h + \overline{g}$ in the open unit disk U such that $h(0) = g(0) = h'(0) - 1 = 0$. Therefore, a function $f = h + \overline{g}$ in the class $\mathcal{H}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n. \tag{1}$$

We also denote the class of analytic functions f in U with $f(0) = f'(0) - 1 = 0$ by $\mathcal{A}$ so that $\mathcal{H}$ reduces to $\mathcal{A}$ when the co-analytic part g of $f = h + \overline{g} \in \mathcal{H}$ vanishes identically in U.

The class of functions $f \in \mathcal{H}$ that are sense-preserving and univalent in U is denoted by $\mathcal{S}_H$. Two

*Correspondence: syalcin@uludag.edu.tr
2010 *AMS Mathematics Subject Classification:* 30C45, 30C50, 30H10

interesting subsets of $\mathcal{S}_H$ are

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H \ : \ b_1 = f_{\overline{z}}(0) = 0\}, \quad \mathcal{S} = \{f \in \mathcal{S}_H \ : \ g(z) \equiv 0\}.$$

In recent years, properties of the class $\mathcal{S}_H$ together with its interesting geometric subclasses have been the subject of investigations. We refer to the pioneering works of Clunie and Sheil-Small [4], the book of Duren [5], and the recent survey article of Bshouty and Hengartner [3].

We denote by $\mathcal{K}_H^0$ and S_H^{*0} the subclasses of S_H^0 whose functions map U onto convex and starlike domains.

Set

$$A = \sup_{\mathcal{S}_H} |a_2|. \tag{2}$$

An analytic function k in U is called Bloch if

$$\sup_{z\in U}(1-|z|^2)|k'(z)| < \infty.$$

Using the Koebe transform of f and the compactness of $\mathcal{S}_H$ in the topology of almost uniform convergence, Abu Muhanna and Lyzzaik [2] proved the following result:

Theorem 1.1 *Let $f = h + \overline{g} \in \mathcal{S}_H$. Then $\log h'$ is a Bloch function; that is,*

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \frac{2A+2}{1-r}, \quad (z = re^{i\theta}),$$

where A is as defined in (2). Moreover, if $\alpha > 0$, then

$$\lim_{r\to 1^-}(1-r)^\alpha h'(re^{i\theta}) = 0,$$

for almost all θ.

As a consequence of this result, Abu Muhanna and Lyzzaik [2] concluded the boundary functions of h, g, and f exist almost everywhere.

Theorem 1.2 *Let $f = h + \overline{g} \in \mathcal{S}_H$. Then the integrals*

$$\int_0^1 |h'(re^{i\theta})|dr, \ \int_0^1 |g'(re^{i\theta})|dr, \ \text{and} \ \int_0^1 |f_r'(re^{i\theta})|dr$$

converge for almost all θ, and the boundary function

$$\hat{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$$

exists almost everywhere.

More generally, Abu Muhanna and Lyzzaik [2] established that h, g, and f belong to Hardy spaces.

Theorem 1.3 *Let $f = h + \overline{g} \in \mathcal{S}_H$. Then $h, g \in H^p$ and $f \in h^p$ for every $p, 0 < p < (2A+2)^{-2}$, where A is as defined in (2).*

This result was subsequently improved by Nowak [10] by showing that $h, g \in H^p$ and $f \in h^p$ for every $p, 0 < p < A^{-2}$.

In this paper, by using a different method, we improve the result obtained by Nowak.
Here, we recall the notions of linear and affine invariance. Linear invariance was first studied by Pommerenke [11] for families of locally univalent analytic functions. Sheil-Small [14] then generalized the notion to families of harmonic mappings. A family $\mathcal{F} \subseteq \mathcal{S}_H$ of harmonic mappings is said to be linearly invariant if $f = h + \overline{g} \in \mathcal{F}$ implies that

$$\frac{f((z+z_0)/(1+\overline{z_0}z)) - f(z_0)}{(1-|z_0|^2)h'(z_0)} \in \mathcal{F}, \quad z_0 \in U.$$

The family $\mathcal{F}$ is affine invariant if $f \in \mathcal{F}$ implies that

$$\frac{f(z) + \varepsilon \overline{f(z)}}{1 + \varepsilon g'(0)} \in \mathcal{F}, \quad \varepsilon \in U.$$

The full family $\mathcal{S}_H$ is both linearly and affine invariant.
The order of a family $\mathcal{F} \subseteq \mathcal{S}_H$ is defined by

$$\alpha = \alpha(\mathcal{F}) = \sup\left\{ \frac{|h''(0)|}{2} \ : \ f = h + \overline{g} \in \mathcal{F} \right\}.$$

In view of the maximum principle and the fact that h is locally univalent, we see that $\alpha(\mathcal{F}) \geq 1$ (cf. [11]). Thus, $A = \alpha(\mathcal{S}_H) \geq 1$. Bieberbach's theorem says that $\alpha(\mathcal{S}) = 2$. It has long been conjectured that $\alpha(\mathcal{S}_H) = 3$, but this is still an open question.

2. Main results

In order to state main result, we need the following definitions:

Definition 2.1 *For $0 < p < \infty$, the Hardy space H^p is the set of all functions f analytic in U for which*

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}$$

is bounded on $0 < r < 1$.

The space h^p consists of all harmonic mappings f in U for which $M_p(r, f)(0 < r < 1)$ are bounded (cf. [5]).
Let $BMOA$ ($resp.$ $BMOH$) denote the class of analytic functions (resp. harmonic mappings) that have bounded mean oscillation on the unit disk U (cf.[1]).

Definition 2.2 *(see [12]) Let*

$$\mathcal{B}_H(\lambda) = \{ f = h + \overline{g} \in \mathcal{H} \ : \ ||T_f|| \leq 2\lambda \},$$

with

$$||T_f|| := \sup_{z\in U,\theta\in[0,2\pi]} (1-|z|^2) \left| \frac{h''(z)+e^{i\theta}g''(z)}{h'(z)+e^{i\theta}g'(z)} \right|,$$

where λ *is a positive real number.*

In [8], the authors discussed the set $\mathcal{B}(\lambda) := \mathcal{A} \cap \mathcal{B}_H(\lambda)$ and obtained distortion estimates for analytic functions in $\mathcal{B}(\lambda)$ in terms of λ, and characterization for functions in $\mathcal{B}(\lambda)$. In [7], Kim proved the following result for analytic functions.

Theorem D.

(1) If $\lambda < 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset H^\infty$,

(2) If $\lambda = 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset BMOA$,

(3) If $\lambda > 1$, $\mathcal{B}(\lambda) \cap \mathcal{S} \subset H^p$ for every $0 < p < 1/(\lambda-1)$.

Recently, Ponnusamy et al. [12] generalized this result to harmonic mappings as follows.

Theorem E.

(1) If $\lambda < 1$, $\mathcal{B}_H(\lambda) \cap \mathcal{S}_H \subset h^\infty$,

(2) If $\lambda = 1$, $\mathcal{B}_H(\lambda) \cap \mathcal{S}_H \subset BMOH$,

(3) If $\lambda > 1$, $\mathcal{B}_H(\lambda) \cap \mathcal{S}_{H_k} \subset H^p$ for every $0 < p < 1/(\lambda-1)$.

Note that in the above result,

$$\mathcal{S}_{H_k} = \{f = h + \overline{g} \in \mathcal{S}_H \;:\; f \text{ is } k\text{-quasiconformal}\}$$

for $0 \le k < 1$. We recall that a sense-preserving harmonic mapping $f = h+\overline{g}$ in domain Ω is a k-quasiconformal mapping if $|w_f(z)| \le k$ holds in Ω.

In [6], Hernandez and Martin studied stable harmonic and analytic univalent functions. The sense-preserving harmonic mapping $f = h + \overline{g}$ is stable harmonic univalent or SHU in the open unit disk if all the mappings $f_\mu = h + \mu\overline{g}$ with $|\mu| = 1$ are univalent. Also, the analytic function $h + g$ is stable analytic univalent, or SAU, if all the mappings $F_\mu = h + \mu g$ with $|\mu| = 1$ are univalent. They proved that for all $|\mu| = 1$, the functions $f_\mu = h + \mu\overline{g}$ are univalent (resp. close to convex, starlike, or convex) if and only if the analytic functions $F_\mu = h + \mu g$ are univalent (resp. close to convex, starlike, or convex). We use this statement to prove the following theorems.

The following result is a generalization of statement (3) in Theorem D for harmonic mappings without quasiconformality condition, which is remarkable.

Theorem 2.3 *If* $\lambda > 1$, $\mathcal{B}_H(\lambda) \cap \mathcal{S}_H^0 \subset h^p$ *for every* $0 < p < 1/(\lambda-1)$.

Proof Let $f \in \mathcal{B}_H(\lambda) \cap \mathcal{S}_H^0$. Since $f = h + \overline{g} \in \mathcal{S}_H^0$ and $f = h + \mu\overline{\mu g}$ for all $|\mu| = 1$, in view of [6, Proposition 2.1], $h + \mu^2 g$ is univalent and normalized for all $|\lambda| = 1$. That is, $h + \mu^2 g \in \mathcal{S}$ for all such μ. In particular, for $\mu = e^{i\varphi/2}$ and $\mu = e^{i(\varphi+\pi)/2}$, where $\varphi \in [0, 2\pi)$, we observe that there exists at least one $\varphi \in [0, 2\pi)$ such that

$h + e^{i\varphi}g, h - e^{i\varphi}g \in \mathcal{S}$. On the other hand, from $f \in \mathcal{B}_H(\lambda)$, clearly, it follows that $\dfrac{h + e^{i\theta}g}{1 + e^{i\theta}b_1} \in \mathcal{B}(\lambda)$ for each $\theta \in [0, 2\pi]$. Since $b_1 = 0$, so $h + e^{i\theta}g \in \mathcal{B}(\lambda)$ for each $\theta \in [0, 2\pi]$. In particular, for $\theta = \varphi$ and

$$\theta = \begin{cases} \varphi + \pi, & 0 \le \varphi \le \pi, \\ \varphi - \pi, & \pi < \varphi < 2\pi, \end{cases}$$

we have $h + e^{i\varphi}g, h - e^{i\varphi}g \in \mathcal{B}(\lambda)$. Now, from Theorem D, it follows that $h + e^{i\varphi}g \in H^p$ and $h - e^{i\varphi}g \in H^p$ with $0 < p < 1/(\lambda - 1)$, which implies that $f \in h^p$. □

Theorem 2.4 *Let $f = h + \overline{g} \in \mathcal{S}_H^0$. Then $h, g \in H^p$ and $f \in h^p$ for every $p, 0 < p < A^{-1}$, where A is as defined in* (2).

Proof We assume that $F = H + \overline{G}$ is univalent and sense-preserving in U. Let

$$F_0(\zeta) = \frac{H(\zeta) - H(0)}{H'(0)} + \overline{\frac{G(\zeta) - G(0)}{H'(0)}} = H_0(\zeta) + \overline{G_0(\zeta)}.$$

Clearly, $F_0 \in \mathcal{S}_H$. For $\zeta \in U$, set

$$F_1(z) = \frac{F_0\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - F_0(\zeta)}{(1 - |\zeta|^2)H_0'(\zeta)} = H_1(z) + \overline{G_1(z)},$$

which again belongs to $\mathcal{S}_H$. The analytic function $H_1(z)$ has the form

$$H_1(z) = z + A_2(\zeta)z^2 + A_3(\zeta)z^3 + \dots,$$

and a direct computation shows that

$$A_2(\zeta) = \frac{1}{2}\left\{(1 - |\zeta|^2)\frac{H_0''(\zeta)}{H_0'(\zeta)} - 2\overline{\zeta}\right\} = \frac{1}{2}\left\{(1 - |\zeta|^2)\frac{H''(\zeta)}{H'(\zeta)} - 2\overline{\zeta}\right\}.$$

Let

$$A = \sup\left\{|a_2| \ : \ f(z) = \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k}\overline{z}^k \in \mathcal{S}_H\right\}.$$

Since $F_1 \in \mathcal{S}_H$, we must have $|A_2(\zeta)| \le A$, and therefore,

$$(1 - |\zeta|^2)\left|\frac{H''(\zeta)}{H'(\zeta)}\right| < 2(A + 1), \quad \zeta \in U.$$

For each $c \in U$, the composition of sense-preserving affine mapping $\phi(\omega) = \omega + c\overline{\omega}$ with F, namely the function $F + c\overline{F}$, is univalent and sense-preserving in U. Then by what we have just proved, we obtain

$$(1 - |\zeta|^2)\left|\frac{H''(\zeta) + cG''(\zeta)}{H'(\zeta) + cG'(\zeta)}\right| < 2(A + 1), \quad \zeta \in U,$$

which in particular implies that, for each $\theta \in [0, 2\pi]$,

$$(1-|\zeta|^2)\left|\frac{H''(\zeta)+e^{i\theta}G''(\zeta)}{H'(\zeta)+e^{i\theta}G'(\zeta)}\right| < 2(A+1), \quad \zeta \in U.$$

Then for $f = h + \overline{g} \in \mathcal{S}_H^0$, we have

$$A(\theta) = \sup_{z\in U}(1-|z|^2)\left|\frac{h''(z)+e^{i\theta}g''(z)}{h'(z)+e^{i\theta}g'(z)}\right| \leq 2(A+1).$$

Since $A(\theta)$ is a continuous function of θ in $[0, 2\pi]$, it follows that

$$\sup_{z\in U,\ \theta\in[0,2\pi]}(1-|z|^2)\left|\frac{h''(z)+e^{i\theta}g''(z)}{h'(z)+e^{i\theta}g'(z)}\right| \leq 2(A+1).$$

Then, by the definition, $f \in \mathcal{B}_H(\lambda_0)$ with $\lambda_0 = A+1$. Now Theorem 2.3 implies that $f \in h^p$ with $0 < p < A^{-1}$. Also, from $f \in \mathcal{B}_H(\lambda_0)$, we have $h + e^{i\theta}g \in \mathcal{B}(\lambda_0)$ for each $\theta \in [0, 2\pi]$. Now, in view of Theorem D, from the proof of Theorem 2.3 it follows that $h + e^{i\varphi}g \in H^p$ and $h - e^{i\varphi}g \in H^p$ with $0 < p < A^{-1}$, for some $\varphi \in [0, 2\pi)$, which implies that $h, g \in H^p$ and $f \in h^p$. □

Theorem 2.5 *Let $f = h + \overline{g} \in \mathcal{S}_H$. Then $h, g \in H^p$ and $f \in h^p$ for every $p, 0 < p < A^{-1}$, where A is as defined in (2).*

Proof Since $f = h + \overline{g} \in \mathcal{S}_H$, thus

$$\frac{f(z) - \overline{b_1 f(z)}}{1-|b_1|^2} = \frac{1}{1-|b_1|^2}[h - \bar{b}_1 g + \overline{g - b_1 h}] \in \mathcal{S}_H^0.$$

Using Theorem 2.4, we have $h - \bar{b}_1 g \in H^p$ and $g - b_1 h \in H^p$ with $0 < p < A^{-1}$. From this, we can easily obtain $h - g \in H^p$ and $g \in H^p$, which implies the desired result. □

Since $A \geq 1$, the above result improves the result obtained by Nowak [10].

By using the distortion theorem for univalent convex and starlike functions, Sheil-Small obtained the following result:

Proposition 2.6 *(see [13]) Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be a regular, starlike univalent function for $|z| < 1$. Let $C(r, \theta) = \{f(\rho e^{i\theta}), 0 \leq \rho \leq r\}$ and let $l(r, \theta)$ be the length of $C(r, \theta)$. Then*

$$l(r, \theta) = \int_0^r |f'(\rho e^{i\theta})| d\rho < A|f'(\rho e^{i\theta})|.$$

If $f(z)$ is univalent convex then we have

$$l(r, \theta) < B|f'(\rho e^{i\theta})|,$$

where $2 \leq A \leq 1 + \log 4$ and $\pi/2 \leq B \leq 1 + \log 2$.

Here we generalize the above result to univalent harmonic mappings in two different methods. Suppose that $f = h + \overline{g} \in \mathcal{S}_H$. Let $C(r,\theta)$ be the image of the ray joining 0 and $z = re^{i\theta}$ under f, and let

$$L_\eta(r,\theta) = \int_0^r |h'(\rho e^{i\theta}) + \eta g'(\rho e^{i\theta})| d\rho,$$

where, $|\eta| = 1$.

Theorem 2.7 *Let $f \in \mathcal{S}_H$ and $Reh(g - \overline{\eta g}) \geq 0$. Then*

$$L_\eta(r,\theta) \leq B|f(re^{i\theta})|, \quad |z| = r < A - \sqrt{A^2 - 1},$$

where B is given by Proposition 2.6.

Proof If $f = h + \overline{g} \in S_H$, from the proof of Theorem 2.4 we have

$$(1 - |z|^2) \left| \frac{h''(z)}{h'(z)} \right| < 2(A + 1), \quad z \in U.$$

Thus,

$$Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \frac{1 - 2A|z| + |z|^2}{1 - |z|^2}.$$

Therefore, h is univalent convex in the disk $|z| < A - \sqrt{A^2 - 1}$. From the affine invariance of the class S_H, $h + \varepsilon g$ is univalent convex in the disk $|z| < A - \sqrt{A^2 - 1}$ for $|\varepsilon| < 1$. Thus, this remains true for $|\varepsilon| = 1$. Applying Proposition 2.6 with $\eta = \varepsilon$ we deduce that in the disk $|z| < A - \sqrt{A^2 - 1}$,

$$\begin{aligned} L(r,\theta) &= \int_0^r |h'(\rho e^{i\theta}) + \eta g'(\rho e^{i\theta})| d\rho \\ &\leq B|h(\rho e^{i\theta}) + \eta g(\rho e^{i\theta})| \\ &\leq B|h(\rho e^{i\theta}) + \overline{g}(\rho e^{i\theta})| \\ &= B|f(re^{i\theta})|. \end{aligned}$$

□

Theorem 2.8 *Let $f = h + \overline{g} \in \mathcal{S}_H^{*0}$ and $Reh(g - \overline{\eta g}) \geq 0$. Then*

$$L_\eta(r,\theta) \leq A|f(re^{i\theta})|,$$

where A is given by Proposition 2.6.

Proof Let $f = h + \overline{g} = h + \mu\overline{\mu g} \in \mathcal{S}_H^{*0}$, where $|\mu| = 1$. In view of [6, Theorem 4.2], $h + \mu^2 g$ is a univalent convex function. Applying Proposition 2.6 with $\eta = \mu^2$, we obtain

$$\begin{aligned} L_\eta(r,\theta) &= \int_0^r |h'(\rho e^{i\theta}) + \eta g'(\rho e^{i\theta})| d\rho \\ &\leq A|h(\rho e^{i\theta}) + \eta g(\rho e^{i\theta})| \\ &\leq A|h(\rho e^{i\theta}) + \overline{g}(\rho e^{i\theta})| \\ &= A|f(re^{i\theta})|. \end{aligned}$$

□

Similarly, by using [6, Theorem 3.1] we have the following result.

Theorem 2.9 *Let* $f = h + \overline{g} \in \mathcal{K}_H^0$ *and* $Reh(g - \overline{\eta g}) \geq 0$. *Then*

$$L_\eta(r, \theta) \leq B|f(re^{i\theta})|,$$

where B *is given by Proposition 2.6.*

In the sequel, we suppose that

$$L(r, \theta) = \int_0^r \sqrt{J_f(\rho e^{i\theta})} d\rho.$$

Recall that $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. Then we have the following results.

Theorem 2.10 *Let* $f \in \mathcal{S}_H$ *and* $Re(hg) \geq 0$. *Then*

$$L(r, \theta) \leq B|f(re^{i\theta})|, \quad |z| = r < A - \sqrt{A^2 - 1},$$

where B *is given by Proposition 2.6.*

Proof Let $f = h + \overline{g} \in S_H$. From the proof of Theorem 2.7, it follows that h is univalent convex in the disk $|z| = r < A - \sqrt{A^2 - 1}$. Therefore, by using Proposition 2.6 in this disk, we have

$$\begin{aligned} L(r, \theta) &= \int_0^r \sqrt{J_f(\rho e^{i\theta})} d\rho \\ &\leq \int_0^r |h'(\rho e^{i\theta})| d\rho \\ &\leq |h(\rho e^{i\theta})| \\ &\leq |h + \overline{g}| = |f|. \end{aligned}$$

□

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the horizontal direction (CHD) if its intersection with each horizontal line is connected or empty.

Lemma 2.11 *(see [5]) Let* $f = h + \overline{g}$ *be harmonic and locally univalent in the open unit disk. Then* f *is univalent and its range is convex if and only if for each choice of* $\alpha (0 \leq \alpha < 2\pi)$ *the analytic function* $e^{i\alpha} h - e^{-i\alpha} g$ *is univalent and its range is* CHD.

Corollary 2.12 *([5]) If* $f = h + \overline{g}$ *is a convex harmonic mapping, then the function* $h + e^{i\beta} g$ *is univalent for each* β, $0 \leq \beta < 2\pi$.

Theorem 2.13 *Let* $f = h + \overline{g} \in \mathcal{K}_H^0$, $Re(hg) \geq 0$, *and* $h + e^{i\varphi} g(U) \subset \Omega$, $h - e^{i\varphi} g(U) \subset \Omega$ *for* $\varphi \in [0, 2\pi)$, *where* Ω *is a simply connected domain. Then*

$$L(r, \theta) \leq B|f(re^{i\theta})|,$$

where B *is given by Proposition 2.6.*

Proof Let $f = h + \overline{g} = h + \mu\overline{\mu g} \in \mathcal{K}_H^0$, where $|\mu| = 1$. In view of [6, Theorem 3.1], $h + \mu^2 g$ is a univalent convex function. For $\mu = e^{i\varphi/2}$ and $\mu = e^{i(\varphi+\pi)/2}$, where $\varphi \in [0, 2\pi)$, we observe that there exists at least one $\varphi \in [0, 2\pi)$ such that the functions $h + e^{i\varphi}g$ and $h - e^{i\varphi}g$ are univalent convex. Now, fix $z \in U$ and let $h + e^{i\varphi}g(z) = t_1 \in \Omega$, $h - e^{i\varphi}g(z) = t_2 \in \Omega$. Then we have $h(z) = \dfrac{t_1 + t_2}{2} \in \Omega$. Since Ω is a convex domain, h is a convex function. On the other hand, Corollary 2.12 implies that $h + e^{i\beta}g$ is univalent for each β, $0 \leq \beta < 2\pi$ or equivalently $h + \mu g$ is univalent for each $|\mu| = 1$. That is, f is a SHU mapping in the open unit disk. Therefore, by using [6, Theorem 7.1], it follows that h is univalent. Thus, from Proposition 2.6 we get

$$\begin{aligned} L(r,\theta) &= \int_0^r \sqrt{J_f(\rho e^{i\theta})}d\rho \\ &\leq \int_0^r |h'(\rho e^{i\theta})|d\rho \\ &\leq |h(\rho e^{i\theta})| \\ &\leq |h + \overline{g}| = |f|. \end{aligned}$$

□

References

[1] Abu-Muhanna Y. Bloch, BMO and harmonic univalent functions. Complex Variables Theory Appl 1996; 31: 271-279.

[2] Abu-Muhanna Y, Lyzzaik A. The boundary behavior of harmonic univalent maps. Pac J Math 1990; 141: 1-20.

[3] Bshouty D, Hengartner W. Univalent harmonic mappings in the plane. In: Kuhnau R, editor. Handbook of Complex Analysis: Geometric Function Theory. Vol. 2. Amsterdam, the Netherlands: Elsevier, 2005, pp. 479-506.

[4] Clunie J, Sheil-Small T. Harmonic univalent functions. Ann Acad Sci Fenn Ser AI 1984; 9: 3-25.

[5] Duren P. Harmonic Mappings in the Plane. Tracts in Mathematics, 156. Cambridge, UK: Cambridge University Press, 2004.

[6] Hernandez R, Martin MJ. Stable geometric properties of analytic and harmonic functions. Math Proc Camb Phil Soc 2013; 155: 343-359.

[7] Kim YC. Some inequalities for uniformly locally univalent functions on the unit disk. Math Inequal Appl 2007; 10: 805-809.

[8] Kim YC, Sugawa T. Growth and coefficient estimates for uniformly locally univalent functions on the unit disk. Rocky Mountain J Math 2002; 32: 179-200.

[9] Lewy H. On the non-vanishing of the Jacobian in certain one-to-one mappings. B Am Math Soc 1936; 42: 689-692.

[10] Nowak M. Integral means of univalent harmonic maps. Ann Univ Mariae Curie-Sklodwska 1996; 50: 155-162.

[11] Pommerenke C. Linear-invariante Familien analytischer Funktionen I. Math Ann 1964; 155: 108-154 (in German).

[12] Ponnusamy S, Qiao J, Wang X. Uniformly locally univalent harmonic mappings. arXiv:1601.01139v1 [math.CV], 2016.

[13] Sheil-Small T. Some conformal mapping inequalities for starlike and convex functions. J London Math Soc 1969; 2-1: 577-587.

[14] Sheil-Small T. Constants for planar harmonic mappings. J London Math Soc 1990; 42: 237-248.