

Enveloping algebras of color hom-Lie algebras

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Abstract: In this paper, the universal enveloping algebra of color hom-Lie algebras is studied. A construction of the free involutive hom-associative color algebra on a hom-module is described and applied to obtain the universal enveloping algebra of an involutive hom-Lie color algebra. Finally, the construction is applied to obtain the well-known Poincaré–Birkhoff–Witt theorem for Lie algebras to the enveloping algebra of an involutive color hom-Lie algebra.

Key words: Color hom-Lie algebra, enveloping algebra

1. Introduction

The investigations of various quantum deformations (or q -deformations) of Lie algebras started a period of rapid expansion in 1980s, stimulated by the introduction of quantum groups motivated by applications to the quantum Yang–Baxter equation, quantum inverse scattering methods, and constructions of the quantum deformations of universal enveloping algebras of semisimple Lie algebras. Since then, several other versions of q -deformed Lie algebras have appeared, especially in physical contexts such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory in the context of deformations of infinite-dimensional algebras, primarily the Heisenberg algebras, oscillator algebras, and Witt and Virasoro algebras [3, 16–19, 21–23, 28, 30, 41–43]. In these pioneering works, it has been discovered in particular that in these q -deformations of Witt and Virasoro algebras and some related algebras, some interesting q -deformations of Jacobi identities, extending Jacobi identity for Lie algebras, are satisfied. This has been one of the initial motivations for the development of general quasideformations and discretizations of Lie algebras of vector fields using more general σ -derivations (twisted derivations) in [25], and introduction of abstract quasi-Lie algebras and subclasses of quasi-hom-Lie algebras and hom-Lie algebras as well as their general colored (graded) counterparts in [25, 36, 37, 39, 58]. These generalized Lie algebra structures with (graded) twisted skew-symmetry and twisted Jacobi conditions by linear maps are tailored to encompass within the same algebraic framework such quasideformations and discretizations of Lie algebras of vector fields using σ -derivations, describing general discretizations and deformations of derivations with twisted Leibniz rule, and the well-known generalizations of Lie algebras such as color Lie algebras, which are the natural generalizations of Lie algebras and Lie superalgebras.

Quasi-Lie algebras are nonassociative algebras for which the skew-symmetry and the Jacobi identity

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are twisted by several deforming twisting maps and also the Jacobi identity in quasi-Lie and quasi-hom-Lie algebras in general contains six twisted triple bracket terms. Hom-Lie algebras is a special class of quasi-Lie algebras with the bilinear product satisfying the nontwisted skew-symmetry property as in Lie algebras, whereas the Jacobi identity contains three terms twisted by a single linear map, reducing to the Jacobi identity for ordinary Lie algebras when the linear twisting map is the identity map. Subsequently, hom-Lie admissible algebras were considered in [45], where the hom-associative algebras were also introduced and shown to be hom-Lie admissible natural generalizations of associative algebras corresponding to hom-Lie algebras. In [45], moreover, several other interesting classes of hom-Lie admissible algebras generalizing some nonassociative algebras, as well as examples of finite-dimensional hom-Lie algebras were described. Since these pioneering works [25, 36, 37, 39, 40, 45], hom-algebra structures have become a popular area with increasing number of publications in various directions.

Hom-Lie algebras, hom-Lie superalgebras, and color hom-Lie algebras are important special classes of color (Γ -graded) quasi-Lie algebras introduced first by Larsson and Silvestrov in [37, 39]. Hom-Lie algebras and hom-Lie superalgebras were studied further in different aspects by Makhoul, Silvestrov, Sheng, Ammar, Yau and other authors [12, 38, 44–48, 51, 52, 57, 59–62, 64–66, 68, 69], and color hom-Lie algebras were considered, for example, in [1, 13, 14, 68]. In [4], the constructions of hom-Lie and quasi-hom-Lie algebras based on twisted discretizations of vector fields [25] and hom-Lie admissible algebras were extended to hom-Lie superalgebras, a subclass of graded quasi-Lie algebras [37, 39]. We also wish to mention that \mathbb{Z}_3 -graded generalizations of supersymmetry, \mathbb{Z}_3 -graded algebras, ternary structures, and related algebraic models for classifications of elementary particles and unification problems for interactions, quantum gravity and noncommutative gauge theories [2, 31–34] also provide interesting examples related to hom-associative algebras, graded hom-Lie algebras, twisted differential calculi, and n -ary hom-algebra structures. It would be a project of great interest to extend and apply all the constructions and results in the present paper in the relevant contexts of the articles [2, 4, 6, 8–10, 31–33, 37, 39, 45].

An important direction with many fundamental open problems in the theory of (color) quasi-Lie algebras and in particular (color) quasi-hom-Lie algebras and (color) hom-Lie algebras is the development of comprehensive fundamental theory, explicit constructions, examples and algorithms for enveloping algebraic structures, expanding the corresponding more developed fundamental theory and constructions for enveloping algebras of Lie algebras, Lie superalgebras, and general color Lie algebras [11, 20, 29, 49, 50, 53–56]. Several authors have tried to construct the enveloping algebras of hom-Lie algebras. For instance, Yau has constructed the enveloping hom-associative algebra $U_{HLie}(L)$ of a hom-Lie algebra L in [64] as the left adjoint functor of $HLie$ using combinatorial objects of weighted binary trees, i.e. planar binary trees in which the internal vertices are equipped with weights of nonnegative integers. This is analogous to the fact that the functor Lie admits a left adjoint U , the enveloping algebra functor. He also introduced construction of the counterpart functors $HLie$ and U_{HLie} for hom-Leibniz algebras. In [26], for hom-associative algebras and hom-Lie algebras, the envelopment problem, operads, and the Diamond Lemma, and Hilbert series for the hom-associative operad and free algebra were studied. Recently, making use of free involutive hom-associative algebras, the authors in [24] have found an explicit constructive way to obtain the universal enveloping algebras of hom-Lie algebras in order to prove the Poincaré–Birkhoff–Witt theorem.

In this paper, we will give a brief review of well-known facts about hom-Lie algebras and their enveloping algebras. We will then present some new results, in the hope that they may eventually have a bearing on representation theory of color hom-Lie algebras. In Section 2, some necessary notions and definitions are

presented as an introduction to color hom-Lie algebras. In Section 3 the notion of universal enveloping algebras of color hom-Lie algebras is given and several useful result about involutive color hom-Lie algebras are proven. In Section 4, we prove the analogous of the well known Poincaré–Birkhoff–Witt theorem for color hom-Lie algebras, using the definitions and results of Sections 2 and 3. Finally, due to the importance of hom-Lie superalgebras, we present the most important results of the paper in hom-Lie superalgebras case.

2. Basic concepts on hom-Lie algebras and color quasi-Lie algebras

We start by recalling some basic concepts from [24, 45, 47]. We use \mathbf{k} to denote a commutative unital ring (for example a field).

Definition 2.1 (i) A hom-module is a pair (M, α) consisting of a \mathbf{k} -module M and a linear operator $\alpha : M \rightarrow M$.

(ii) A hom-associative algebra is a triple (A, \cdot, α) consisting of a \mathbf{k} -module A , a linear map $\cdot : A \otimes A \rightarrow A$ called the multiplication and a multiplicative linear operator $\alpha : A \rightarrow A$ which satisfies the hom-associativity condition, namely

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z),$$

for all $x, y, z \in A$.

(iii) A hom-associative algebra or a hom-module is called involutive if $\alpha^2 = id$.

(iv) Let (M, α) and (N, β) be two hom-modules. A \mathbf{k} -linear map $f : M \rightarrow N$ is called a morphism of hom-modules if

$$f(\alpha(x)) = \beta(f(x)),$$

for all $x \in M$.

(v) Let (A, \cdot, α) and (B, \bullet, β) be two hom-associative algebras. A \mathbf{k} -linear map $f : A \rightarrow B$ is called a morphism of hom-associative algebras if

$$(1) f(x \cdot y) = f(x) \bullet f(y),$$

$$(2) f(\alpha(x)) = \beta(f(x)), \text{ for all } x, y \in A.$$

(vi) If (A, \cdot, α) is a hom-associative algebra, then $B \subseteq A$ is called a hom-associative subalgebra of A if it is closed under the multiplication \cdot and $\alpha(B) \subseteq B$. A submodule I is called a hom-ideal of A if $x \cdot y \in I$ and $x \cdot y \in I$ for all $x \in I$ and $y \in A$, and also $\alpha(I) \subseteq I$.

One can find various examples of hom-associative algebras and their properties in [45, 47].

Definition 2.2 Let (M, α) be an involutive hom-module. A free involutive hom-associative algebra on M is an involutive hom-associative algebra (F_M, \star, β) together with a morphism of hom-modules $j : M \rightarrow F_M$ with the property that for any involutive hom-associative algebra A together with a morphism $f : M \rightarrow A$ of hom-modules, there is a unique morphism $\bar{f} : F_M \rightarrow A$ of hom-associative algebras such that $f = \bar{f} \circ j$.

Our next goal is to recall the definition of an involutive hom-associative algebra on an involutive hom-module (M, α) from [24], which is known as the hom-tensor algebra and is denoted here by $T(M)$. Note that as an R -module, $T(M)$ is the same as the tensor algebra, i.e.

$$T(M) = \bigoplus_{i \geq 1} M^{\otimes i},$$

on which we have the following multiplication in order to obtain a hom-associative algebra. First, the linear map α on M is extended to a linear map α_T on $M^{\otimes i}$ by the tensor multiplicativity, i.e.

$$\alpha_T(x) = \alpha_T(x_1 \otimes \cdots \otimes x_i) := \alpha(x_1) \otimes \cdots \otimes \alpha(x_i),$$

for all pure tensors $x := x_1 \otimes \cdots \otimes x_i \in M^{\otimes i}$, $i \geq 1$. One can see that α_T has the following properties:

- (i) $\alpha_T(x \otimes y) = \alpha_T(x) \otimes \alpha_T(y)$, for all $x \in M^{\otimes i}$, $y \in M^{\otimes j}$.
- (ii) $\alpha_T^2 = id$.

Now, the binary operation on $T(M)$ is defined as follows:

$$x \odot y := \alpha_T^{i-1}(x) \otimes y_1 \otimes \cdots \otimes \alpha_T(y_2 \otimes \cdots \otimes y_i), \tag{2.1}$$

for all $x \in M^{\otimes i}$ and $y \in M^{\otimes j}$.

Theorem 2.3 [24] *Let (M, α) be an involutive hom-module. Then,*

- (i) *The triple $T(M) := (T(M), \odot, \alpha_T)$ is an involutive hom-associative algebra.*
- (ii) *The quadruple $(T(M), \odot, \alpha_T, i_M)$ is the free involutive hom-associative algebra on M .*

It is now convenient to recall some definitions for hom-Lie algebras [25, 36, 37, 39, 45].

Definition 2.4 *A hom-Lie algebra is a triple $(\mathfrak{g}, [,], \alpha)$, where \mathfrak{g} is a vector space equipped with a skew-symmetric bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ such that*

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all $x, y, z \in \mathfrak{g}$, which is called hom-Jacobi identity.

A hom-Lie algebra is called a multiplicative hom-Lie algebra if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{g}$,

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

We call a hom-Lie algebra regular if α is an automorphism. Moreover, it is called involutive if $\alpha^2 = id$.

A subvector space $\mathfrak{h} \subset \mathfrak{g}$ is a hom-Lie subalgebra of $(\mathfrak{g}, [,], \alpha)$ if $\alpha(\mathfrak{h}) \subset \mathfrak{h}$ and \mathfrak{h} is closed under the bracket operation, i.e.

$$[x_1, x_2]_{\mathfrak{g}} \in \mathfrak{h},$$

for all $x_1, x_2 \in \mathfrak{h}$. Let $(\mathfrak{g}, [,], \alpha)$ be a multiplicative hom-Lie algebra. Let α^k denote the k -times composition of α by itself, for any nonnegative integer k , i.e.

$$\alpha^k = \alpha \circ \dots \circ \alpha \quad (k - \text{times}),$$

where we define $\alpha^0 = Id$ and $\alpha^1 = \alpha$. For a regular hom-Lie algebra \mathfrak{g} , let

$$\alpha^{-k} = \alpha^{-1} \circ \dots \circ \alpha^{-1} \quad (k - \text{times}).$$

We now recall the notion of a color hom-Lie algebra step by step in order to indicate them as a generalization of Lie color algebras.

Definition 2.5 [11, 39, 49, 53, 54] *Given a commutative group Γ (referred to as the grading group), a commutation factor on Γ with values in the multiplicative group $K \setminus \{0\}$ of a field K of characteristic 0 is a map*

$$\varepsilon : \Gamma \times \Gamma \rightarrow K \setminus \{0\},$$

satisfying three properties:

- (i) $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$,
- (ii) $\varepsilon(\alpha, \gamma + \beta) = \varepsilon(\alpha, \gamma)\varepsilon(\alpha, \beta)$,
- (iii) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$.

A Γ -graded ε -Lie algebra (or a Lie color algebra) is a Γ -graded linear space

$$X = \bigoplus_{\gamma \in \Gamma} X_\gamma,$$

with a bilinear multiplication (bracket) $[.,.] : X \times X \rightarrow X$ satisfying the following properties:

- (i) **Grading axiom:** $[X_\alpha, X_\beta] \subseteq X_{\alpha+\beta}$,
- (ii) **Graded skew-symmetry:** $[a, b] = -\varepsilon(\alpha, \beta)[b, a]$,
- (iii) **Generalized Jacobi identity:**
 $\varepsilon(\gamma, \alpha)[a, [b, c]] + \varepsilon(\beta, \gamma)[c, [a, b]] + \varepsilon(\alpha, \beta)[b, [c, a]] = 0$,

for all $a \in X_\alpha, b \in X_\beta, c \in X_\gamma$ and $\alpha, \beta, \gamma \in \Gamma$. The elements of X_γ are called homogenous of degree γ , for all $\gamma \in \Gamma$.

Analogous to the other kinds of definitions of hom-algebras, the definition of a color hom-Lie algebra can be given as follows [1, 13, 14, 39, 68].

Definition 2.6 *A color hom-Lie algebra is a quadruple $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ consisting of a Γ -graded vector space \mathfrak{g} , a bi-character ε , an even bilinear mapping*

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

(i.e. $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, for all $a, b \in \Gamma$) and an even homomorphism $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ such that for homogeneous elements $x, y, z \in \mathfrak{g}$, we have

1. ε -**skew symmetric**: $[x, y] = -\varepsilon(x, y)[y, x]$,
2. ε -**Hom-Jacobi identity**: $\sum_{cyclic\{x,y,z\}} \varepsilon(z, x)[\alpha(x), [y, z]] = 0$.

Color hom-Lie algebras are a special class of general color quasi-Lie algebras (Γ -graded quasi-Lie algebras) defined first by Larsson and Silvestrov in [39].

Let $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ and $\mathfrak{h} = \bigoplus_{\gamma \in \Gamma} \mathfrak{h}_\gamma$ be two Γ -graded color Lie algebras. A linear mapping $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is said to be homogenous of the degree $\mu \in \Gamma$ if

$$f(\mathfrak{g}_\gamma) \subseteq \mathfrak{h}_{\gamma+\mu},$$

for all $\gamma \in \Gamma$. If in addition, f is homogenous of degree zero, i.e.

$$f(\mathfrak{g}_\gamma) \subseteq \mathfrak{h}_\gamma,$$

holds for any $\gamma \in \Gamma$, then f is said to be even.

Let $(\mathfrak{g}, [,], \varepsilon, \alpha)$ and $(\mathfrak{g}', [,]', \varepsilon', \alpha')$ be two color hom-Lie algebras. A homomorphism of degree zero $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ is said to be a morphism of color hom-Lie algebras if

1. $[f(x), f(y)]' = f([x, y])$, for all $x, y \in \mathfrak{g}$,
2. $f \circ \alpha = \alpha' \circ f$.

In particular, if α is a morphism of color Lie algebras, then we call $(\mathfrak{g}, [,], \varepsilon, \alpha)$, a multiplicative color hom-Lie algebra.

Example 2.7 [1] *As in case of hom-associative and hom-Lie algebras, examples of multiplicative color hom-Lie algebras can be constructed for example by the standard method of composing multiplication with algebra morphism.*

Let $(\mathfrak{g}, [,], \varepsilon)$ be a color Lie algebra and α be a color Lie algebra morphism. Then $(\mathfrak{g}, [,]_\alpha := \alpha \circ [,], \varepsilon, \alpha)$ is a multiplicative color hom-Lie algebra.

Definition 2.8 *A hom-associative color algebra is a triple (V, μ, α) consisting of a color space V , an even bilinear map $\mu : V \times V \rightarrow V$ and an even homomorphism $\alpha : V \rightarrow V$ satisfying*

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)),$$

for all $x, y, z \in V$.

A hom-associative color algebra or a color hom-Lie algebra is said to be involutive if $\alpha^2 = id$.

As in the case of an associative algebra and a Lie algebra, a hom-associative color algebra (V, μ, α) gives a color hom-Lie algebra by antisymmetrization. We denote this color hom-Lie algebra by $(A, [,]_A, \beta_A)$, where $\beta_A = \alpha$ and $[x, y]_A = xy - yx$, for all $x, y \in A$.

3. The universal enveloping algebra

In this section, we introduce the notion of the universal enveloping algebra of a color hom-Lie algebra. Moreover, we prove a new result on the free involutive hom-associative color algebra on an involutive hom-module.

Definition 3.1 Let (V, α_V) be an involutive hom-module. A free involutive hom-associative color algebra on V is an involutive hom-associative color algebra $(F(V), *, \alpha_F)$ together with a morphism of hom-modules

$$j_V : (V, \alpha_V) \rightarrow (F(V), \alpha_F),$$

with the property that, for any involutive hom-associative color algebra (A, \cdot, α_A) together with a morphism $f : (V, \alpha_V) \rightarrow (A, \alpha_A)$ of hom-modules, there is a unique morphism $\bar{f} : (F(V), *, \alpha_F) \rightarrow (A, \cdot, \alpha_A)$ of hom-associative color algebras such that $f = \bar{f} \circ j_V$.

Definition 3.2 Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a color hom-Lie algebra. A universal enveloping hom-associative color algebra of \mathfrak{g} is a hom-associative color algebra

$$U(\mathfrak{g}) := (U(\mathfrak{g}), \mu_U, \alpha_U),$$

together with a morphism $\varphi_{\mathfrak{g}} : \mathfrak{g} \rightarrow U(\mathfrak{g})$ of color hom-Lie algebras such that for any hom-associative color algebra (A, μ, α_A) and any color hom-Lie algebra morphism $\phi : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \rightarrow (A, \mu, \alpha_A)$, there exists a unique morphism $\bar{\phi} : U(\mathfrak{g}) \rightarrow A$ of hom-associative color algebras such that $\bar{\phi} \circ \varphi_{\mathfrak{g}} = \phi$.

The following lemma shows an easy way to construct the universal algebra when we have an involutive color hom-Lie algebra.

Lemma 3.3 Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive color hom-Lie algebra.

(i) Let (A, \cdot, α_A) be a hom-associative algebra. Let

$$f : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \rightarrow (A, [\cdot, \cdot]_A, \beta_A)$$

be a morphism of color hom-Lie algebras and let B be the hom-associative subcolor algebra of A generated by $f(\mathfrak{g})$. Then B is involutive.

(ii) The universal enveloping hom-associative algebra $(U(\mathfrak{g}), \varphi_{\mathfrak{g}})$ of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ is involutive.

(iii) In order to verify the universal property of $(U(\mathfrak{g}), \varphi_{\mathfrak{g}})$ in Definition 3.2, we only need to consider involutive hom-associative algebras $A := (A, \cdot_A, \alpha_A)$.

Proof

(i) Let

$$S = \{x \in A \mid \alpha_A^2(x) = x\}.$$

One can easily check that S is a submodule. Also, for $x, y \in S$, we have $xy \in S$, since $\alpha_A^2(xy) = \alpha_A^2(x)\alpha_A^2(y) = xy$. Moreover, we have

$$\alpha_A^2(\alpha_A(x)) = \alpha_A(\alpha_A^2(x)) = \alpha_A(x),$$

which shows that $\alpha_A(x) \in S$, for all $x \in S$. Thus, S is a hom-associative subalgebra of A since f is a morphism.

(ii) Since $U(\mathfrak{g})$ is generated by $\varphi_{\mathfrak{g}}(\mathfrak{g})$ as a hom-associative algebra, the statement follows from (i).

(iii) We should prove that, assuming that the universal property of $U(\mathfrak{g})$ holds for involutive hom-associative algebras, then it holds for all hom-associative algebras. Let (A, \cdot, α_A) be a hom-associative algebra and let

$$\psi : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \rightarrow (A, [\cdot, \cdot]_A, \beta_A)$$

be a morphism of color hom-Lie algebras. Let

$$S = \{x \in A \mid \alpha_A^2(x) = x\}$$

be the involutive hom-associative subalgebra of A defined in the proof of (i). Since $im(\psi)$ is contained in S , ψ is the composition of a morphism $\psi_S : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \rightarrow (S, [\cdot, \cdot]_S, \beta_S)$ of hom-associative color algebras with the inclusion $i : S \rightarrow A$. By assumption, there is a morphism $\bar{\psi}_S : U(\mathfrak{g}) \rightarrow A$ of hom-associative color algebras such that $\hat{\psi}_S \circ \varphi_{\mathfrak{g}} = \psi_S$. Then composing with the inclusion $i : S \rightarrow A$, we obtain a morphism $\bar{\psi} : U(\mathfrak{g}) \rightarrow A$ of hom-associative color algebras such that $\bar{\psi} \circ \varphi_{\mathfrak{g}} = \psi$.

Now, let $\bar{\psi}' : U(\mathfrak{g}) \rightarrow A$ be another morphism of hom-associative color algebras such that $\bar{\psi}' \circ \varphi_{\mathfrak{g}} = \psi$. By (ii), $im(\bar{\psi}')$ is involutive. So $\bar{\psi}'$ is the composition of a morphism $\bar{\psi}'_S : U(\mathfrak{g}) \rightarrow S$ with the inclusion $i : S \rightarrow A$ and $\bar{\psi}'_S \circ \varphi_{\mathfrak{g}} = \psi_S$. Since S is involutive, the morphisms $\bar{\psi}'_S$ and $\bar{\psi}_S$ coincide. As a consequence, $\bar{\psi}'$ and $\bar{\psi}$ coincide, which completes the proof. □

We can now give the construction of the universal enveloping hom-associative color algebra of an involutive color hom-Lie algebra.

Theorem 3.4 *Let $\mathfrak{g} := (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive color hom-Lie algebra. Let*

$$T(\mathfrak{g}) := (T(\mathfrak{g}), \odot, \alpha_T)$$

be the free hom-associative algebra on the hom-module underlying \mathfrak{g} obtained in Theorem 2.3. Let I be the hom-ideal of $T(\mathfrak{g})$ generated by the set

$$\{a \otimes b - \varepsilon(a, b)b \otimes a - [a, b]\} \tag{3.1}$$

and let

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I}$$

be the quotient hom-associative algebra. Let ψ be the composition of the natural inclusion $i : \mathfrak{g} \rightarrow T(\mathfrak{g})$ with the quotient map $\pi : T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. Then $(U(\mathfrak{g}), \psi)$ is a universal enveloping hom-associative algebra of \mathfrak{g} . Also, the universal enveloping hom-associative algebra of \mathfrak{g} is unique up to isomorphism.

Proof The multiplication in $U(\mathfrak{g})$ is denoted by $*$. The map ψ is a morphism of hom-modules since it is the composition of two hom-module morphisms. We have

$$\begin{aligned} \psi([x, y]_{\mathfrak{g}}) &= \pi([x, y]_{\mathfrak{g}}) = \pi(x \otimes y - \varepsilon(x, y)y \otimes x) \\ &= \pi(x \odot y - \varepsilon(x, y)y \odot x) = \pi(x) * \pi(y) - \varepsilon(x, y)\pi(y) * \pi(x) \\ &= \psi(x) * \psi(y) - \varepsilon(x, y)\psi(y) * \psi(x) = [\psi(x), \psi(y)]_{\mathfrak{g}}, \end{aligned}$$

for all $x, y \in \mathfrak{g}$, since $x \otimes y - \varepsilon(x, y)y \otimes x - [x, y]_{\mathfrak{g}}$ is in $I = \ker(\pi)$. Therefore, ψ is a morphism of color hom-Lie algebras. Now, using Lemma 3.3 (iii), we consider an arbitrary involutive hom-associative color algebra $A := (A, \cdot_A, \alpha_A)$. Let $\xi : (\mathfrak{g}, [\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \rightarrow (A, [\cdot]_A, \beta_A)$ be a morphism of color hom-Lie algebras. Since $T(\mathfrak{g})$ is the free involutive hom-associative color algebra on the underlying hom-module of \mathfrak{g} , according to Theorem 2.3, there exists a hom-associative color algebra morphism $\tilde{\xi} : T(\mathfrak{g}) \rightarrow A$ of hom-associative color algebras such that $\tilde{\xi} \circ i_{\mathfrak{g}} = \xi$. We have

$$\begin{aligned} \tilde{\xi}(x \otimes y - \varepsilon(x, y)y \otimes x) &= \tilde{\xi}(x \odot y - \varepsilon(x, y)y \odot x) \\ &= \tilde{\xi}(x) \cdot_A \tilde{\xi}(y) - \varepsilon(x, y)\tilde{\xi}(y) \cdot_A \tilde{\xi}(x) \\ &= \xi(x) \cdot_A \xi(y) - \varepsilon(x, y)\xi(y) \cdot_A \xi(x) = [\xi(x), \xi(y)]_A \\ &= \xi([x, y]_A) = \tilde{\xi}([x, y]_A). \end{aligned}$$

So I is contained in $\ker(\tilde{\xi})$ and $\tilde{\xi}$ induces a morphism $\bar{\xi} : U(\mathfrak{g}) \rightarrow A$ of hom-associative color algebras such that $\tilde{\xi} = \bar{\xi} \circ \pi$. Therefore, $\bar{\xi} \circ \psi = \bar{\xi} \circ \pi \circ i = \tilde{\xi} \circ i = \xi$.

Let $\bar{\xi}' : U(\mathfrak{g}) \rightarrow A$ be another morphism of hom-associative color algebras such that $\bar{\xi}' \circ \psi = \xi$. We must show that $\bar{\xi}(u) = \bar{\xi}'(u)$, for all $u \in U(\mathfrak{g})$. It is sufficient to show that $\bar{\xi}\pi(\mathbf{a}) = \bar{\xi}'\pi(\mathbf{a})$, for $\mathbf{a} \in \mathfrak{g}^{\otimes i}$ with $i \geq 1$, since $T(\mathfrak{g}) = \bigoplus_{i \geq 1} \mathfrak{g}^{\otimes i}$. We do this by using the induction on $i \geq 1$. For $i = 1$, we have

$$(\bar{\xi} \circ \pi)(\mathbf{a}) = (\bar{\xi} \circ \pi \circ i)(\mathbf{a}) = (\tilde{\xi} \circ i)(\mathbf{a}) = \xi(\mathbf{a}) = (\bar{\xi}' \circ \psi)(\mathbf{a}) = (\bar{\xi}' \circ \pi)(\mathbf{a}).$$

Now, assume that the statement holds for $i \geq 1$. Let $\mathbf{a} = \mathbf{a}' \otimes a_{i+1} \in \mathfrak{g}^{\otimes(i+1)}$, where $\mathbf{a}' \in \mathfrak{g}^{\otimes i}$. We have

$$\begin{aligned} (\bar{\xi} \circ \pi)(\mathbf{a}) &= (\bar{\xi} \circ \pi)(\mathbf{a}' \odot a_{i+1}) = \bar{\xi}(\pi(\mathbf{a}')) \cdot_A \bar{\xi}(\pi(a_{i+1})) \\ &= \bar{\xi}'(\pi(\mathbf{a}')) \cdot_A \bar{\xi}'(\pi(a_{i+1})) = (\bar{\xi}' \circ \pi)(\mathbf{a}' \odot a_{i+1}) = (\bar{\xi}' \circ \pi)(\mathbf{a}). \end{aligned}$$

Thus, the uniqueness of $\bar{\xi}$ makes $(U(\mathfrak{g}), \psi)$ a universal enveloping algebra of \mathfrak{g} .

The only thing left to prove is the uniqueness of $U(\mathfrak{g})$ up to isomorphism. Let $(U(\mathfrak{g})_1, \psi_1)$ be another universal enveloping algebra of \mathfrak{g} . By the definition of universal algebra, there exist homomorphisms

$$f : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})_1$$

and

$$f_1 : U(\mathfrak{g})_1 \rightarrow U(\mathfrak{g})$$

of hom-associative color algebras such that $f \circ \psi = \psi_1$ and $f_1 \circ \psi_1 = \psi$. Therefore,

$$f_1 \circ f \circ \psi = \psi = id_{U(\mathfrak{g})} \circ \psi.$$

Since id and $f_1 \circ f$ are both hom-associative homomorphisms, by the uniqueness in the universal property of $(U(\mathfrak{g}), \psi)$, we have $f_1 \circ f = id_{U(\mathfrak{g})}$, and hence

$$f \circ f_1 = id_{U(\mathfrak{g})_1},$$

which completes the proof. □

4. The Poincaré–Birkhoff–Witt theorem

In this section we prove a Poincaré–Birkhoff–Witt like type theorem for involutive color hom-Lie algebras.

Let \mathfrak{g} be a Lie color algebra with an ordered basis

$$X = \{x_n | n \in H\},$$

where H is a well- and totally ordered set. Let I be the ideal of the free associative algebra $T(\mathfrak{g})$ on \mathfrak{g} , which was given in Theorem 3.4, so that $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . The Poincaré–Birkhoff–Witt theorem states that the linear subspace I of $T(\mathfrak{g})$ has a canonical linear complement which has a basis given by

$$W := \{x_{n_1} \otimes \cdots \otimes x_{n_i} | n_1 \geq \cdots \geq n_i, i \geq 0\}, \tag{4.1}$$

called the Poincaré–Birkhoff–Witt basis of $U(\mathfrak{g})$.

To simplify the notations, we denote $\bar{x} := \beta_{\mathfrak{g}}(x)$ for $x \in \mathfrak{g}$ and $\bar{\mathfrak{r}} := \alpha_T(\mathfrak{r})$ for $\mathfrak{r} \in \mathfrak{g}^{\otimes i}$, $i \geq 1$. There exists a linear operator which is introduced in [24].

$$\theta : \mathfrak{g}^{\otimes i} \rightarrow \mathfrak{g}^{\otimes i}, \tag{4.2}$$

which maps every $x := x_1 \otimes \cdots \otimes x_i$ to $\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_i$, where

$$\tilde{x}_n = \begin{cases} \bar{x}_n & \text{if } n = 2k + 1 \text{ and } k \geq 1, \\ x_n & \text{otherwise.} \end{cases}$$

Now, we can study linear generators of the hom-ideal I and express them in terms of the tensor product. Since $\beta_{\mathfrak{g}}$ is involutive, it is also bijective, so

$$\beta_{\mathfrak{g}}(\mathfrak{g}) = \mathfrak{g},$$

and we have the same argument for θ :

$$\theta(\mathfrak{g}^{\otimes i}) = \mathfrak{g}^{\otimes i}.$$

Let us now review some properties of the linear operator α_T and the multiplication \odot which was stated in [24]. First we have

$$\alpha_T^j(\mathfrak{g}^{\otimes i}) = \mathfrak{g}^{\otimes i}, j \geq 0, i \geq 0.$$

Then for any natural numbers $r, s \geq 1$, we have

$$\mathfrak{g}^{\otimes r} \odot \mathfrak{g}^{\otimes s} = \alpha_T^{s-1}(\mathfrak{g}^{\otimes r}) \otimes \mathfrak{g} \otimes \alpha_T(\mathfrak{g}^{\otimes(s-1)}) = \mathfrak{g}^{\otimes r+s}.$$

In the following lemma, for $\mathfrak{a} = a_1 \otimes \cdots \otimes a_i \in \mathfrak{g}^{\otimes i}$ and $i \geq 1$, we denote $l(\mathfrak{a}) = i$.

Lemma 4.1 *Let $\mathfrak{u} = u_1 \otimes \cdots \otimes u_j \in \mathfrak{g}^{\otimes j}$, $\mathfrak{v} = v_1 \otimes \cdots \otimes v_k \in \mathfrak{g}^{\otimes k}$ and $\mathfrak{w} = w_1 \otimes \cdots \otimes w_l \in \mathfrak{g}^{\otimes l}$. Let θ be defined as in (4.2). Then*

$$(i) \quad \theta(\mathfrak{u}) = u_1 \otimes u_2 \otimes \alpha_T(u_3) \otimes \alpha_T^2(u_4) \cdots \otimes \alpha_T^{j-2}(u_j) = u_1 \otimes \bigotimes_{k=2}^j \alpha_T^{k-2}(u_k),$$

$$(ii) \quad \theta(\alpha_T(\mathfrak{u})) = \alpha_T(\theta(\mathfrak{u})),$$

$$(iii) \quad \theta(\mathfrak{u} \otimes \mathfrak{w}) = \theta(\mathfrak{u}) \otimes \alpha_T^{j-1}(w_1) \otimes \cdots \otimes \alpha_T^{j+l-2}(w_l)$$

(iv) If $l(\mathbf{v}) \geq 1$ and $l(\mathbf{u}) = l(\mathbf{v}) + 1$, i.e. $j = k + 1$, then there is $\mathbf{c} \in \mathfrak{g}^{\otimes j}$ such that

$$\theta(\mathbf{u} \otimes \mathbf{w}) = \theta(\mathbf{u}) \otimes \mathbf{c} \quad \text{and} \quad \theta(\mathbf{v} \otimes \mathbf{w}) = \theta(\mathbf{v}) \otimes \alpha_T(\mathbf{c}).$$

Proof Straightforward calculations. □

Theorem 4.2 Let \mathfrak{g} be an involutive color hom-Lie algebra. Let $\theta : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ be as defined in (4.2). Let I be the hom-ideal of $T(\mathfrak{g})$ as defined in Theorem 3.4. Let

$$J = \sum_{n,m \geq 0} \sum_{\substack{\mathbf{a} \in \mathfrak{g}^{\otimes n} \\ \mathbf{b} \in \mathfrak{g}^{\otimes m}}} \sum_{a,b \in \mathfrak{g}} (\mathbf{a} \otimes (a \otimes b - \varepsilon(a,b)b \otimes a) \otimes \mathbf{b} - \alpha_T(\mathbf{a}) \otimes [a, b]_{\mathfrak{g}} \otimes \alpha_T(\mathbf{b})). \quad (4.3)$$

Then

$$(i) \quad I = \sum_{n,m \geq 0} \sum_{a,b \in \mathfrak{g}} (\mathfrak{g}^{\otimes n} \odot (a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m}$$

$$(ii) \quad \theta(I) = J$$

Proof

(i) Since the hom-ideal I is generated by the elements of the form

$$a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}},$$

for all $a, b \in \mathfrak{g}$, the right-hand side is contained in the left-hand side. To prove the opposite, we just need to prove that the right-hand side is a hom-ideal of $T(\mathfrak{g})$, i.e. it is closed under the left and right multiplication and the operator α_T . Therefore, we should check these one by one. For any natural number $k \geq 0$, we have

$$\begin{aligned} & ((\mathfrak{g}^{\otimes n} \odot (a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m}) \odot \mathfrak{g}^{\otimes k} \\ &= \alpha_T(\mathfrak{g}^{\otimes n} \odot (a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot (\mathfrak{g}^{\otimes m} \odot \alpha_T(\mathfrak{g}^{\otimes k})) \\ &= \alpha_T(\mathfrak{g}^{\otimes n} \odot (a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m+k} \\ &= (\alpha_T(\mathfrak{g}^{\otimes n}) \odot \alpha_T(a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m+k} \\ &= (\mathfrak{g}^{\otimes n} \odot (\alpha_T(a) \otimes \alpha_T(b) - \varepsilon(a,b)\alpha_T(b) \otimes \alpha_T(a) - [\alpha_T(a), \alpha_T(b)]_{\mathfrak{g}})) \\ & \quad \odot \mathfrak{g}^{\otimes m+k}, \end{aligned}$$

which is seen to be contained in

$$\sum_{x,y \in \mathfrak{g}} (\mathfrak{g}^{\otimes n} \odot (x \otimes y - \varepsilon(x,y)y \otimes x - [x, y]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m+k}.$$

Thus the right-hand side is closed under the right multiplication. We can also get

$$\begin{aligned} & \alpha_T((\mathfrak{g}^{\otimes n} \odot (a \otimes b - \varepsilon(a,b)b \otimes a - [a, b]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m}) \\ &= (\mathfrak{g}^{\otimes n} \odot (\alpha_T(a) \otimes \alpha_T(b) - \varepsilon(a,b)\alpha_T(b) \otimes \alpha_T(a) - [\alpha_T(a), \alpha_T(b)]_{\mathfrak{g}})) \\ & \quad \odot \mathfrak{g}^{\otimes m}, \end{aligned}$$

which is contained in

$$\sum_{x,y \in \mathfrak{g}} (\mathfrak{g}^{\otimes n} \odot (x \otimes y - \varepsilon(x,y)y \otimes x - [x,y]_{\mathfrak{g}})) \odot \mathfrak{g}^{\otimes m}.$$

So the right-hand side is a hom-ideal of $(T(\mathfrak{g}), \odot, \alpha_T)$, which also contains the elements of the form

$$(x \otimes y - \varepsilon(x,y)y \otimes x - [x,y]_{\mathfrak{g}})$$

for all $x, y \in \mathfrak{g}$. Therefore, it contains the left-hand side.

- (ii) We first prove that $\theta(I)$ is contained in J . By the first part of the proposition, we just need to verify that the element

$$\theta((\mathfrak{a} \odot (x \otimes y - \varepsilon(x,y)y \otimes x - [x,y]_{\mathfrak{g}})) \odot \mathfrak{b})$$

is contained in J , for $x, y \in \mathfrak{g}$ and $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m}$, respectively, $n, m \geq 0$.

We have

$$\begin{aligned} & (\mathfrak{a} \odot (x \otimes y - \varepsilon(x,y)y \otimes x - [x,y]_{\mathfrak{g}})) \odot \mathfrak{b} \\ &= (\alpha_T(\mathfrak{a}) \otimes (x \otimes \alpha_T(y) - \varepsilon(x,y)y \otimes \alpha_T(x)) - \mathfrak{a} \otimes [x,y]_{\mathfrak{g}}) \odot \mathfrak{b} \\ &= \alpha_T^{m-1}(\alpha_T(\mathfrak{a}) \otimes (x \otimes \alpha_T(y) - \varepsilon(x,y)y \otimes \alpha_T(x)) \otimes b_1 \\ & \otimes \alpha_T(b_2 \otimes \cdots \otimes b_m) \\ & - \alpha_T^{m-1}(\mathfrak{a} \otimes [x,y]_{\mathfrak{g}}) \otimes b_1 \otimes \alpha_T(b_2 \otimes \cdots \otimes b_m), \end{aligned}$$

by the definition of \odot . Furthermore, according to Lemma 4.1 (iv), there exists $\mathfrak{c} \in \mathfrak{g}^{\otimes n}$ such that

$$\begin{aligned} & \theta((\mathfrak{a} \odot (x \otimes y - \varepsilon(x,y)y \otimes x - [x,y]_{\mathfrak{g}})) \odot \mathfrak{b}) \\ &= \theta(\alpha_T^{m-1}(\alpha_T(\mathfrak{a}) \otimes (x \otimes \alpha_T(y) - \varepsilon(x,y)y \otimes \alpha_T(x)))) \otimes \mathfrak{c} \\ & - \theta(\alpha_T^{m-1}(\mathfrak{a} \otimes [x,y]_{\mathfrak{g}})) \otimes \alpha_T(\mathfrak{c}) \\ &= \theta(\alpha_T^{m-1}(\mathfrak{a}) \otimes (\alpha_T^{m-1}(x) \otimes (x \otimes \alpha_T(y) - \varepsilon(x,y)y \otimes \alpha_T(x)))) \otimes \mathfrak{c} \\ & - \theta(\alpha_T^{m-1}(\mathfrak{a}) \otimes [\alpha_T^{m-1}(x), \alpha_T^{m-1}(y)]_{\mathfrak{g}}) \otimes \alpha_T(\mathfrak{c}) \\ &= \theta(\alpha_T^m(\mathfrak{a})) \otimes (\alpha_T^{n-1}(\alpha_T^{m-1}(x)) \otimes \alpha_T^n(\alpha_T^m(y)) - \varepsilon(x,y)\alpha_T^{n-1}(\alpha_T^{m-1}(y)) \\ & \otimes \alpha_T^n(\alpha_T^m(x))) \otimes \mathfrak{c} - \theta(\alpha_T^{m-1}(\mathfrak{a}) \otimes [\alpha_T^{m-1}(x), \alpha_T^{m-1}(y)]_{\mathfrak{g}}) \otimes \alpha_T(\mathfrak{c}) \\ &= \theta(\alpha_T^m(\mathfrak{a})) \otimes (\alpha_T^{n+m}(x) \otimes \alpha_T^{n+m}(y) - \varepsilon(x,y)\alpha_T^{n+m}(y) \otimes \alpha_T^{n+m}(x)) \otimes \mathfrak{c} \\ & - \alpha_T(\theta(\alpha_T^m(\mathfrak{a}))) \otimes [\alpha_T^{n+m}(x), \alpha_T^{n+m}(y)]_{\mathfrak{g}} \otimes \alpha_T(\mathfrak{c}). \end{aligned}$$

This is an element in

$$\sum_{\substack{\mathfrak{u} \in \mathfrak{g}^{\otimes n} \\ \mathfrak{v} \in \mathfrak{g}^{\otimes m}}} \sum_{s,t \in \mathfrak{g}} \mathfrak{u} \otimes (s \otimes t - \varepsilon(s,t)t \otimes s) \otimes \mathfrak{v} - \alpha_T(\mathfrak{u}) \otimes [s,t]_{\mathfrak{g}} \otimes \alpha_T(\mathfrak{v})$$

if we simply take $\mathfrak{u} := \theta(\alpha_T^m(\mathfrak{a}))$, $s := \alpha_T^{m+n}(x)$ and $t := \alpha_T^{m+n}(y)$. Therefore, $\theta(I)$ is contained in J . Conversely, since θ and α_T are bijective, the above argument shows that any term

$$\mathfrak{u} \otimes (s \otimes t - \varepsilon(s,t)t \otimes s) \otimes \mathfrak{v} - \alpha_T(\mathfrak{u}) \otimes [s,t]_{\mathfrak{g}} \otimes \alpha_T(\mathfrak{v}),$$

in the previous sum can be expressed in the form

$$\theta((\mathbf{a} \odot (x \otimes y - \varepsilon(x, y)y \otimes x - [x, y]_{\mathfrak{g}})) \odot \mathbf{b}),$$

which shows the surjectivity of θ and completes the proof. □

In the next theorem, we suppose that \mathfrak{g} is an involutive color hom-Lie algebra with a basis $X = \{x_n | n \in \omega\}$ for a well-ordered set ω .

Theorem 4.3 *Let $\mathfrak{g} := (\mathfrak{g}, [,]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive color hom-Lie algebra such that $\beta_{\mathfrak{g}}(X) = X$. Let W be the one defined in (4.1) and let $\mu \in k$ be given. If we define*

$$J_{\mu} := \sum_{n,m \geq 0} \sum_{\substack{\mathbf{a} \in \mathfrak{g}^{\otimes n} \\ \mathbf{b} \in \mathfrak{g}^{\otimes m}}} \sum_{\mathbf{a}, \mathbf{b} \in \mathfrak{g}} (\mathbf{a} \otimes (a \otimes b - \varepsilon(a, b)b \otimes a) \otimes \mathbf{b} - \mu^{n+m} \alpha_T(\mathbf{a}) \otimes [a, b]_{\mathfrak{g}} \otimes \alpha_T(\mathbf{b})), \quad (4.4)$$

then we can have the linear decomposition

$$T(\mathfrak{g}) = J_{\mu} \oplus kW.$$

Proof Let us first introduce some notations. For $i \geq 2$ let

$$\mathfrak{r} := x_{n_1} \otimes x_{n_2} \otimes \cdots \otimes x_{n_i} \in X^{\otimes i} \subseteq \mathfrak{g}^{\otimes i}.$$

Define the index of \mathfrak{r} to be

$$d := |\{(r, s) | r < s, n_r < n_s, 1 \leq r, s \leq i\}|.$$

Let $\mathfrak{g}_{i,d}$ be the linear span of all pure tensors \mathfrak{r} of degree i and index d . Then we have

$$\mathfrak{g}^{\otimes i} = \bigoplus_{d \geq 0} \mathfrak{g}_{i,d}.$$

In particular, $\mathfrak{g}_{i,0} = kW^{(i)}$, where

$$W^{(i)} := \{x_{n_1} \otimes x_{n_2} \otimes \cdots \otimes x_{n_i} \in X^{\otimes i} | n_1 \geq n_2 \geq \cdots \geq n_i\}.$$

In order to prove $T(\mathfrak{g}) = J_{\mu} \oplus kW$, we need to prove that

$$\mathfrak{g}^{\otimes i} \subseteq J_{\mu} \oplus \sum_{1 \leq q \leq i} kW^{(q)},$$

using induction on $i \geq 1$. For $i = 1$, we have $kW^{(1)} = \mathfrak{g}$. So $\mathfrak{g} \subseteq J_{\mu} \oplus kW^{(1)}$. Suppose that the above equation is true for $n \geq 1$. Since $\mathfrak{g}^{\otimes(i+1)} = \sum_{d \geq 0} \mathfrak{g}_{i+1,d}$, we just need to prove that

$$\mathfrak{g}_{i+1,d} \subseteq J_{\mu} \oplus \sum_{1 \leq q \leq i+1} kW^{(q)},$$

for all $d \geq 0$. We use induction on d . For $d = 0$, we have $\mathfrak{g}_{i+1,0} = kW^{(i+1)}$. Suppose for $l \geq 0$ we have

$$\mathfrak{g}_{i+1,l} \subseteq J_{\mu} \oplus \sum_{1 \leq q \leq i+1} kW^{(q)}.$$

Let $\mathfrak{x} = x_{n_1} \otimes x_{n_2} \otimes \cdots \otimes x_{n_i} \in X^{\otimes(i+1)} \cap \mathfrak{q}_{i+1, l+1}$. Since $l + 1 \geq 1$, we can choose an integer $1 \leq r \leq i$ such that $n_r \leq n_{r+1}$. Let

$$\mathfrak{x}' = x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}$$

be the pure tensor formed by interchanging x_{n_r} by $x_{n_{r+1}}$ in \mathfrak{x} . Then

$$\mathfrak{x}' \in \mathfrak{g}_{i+1, l} \subseteq J_\mu \oplus \sum_{1 \leq q \leq i+1} kW^{(q)}.$$

Since the definition of J_μ gives

$$\mathfrak{x} - \mathfrak{x}' \equiv \mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_i}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}}) \pmod{J_\mu},$$

by the induction hypothesis on i , we have

$$\mathfrak{x} \in J_\mu \oplus \sum_{1 \leq q \leq i+1} kW^{(q)} \oplus \sum_{1 \leq q \leq i} kW^{(q)}.$$

So \mathfrak{x} is in $J_\mu \oplus \sum_{1 \leq q \leq i+1} kW^{(q)}$. This proves that $\mathfrak{g}_{i+1, l+1} \subseteq J_\mu \oplus \sum_{1 \leq q \leq i+1} kW^{(q)}$. Hence, $\mathfrak{g}^{\otimes i+1} \subseteq J_\mu \oplus \sum_{1 \leq q \leq i+1} kW^{(q)}$ which completes the induction steps on i .

Now, we want to show that $J_\mu \cap kW = 0$. Let S be an operator on $T(\mathfrak{g})$ such that

$$\begin{aligned} \text{(i)} \quad & S(t) = t, \text{ for all } t \in W. \\ \text{(ii)} \quad & \text{if } p \geq 2, 1 \leq s \leq p-1, \text{ and } n_s < n_{s+1}, \text{ then} \\ & S(x_{n_1} \otimes \cdots \otimes x_{n_s} \otimes x_{n_{s+1}} \otimes \cdots \otimes x_{n_p}) \\ & = S(x_{n_1} \otimes \cdots \otimes x_{n_{s+1}} \otimes x_{n_s} \otimes \cdots \otimes x_{n_p}) \\ & + S(\mu^{p-2} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{s-1}}) \otimes [x_{n_s}, x_{n_{s+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{s+2}} \otimes \cdots \otimes x_{n_p})). \end{aligned} \tag{4.5}$$

We define S on $\sum_{1 \leq q \leq i} \mathfrak{g}^{\otimes q}$ by induction on i . For $i = 1$, we define $S := Id_{\mathfrak{g}}$. Let $n \geq 2$ and let S be an operator on $\sum_{1 \leq q \leq i} \mathfrak{g}^{\otimes q}$ satisfying (4.5) for all tensors of degree i . Note that $\mathfrak{g}^{\otimes i+1} = \sum_{d \leq 0} \mathfrak{g}_{i+1, d}$. For a pure tensor

$$\mathfrak{x} = x_{n_1} \otimes x_{n_2} \otimes \cdots \otimes x_{n_{i+1}} \in X^{\otimes i+1} \subseteq \mathfrak{g}^{\otimes i+1},$$

we use induction again on d which is the index of \mathfrak{x} , in order to extend S to an operator on $\sum_{1 \leq q \leq i+1} \mathfrak{g}^{\otimes q}$. For $d = 0$, define $S(\mathfrak{x}) = \mathfrak{x}$. For $l \geq 0$, suppose that $S(\mathfrak{x})$ has been defined for

$$\mathfrak{x} \in \sum_{1 \leq p \leq l} \mathfrak{g}_{i+1, p}.$$

Let $\mathfrak{x} \in \mathfrak{g}_{i+1, l+1}$. Let $1 \leq r \leq i$ be an integer such that $n_r < n_{r+1}$. Then

$$\begin{aligned} S(\mathfrak{x}) & := S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})). \end{aligned}$$

We should show that S is well-defined and it is independent of the choice of r . Therefore, let r' be another integer, $1 \leq r' \leq i$, such that $n_{r'} < n_{r'+1}$. Consider

$$\begin{aligned} u := & S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})), \end{aligned}$$

and

$$\begin{aligned} v := & S(x_{n_1} \otimes \cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r'-1}}) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r'+2}} \otimes \cdots \otimes x_{n_{i+1}})). \end{aligned}$$

We check that $u = v$. There appear two cases:

Case 1: If $|r - r'| \geq 2$, without losing the generality, we assume $r - r' \geq 2$. Since $u, v \in \sum_{0 \leq p \leq l} \mathfrak{g}^{i+1, p} + \sum_{1 \leq q \leq i} \mathfrak{g}^{\otimes q}$, we have

$$\begin{aligned} u = & S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{r'}} \otimes x_{n_{r'+1}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})) \\ = & S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}} \otimes x_{n_r} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \quad \otimes \alpha_T(\cdots \otimes x_{n_{r+1}})) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \quad \otimes \alpha_T(\cdots \otimes x_{n_{r'}} \otimes x_{n_{r'+1}} \otimes \cdots \otimes x_{n_{i+1}})), \\ v = & S(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r-1}} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \quad \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})) \\ = & S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \quad \otimes \alpha_T(\cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'+1}} \otimes x_{n_{i+1}})) \\ & + S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \quad \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})), \end{aligned}$$

by the induction hypothesis. Now, since $\beta_{\mathfrak{g}}(X) = X$ and

$$x_{n_r} \neq x_{n_{r+1}}, \quad x_{n_{r'}} \neq x_{n_{r'+1}},$$

we get $\alpha_T(x_{n_r}), \alpha_T(x_{n_{r+1}}), \alpha_T(x_{n_{r'}}), \alpha_T(x_{n_{r'+1}}) \in X$ and

$$\alpha_T(x_{n_r}) \neq \alpha_T(x_{n_{r+1}}), \quad \alpha_T(x_{n_{r'}}) \neq \alpha_T(x_{n_{r'+1}}).$$

This leads to four different cases among which we consider only the case of $\alpha_T(x_{n_r}) > \alpha_T(x_{n_{r+1}})$ and $\alpha_T(x_{n_{r'}}) < \alpha_T(x_{n_{r'+1}})$ without losing the generality. We obtain

$$\begin{aligned} & S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\ & = S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{2i-3}(x_{n_1} \otimes \cdots \otimes [\alpha_T(x_{n_{r+1}}), \alpha_T(x_{n_r})]_{\mathfrak{g}} \\ & \otimes \cdots \otimes [\alpha_T(x_{n_{r'}}, \alpha_T(x_{n_{r'+1}})]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})), \end{aligned}$$

and

$$\begin{aligned} & S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{r'}} \otimes x_{n_{r'+1}} \otimes \cdots \otimes x_{n_{i+1}})) \\ & = S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{2i-3}(x_{n_1} \otimes \cdots \otimes [\alpha_T(x_{n_r}), \alpha_T(x_{n_{r+1}})]_{\mathfrak{g}} \\ & \otimes \cdots \otimes [\alpha_T(x_{n_{r'}}, \alpha_T(x_{n_{r'+1}})]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})). \end{aligned}$$

If we combine the two former expressions for u , we get

$$\begin{aligned} u & = S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{2i-3}(x_{n_1} \otimes \cdots \otimes [\alpha_T(x_{n_{r+1}}), \alpha_T(x_{n_r})]_{\mathfrak{g}} \otimes \cdots \\ & \otimes [\alpha_T(x_{n_{r'}}, \alpha_T(x_{n_{r'+1}})]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{2i-3}(x_{n_1} \otimes \cdots \otimes [\alpha_T(x_{n_r}), \alpha_T(x_{n_{r+1}})]_{\mathfrak{g}} \otimes \cdots \\ & \otimes [\alpha_T(x_{n_{r'}}, \alpha_T(x_{n_{r'+1}})]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})) \\ & = S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}}) \\ & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots) \otimes [x_{n_{r'}}, x_{n_{r'+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\ & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\ & \otimes \alpha_T(\cdots \otimes x_{n_{r'+1}} \otimes x_{n_{r'}} \otimes \cdots \otimes x_{n_{i+1}})) = v, \end{aligned}$$

by the skew-symmetry condition of the bracket.

Case 2: If $|r - r'| = 1$, once again, without losing the generality, let us suppose $r' = r + 1$. This leads to $n_r < n_{i+1} < n_{i+2}$. We obtain

$$\begin{aligned}
 u &= S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\
 &\quad \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 &= S(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}} \otimes x_{n_{r+2}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}}) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\
 &\quad \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 &= S(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}} \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_r} \otimes \cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}}) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\
 &\quad \otimes \alpha_T(x_{n_{r+2}} \otimes \cdots \otimes x_{n_{i+1}})),
 \end{aligned}$$

and

$$\begin{aligned}
 v &= S(x_{n_1} \otimes \cdots \otimes x_{n_r} \otimes x_{n_{r+2}} \otimes x_{n_{r+1}} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 &= S(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}} \otimes x_{n_r} \otimes x_{n_{r+1}} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+1}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 &= S(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}} \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+1}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 &+ S(\mu^{i-1} \alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})).
 \end{aligned}$$

Moreover, for any $a < b$, $t_1 \in \mathfrak{g}^{\otimes m}$, $t_2 \in \mathfrak{g}^{\otimes k}$, $m + k = i - 2$, we have

$$S(t_1 \otimes a \otimes b \otimes t_2) - S(t_1 \otimes b \otimes a \otimes t_2) = S(\mu^{m+k} \alpha_T(t_1) \otimes [a, b] \otimes \alpha_T(t_2)).$$

So the sum of the last three terms of the previous expression of u is

$$\begin{aligned}
 & S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_r} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+1}}) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{i+2}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & = S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes \alpha_T(x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{2i-3}(x_{n_1} \otimes \cdots \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}}, \alpha_T(x_{n_r}))_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{r+1}}) \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{2i-3}x_{n_1} \otimes \cdots \otimes x_{n_{r-1}} \otimes [\alpha_T(x_{n_{r+1}}), [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}}]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r-1}}) \otimes \alpha_T(x_{n_{r+2}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \\
 & \quad \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{2i-3}x_{n_1} \otimes \cdots \otimes x_{n_{r-1}} \otimes [[x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}}, \alpha_T(x_{n_{r+2}})]_{\mathfrak{g}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & = S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{i+1}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}}))
 \end{aligned}$$

using the hom-Jacobi identity. Thus, we can obtain

$$\begin{aligned}
 u & = S(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}} \otimes x_{n_{r+1}} \otimes x_{n_r} \otimes \cdots \otimes x_{n_{i+1}}) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_r}) \otimes [x_{n_{r+1}}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots) \otimes [x_{n_r}, x_{n_{r+2}}]_{\mathfrak{g}} \otimes \alpha_T(x_{n_{i+1}} \otimes \cdots \otimes x_{n_{i+1}})) \\
 & + S(\mu^{i-1}\alpha_T(x_{n_1} \otimes \cdots \otimes x_{n_{r+2}}) \otimes [x_{n_r}, x_{n_{r+1}}]_{\mathfrak{g}} \otimes \alpha_T(\cdots \otimes x_{n_{i+1}})) \\
 & = v.
 \end{aligned}$$

Now that $u = v$ in either cases, let $\mathfrak{r} \in J_{\mathfrak{g},\beta} \cap kW$. Then $S(\mathfrak{r}) = \mathfrak{r}$ and $S(\mathfrak{r}) = 0$. Therefore $\mathfrak{r} = 0$ and we get that $J_{\mathfrak{g},\beta} \cap kW = 0$ which completes the proof. \square

We are now ready to prove the Poincaré–Birkhoff–Witt theorem for involutive color hom-Lie algebras in the second part of the next theorem.

Theorem 4.4 *Let k be a field whose characteristic is not 2. Let $\mathfrak{g} := (\mathfrak{g}, [,]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive color hom-Lie algebra on k . Let $\theta : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ be as described in (4.2). Let I be the hom-ideal of $T(\mathfrak{g})$ generated by the commutators defined in Theorem 3.4. Let J be as defined in (4.4). Then there is a well-ordered basis X of \mathfrak{g} such that for*

$$W = W_X = \{x_{i_1} \otimes \cdots \otimes x_{i_n} \mid i_1 \geq \cdots \geq i_n, n \geq 0\},$$

the following statements hold.

(i) $T(\mathfrak{g}) = J \oplus kH,$

(ii) $\theta(W)$ is a basis of $U(\mathfrak{g})$.

Proof

(i) Let \mathfrak{g}_+ and \mathfrak{g}_- be the eigenspaces of 1 and -1 of \mathfrak{g} , respectively. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

Let B_+ and B_- be some basis for \mathfrak{g}_+ and \mathfrak{g}_- , respectively. There are two different cases:

If the cardinality of B_+ is more than the cardinality of B_- , fix an injection $\iota : B_- \rightarrow B_+$. Then the following set is a basis of \mathfrak{g}

$$X := \{\iota(x) + x, \iota(x) - x | x \in B_-\} \cup (B_+ \setminus B_-),$$

and $\beta_{\mathfrak{g}}(X) = X$. Let W be defined with a well order on X and choose $\mu = 1$ in (4.4). Then we have

$$T(\mathfrak{g}) = J_{\mathfrak{g}, \beta, 1} \oplus kW.$$

If the cardinality of B_+ is not more than the cardinality of B_+ , it suffices to assume $\gamma := -\beta_{\mathfrak{g}}$ and take $\mu = -1$ as it is in the last case.

(ii) follows directly from Theorem 4.3 and (i).

□

5. Hom-Lie superalgebras case

The study of hom-Lie superalgebras has been widely in the center of interest these last years. The motivation came from the generalization of Lie superalgebras, or in some cases, the generalization of hom-Lie algebras. However, here, we deal with them as a special case of color hom-Lie algebras. One can simply put $\Gamma = \mathbb{Z}_2$ in Definition 2.6 and define ε in such a way that $\varepsilon(x, y) = (-1)^{|x||y|}$ to get the following definition [1, 7, 13, 14, 39, 68].

Definition 5.1 A hom-Lie superalgebra is a triple $(\mathfrak{g}, [,], \alpha)$ consisting of a superspace \mathfrak{g} , a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and a superspace homomorphism $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$, both of which are degree zero satisfying

1. $[x, y] = -(-1)^{|x||y|}[y, x]$,
2. $(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|y||x|}[\alpha(y), [z, x]] + (-1)^{|z||y|}[\alpha(z), [x, y]] = 0$,

for all homogeneous elements $x, y, z \in \mathfrak{g}$.

In particular, one can easily recall from the first section, the notions of a multiplicative hom-Lie superalgebra, a morphism of hom-Lie superalgebras, and a hom-associative superalgebra. Moreover, a hom-associative superalgebra or a hom-Lie superalgebra is said to be involutive if $\alpha^2 = id$.

Again, as in the previous cases, a hom-associative superalgebra (V, μ, α) gives a hom-Lie superalgebra by antisymmetrization. We denote this hom-Lie superalgebra again by $(A, [,]_A, \beta_A)$, where $\beta_A = \alpha$, $[x, y]_A = xy - yx$, for all $x, y \in A$.

Let (V, α_V) be an involutive hom-module. A free involutive hom-associative color algebra on V is an involutive hom-associative super algebra $(F(V), *, \alpha_F)$ together with a morphism of hom-modules $j_V : (V, \alpha_V) \rightarrow (F(V), \alpha_F)$ with the property that, for any involutive hom-associative superalgebra (A, \cdot, α_A) together with a morphism $f : (V, \alpha_V) \rightarrow (A, \alpha_A)$ of hom-modules, there is a unique morphism $\bar{f} : (F(V), *, \alpha_F) \rightarrow (A, \cdot, \alpha_A)$ of hom-associative superalgebras such that $f = \bar{f} \circ j_V$.

The definition of the universal enveloping algebra, as can be predicted, is just a modification of the Definition 3.2.

Definition 5.2 Let $(\mathfrak{g}, [,], \alpha)$ be a hom-Lie superalgebra. A universal enveloping hom-associative superalgebra of \mathfrak{g} is a hom-associative superalgebra

$$U(\mathfrak{g}) := (U(\mathfrak{g}), \mu_U, \alpha_U),$$

together with a morphism $\varphi_{\mathfrak{g}} : \mathfrak{g} \rightarrow U(\mathfrak{g})$ of hom-Lie superalgebras such that for any hom-associative superalgebra (A, μ, α_A) and any hom-Lie superalgebra morphism $\phi : (\mathfrak{g}, [,], \beta_{\mathfrak{g}}) \rightarrow (A, \mu, \alpha_A)$, there exists a unique morphism $\bar{\phi} : U(\mathfrak{g}) \rightarrow A$ of hom-associative superalgebras such that $\bar{\phi} \circ \varphi_{\mathfrak{g}} = \phi$.

The following lemma shows an easy way to construct the universal algebra when we have an involutive hom-Lie superalgebra.

Lemma 5.3 Let $(\mathfrak{g}, [,], \beta_{\mathfrak{g}})$ be an involutive hom-Lie superalgebra.

- (i) Let (A, \cdot, α_A) be a hom-associative algebra, $f : (\mathfrak{g}, [,], \beta_{\mathfrak{g}}) \rightarrow (A, [,]_A, \beta_A)$ be a morphism of hom-Lie superalgebras and B be the hom-associative subsuperalgebra of A generated by $f(\mathfrak{g})$. Then B is involutive.
- (ii) The universal enveloping hom-associative algebra $(U(\mathfrak{g}), \varphi_{\mathfrak{g}})$ of $(\mathfrak{g}, [,], \beta_{\mathfrak{g}})$ is involutive.
- (iii) In order to verify the universal property of $(U(\mathfrak{g}), \varphi_{\mathfrak{g}})$ in Definition 5.2, we need to consider only the involutive hom-associative algebras $A := (A, \cdot_A, \alpha_A)$.

We can now give the construction of the universal enveloping hom-associative superalgebra of an involutive hom-Lie superalgebra.

Theorem 5.4 Let $\mathfrak{g} := (\mathfrak{g}, [,], \beta_{\mathfrak{g}})$ be an involutive hom-Lie superalgebra. Let

$$T(\mathfrak{g}) := (T(\mathfrak{g}), \odot, \alpha_T)$$

be the free hom-associative algebra on the hom-module underlying \mathfrak{g} . Let I be the hom-ideal of $T(\mathfrak{g})$ generated by the set

$$\{a \otimes b - (-1)^{|a||b|} b \otimes a - [a, b]\} \tag{5.1}$$

and let

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I}$$

be the quotient hom-associative algebra. Let ψ be the composition of the natural inclusion $i : \mathfrak{g} \rightarrow T(\mathfrak{g})$ with the quotient map $\pi : T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. Then $(U(\mathfrak{g}), \psi)$ is a universal enveloping hom-associative algebra of \mathfrak{g} . Also, the universal enveloping hom-associative algebra of \mathfrak{g} is unique up to isomorphism.

We can also have a Poincaré–Birkhoff–Witt-like theorem for involutive hom-Lie superalgebras as a special case of color hom-Lie algebras, i.e. if \mathfrak{g} is a Lie superalgebra with an ordered basis $X = \{x_n | n \in H\}$ where H is a well- and totally ordered set, let I be the ideal of the free associative algebra $T(\mathfrak{g})$ on \mathfrak{g} , which was given in Theorem 5.4, so that $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . We also suppose that \mathfrak{g} is an involutive hom-Lie superalgebra with a basis $X = \{x_n | n \in \omega\}$ for a well-ordered set ω . The next theorem, which is a combination of theorems in the previous section, will help us give the Poincaré–Birkhoff–Witt theorem for involutive hom-Lie superalgebras.

Theorem 5.5 *Let $\mathfrak{g} := (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive hom-Lie superalgebra such that $\beta_{\mathfrak{g}}(X) = X$. Let $\theta : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ be as defined in (4.2), I be the hom-ideal of $T(\mathfrak{g})$ as defined in Theorem 5.4, let W be like the one defined in (4.1) and let $\mu \in k$ be given. Moreover, let*

$$J_{\mu} := \sum_{n,m \geq 0} \sum_{\substack{\mathfrak{a} \in \mathfrak{g}^{\otimes n} \\ \mathfrak{b} \in \mathfrak{g}^{\otimes m}}} \sum_{a,b \in \mathfrak{g}} (\mathfrak{a} \otimes (a \otimes b - (-1)^{|a||b|} b \otimes a) \otimes \mathfrak{b} - \mu^{n+m} \alpha_T(\mathfrak{a}) \otimes [a, b]_{\mathfrak{g}} \otimes \alpha_T(\mathfrak{b})), \quad (5.2)$$

Then

$$(i) \quad I = \sum_{n,m \geq 0} \sum_{a,b \in \mathfrak{g}} (\mathfrak{g}^{\otimes n} \odot (a \otimes b - (-1)^{|a||b|} b \otimes a - [a, b]_{\mathfrak{g}}) \odot \mathfrak{g}^{\otimes m}.$$

$$(ii) \quad \theta(I) = J$$

$$(iii) \quad \text{We can have the linear decomposition } T(\mathfrak{g}) = J_{\mu} \oplus kW.$$

Finally, we give the Poincaré–Birkhoff–Witt theorem for involutive hom-Lie superalgebras.

Theorem 5.6 *Let k be a field whose characteristic is not 2. Let $\mathfrak{g} := (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive hom-Lie superalgebra on k . Let $\theta : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ be as described in (4.2). Let I be the hom-ideal of $T(\mathfrak{g})$ generated by the commutators defined in Theorem 5.4. Let J be as defined in (5.2). Then there is a well-ordered basis X of \mathfrak{g} such that for*

$$W = W_X = \{x_{i_1} \otimes \cdots \otimes x_{i_n} | i_1 \geq \cdots \geq i_n, n \geq 0\}$$

the following statements hold.

$$(i) \quad T(\mathfrak{g}) = J \oplus kW,$$

$$(ii) \quad \theta(W) \text{ is a basis of } U(\mathfrak{g}).$$

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