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Harmonic numbers associated with inversion numbers in terms of determinants

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Abstract: It has been known that some numbers, including Bernoulli, Cauchy, and Euler numbers, have such corresponding numbers in terms of determinants of Hessenberg matrices. There exist inversion relations between the original numbers and the corresponding numbers. In this paper, we introduce the numbers related to harmonic numbers in determinants. We also give several of their arithmetical and/or combinatorial properties and applications. These concepts can be generalized in the case of hyperharmonic numbers.

Key words: Harmonic numbers, hyperharmonic numbers, recurrence relations, determinants, convolutions

1. Introduction

It is known that Bernoulli numbers B_n, defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}\,,$$

have determinant expressions such as

$$B_n=(-1)^n n!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \tag{1}$$

([6, p. 53]).

Cauchy numbers c_n, defined by

$$\frac{x}{\ln(1+x)}=\sum_{n=0}^\infty c_n\frac{x^n}{n!}\,,$$

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also have determinant expressions such as

$$c_n = n!\begin{vmatrix} \frac{1}{2} & 1 & 0 & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix}$$

([6, p. 50]).

Similarly, Euler numbers E_n, defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}\,,$$

have determinant expressions such as

$$E_{2n} = (-1)^n (2n)!\begin{vmatrix} \frac{1}{2!} & 1 & 0 & & \\ \frac{1}{4!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}$$

(cf. [6, p. 52]). Note that $E_n = 0$ if n is odd.

In the aspect of determinants of the (lower) Hessenberg matrices, there exist the inversion numbers. See the later section about Trudi's formula.

$$\frac{(-1)^n B_n}{n!} \Longleftrightarrow \frac{1}{(n+1)!}$$

$$\frac{c_n}{n!} \Longleftrightarrow \frac{1}{n+1}$$

$$\frac{(-1)^n E_{2n}}{(2n)!} \Longleftrightarrow \frac{1}{(2n)!}$$

For example, $(-1)^n B_n/n!$ can be expressed in terms of $1/(k+1)!$ $(k = 1, 2, \dots, n)$ in the determinant and vice versa. In addition, it is known that some hypergeometric numbers also have the corresponding inversion numbers (see, e.g., [12]). Recently it was proved that the complementary Euler numbers ([8]) and Lehmer's generalized Euler numbers ([10, 14]) also have the corresponding inversion numbers.

Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ be harmonic numbers. In this paper, we introduce the determinantal harmonic numbers h_n, so that the harmonic numbers H_n appear in determinant expressions. Namely, we have the corresponding inversion relation:

$$H_n \Longleftrightarrow h_n\,.$$

We give several of their arithmetical and/or combinatorial properties and applications. These concepts can be generalized in the case of hyperharmonic numbers $H_n^{(r)}$, defined by $H_n^{(0)} = \frac{1}{n}$ and $H_n^{(r)} = \sum_{k=1}^n H_k^{(r-1)}$ $(r > 0)$.

2. Definitions and preliminary properties

For nonnegative integers n, define determinantal harmonic numbers h_n by

$$\frac{1+x}{1+x-\ln(1+x)}=\sum_{n=0}^\infty h_n x^n \quad (|x|<1)\,. \tag{2}$$

We have the list of the numbers h_n.

n	0	1	2	3	4	5	6	7	8	9	10
h_n	1	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{2}{15}$	$-\frac{23}{360}$	$\frac{241}{2520}$	$-\frac{47}{1680}$	$-\frac{403}{15120}$	$\frac{139}{4725}$

Definition (2) may be obvious or artificial for readers with different backgrounds. However, there are motivations from combinatorics and in particular graph theory. In 1989, Cameron [3] considered the operator A defined on the set of sequences of nonnegative integers as follows: for $x=\{x_n\}_{n\ge 1}$ and $z=\{z_n\}_{n\ge 1}$, set $Ax=z$, where

$$1+\sum_{n=1}^\infty z_n t^n=\left(1-\sum_{n=1}^\infty x_n t^n\right)^{-1}\,. \tag{3}$$

The operator A also plays an important role for free associative (noncommutative) algebras. More motivations and background together with many concrete examples (in particular, for aspects of graph theory) for this operator can be seen in [3].

There is a recurrence relation for determinantal harmonic numbers.

Lemma 1 *For any integer $n\ge 1$,*

$$h_n=\sum_{k=0}^{n-1}(-1)^{n-k-1}H_{n-k}h_k$$

with $h_0=1$.

Proof [Proof of Lemma 1] Notice that the generating function of harmonic numbers H_n is given by

$$\sum_{n=1}^\infty H_n z^n=-\frac{\ln(1-z)}{1-z}\,.$$

By definition (2),

$$\begin{aligned}1&=\left(\sum_{n=0}^\infty h_n x^n\right)\left(1-\frac{\ln(1+x)}{1+x}\right)\\&=\left(\sum_{n=0}^\infty h_n x^n\right)\left(1+\sum_{l=1}^\infty(-1)^l H_l x^l\right)\\&=\sum_{n=0}^\infty h_n x^n+\sum_{n=1}^\infty\sum_{k=0}^{n-1}h_k(-1)^{n-k}H_{n-k}x^n\,.\end{aligned}$$

Comparing the coefficients on both sides, we have $h_0 = 1$ and for $n \ge 1$

$$h_n + \sum_{k=0}^{n-1} (-1)^{n-k} H_{n-k} h_k = 0 \,.$$

□

The determinantal harmonic numbers are expressed in terms of harmonic numbers in the determinant.

Theorem 1 *For any integer* $n \ge 1$*,*

$$h_n = \begin{vmatrix} H_1 & 1 & 0 & & \\ H_2 & H_1 & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ H_{n-1} & H_{n-2} & \cdots & H_1 & 1 \\ H_n & H_{n-1} & \cdots & H_2 & H_1 \end{vmatrix} \,. \tag{4}$$

Proof [Proof of Theorem 1] For $n = 1$, $h_1 = 1 = H_1$. Assume that the result is valid up to $n-1$. By Lemma 1, expanding at the first row of the determinant, we have

$$\begin{aligned} & H_1 h_{n-1} - \begin{vmatrix} H_2 & 1 & 0 & & \\ H_3 & H_1 & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ H_{n-1} & H_{n-3} & \cdots & H_1 & 1 \\ H_n & H_{n-2} & \cdots & H_2 & H_1 \end{vmatrix} \\ & = H_1 h_{n-1} - H_2 h_{n-2} + \cdots + (-1)^{n-2} \begin{vmatrix} H_{n-1} & 1 \\ H_n & H_1 \end{vmatrix} \\ & = \sum_{k=0}^{n-1} (-1)^{n-k-1} H_{n-k} h_k \ = h_n \,. \end{aligned}$$

□

The determinantal harmonic numbers have an explicit expression.

Theorem 2 *For any integer* $n \ge 1$*,*

$$h_n = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \ge 1}} H_{i_1} \cdots H_{i_k} \,.$$

Proof [Proof of Theorem 2] When $n = 1$, it is easy to see that $h_1 = H_1$. Assume that the result is valid up to $n-1$. Then by Lemma 1, we have

$$
\begin{aligned}
h_n &= \sum_{l=0}^{n-1}(-1)^{n-l-1}H_{n-l}h_l\\
&= (-1)^{n-1}H_n\\
&\quad + \sum_{l=1}^{n-1}(-1)^{n-l-1}H_{n-l}\sum_{k=1}^{l}(-1)^{l-k}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge 1}}H_{i_1}\cdots H_{i_k}\\
&= (-1)^{n-1}H_n\\
&\quad + \sum_{k=1}^{n-1}\sum_{l=k}^{n-1}(-1)^{n-k-1}H_{n-l}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge 1}}H_{i_1}\cdots H_{i_k}\\
&= (-1)^{n-1}H_n\\
&\quad + \sum_{k=2}^{n}\sum_{l=k-1}^{n-1}(-1)^{n-k}H_{n-l}\sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_{k-1}\ge 1}}H_{i_1}\cdots H_{i_{k-1}}\\
&= (-1)^{n-1}H_n\\
&\quad + \sum_{k=2}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_k\ge 1}}H_{i_1}\cdots H_{i_k}\quad (n-l=i_k)\\
&= \sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}H_{i_1}\cdots H_{i_k}\,.
\end{aligned}
$$

□

3. Applications by Trudi's formula

Such forms of determinants are very useful, though there are many expressions for Bernoulli, Euler, and other numbers in determinants.

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers h_n.

Lemma 2 *For a positive integer n, we have*

$$
\begin{vmatrix}
a_1 & a_0 & 0 & \cdots & \\
a_2 & a_1 & \ddots & & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0\\
a_{n-1} & & \cdots & a_1 & a_0\\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix}
= \sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-a_0)^{n-t_1-\cdots-t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},
$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [17, Vol. 3, p. 214] [21] and the case $a_0=1$ of this formula is known as Brioschi's formula [2] [17, Vol. 3, pp. 208–209].

In addition, there exists the following inversion formula (see, e.g., [11]), which is based upon the following relation:

$$\sum_{k=0}^{n}(-1)^{n-k}\alpha_k D(n-k)=0\quad(n\ge 1)\,.$$

Lemma 3 *If* $\{\alpha_n\}_{n\ge 0}$ *is a sequence defined by* $\alpha_0=1$ *and*

$$\alpha_n=\begin{vmatrix}D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1\\ D(n) & \cdots & D(2) & D(1)\end{vmatrix},\ \textit{then}\ D(n)=\begin{vmatrix}\alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1\\ \alpha_n & \cdots & \alpha_2 & \alpha_1\end{vmatrix}.$$

By Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}D(1)^{t_1}D(2)^{t_2}\cdots D(n)^{t_n}\,.$$

By applying these lemmas to Theorem 1, we obtain an explicit expression for shifted harmonic numbers.

Theorem 3 *For* $n\ge 1$, *we have*

$$h_n=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}H_1^{t_1}\cdots H_n^{t_n}\,.$$

By applying the inversion relation in Lemma 3 to Theorem 1, we have the following.

Theorem 4 *For* $n\ge 1$, *we have*

$$H_n=\begin{vmatrix}h_1 & 1 & 0 & & \\ h_2 & h_1 & & & \\ \vdots & \vdots & \ddots & 1 & 0\\ h_{n-1} & h_{n-2} & \cdots & h_1 & 1\\ h_n & h_{n-1} & \cdots & h_2 & h_1\end{vmatrix}.$$

Therefore, we also have the inversion relations in Theorem 2 and Theorem 3.

Theorem 5 *For* $n\ge 1$, *we have*

$$\begin{aligned}H_n&=\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}h_{i_1}\cdots h_{i_k}\\ &=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}h_1^{t_1}\cdots h_n^{t_n}\,.\end{aligned}$$

4. Convolution identities

There are many identities involving harmonic numbers (see, e.g., [4, 20] and references therein). In particular, the sums of products of two harmonic numbers (cf. [20, p. 861]) are given as follows:

$$\sum_{k=0}^{n} H_k H_{n-k} = (n+1)\big((H_{n+1}-1)^2 - \mathcal{H}_{n+1}^{(2)} + 1\big)\,,$$

where

$$\mathcal{H}_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}$$

are the generalized harmonic numbers with $H_n = \mathcal{H}_n^{(1)}$. The sums of products have been extensively studied for many numbers, including Bernoulli, Euler, Stirling, and Cauchy and their generalized numbers, by many authors. The famous Euler's formula can be written as

$$\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} = -nB_{n-1} - (n-1)B_n \quad (n \ge 1)\,,$$

where B_n are the Bernoulli numbers, defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\,,$$

and this formula has been generalized in various ways (see, e.g., [1]).

The structure of the determinantal harmonic numbers is not as simple as that of harmonic numbers. Nevertheless, we can find the sums of products of two determinantal harmonic numbers.

Theorem 6 *For* $n \ge 0$*,*

$$\sum_{k=0}^{n} h_k h_{n-k} = -(n+2)h_{n+2} - (2n+1)h_{n+1} - (n-1)h_n\,.$$

Proof Put

$$h(x) := \sum_{n=0}^{\infty} h_n x^n = \left(1 - \frac{\ln(1+x)}{1+x}\right)^{-1}\,.$$

Then

$$\begin{aligned} h'(x) &= h(x)^2 \left(\frac{1}{(1+x)^2} - \frac{\ln(1+x)}{(1+x)^2}\right) \\ &= \frac{h(x)^2}{(1+x)^2} - \frac{h(x)^2}{1+x}\big(1 - h(x)^{-1}\big) \\ &= -\frac{x}{(1+x)^2} h(x)^2 + \frac{h(x)}{1+x}\,. \end{aligned}$$

Hence,

$$\begin{aligned}h(x)^2 &= -\frac{(1+x)^2}{x}h'(x) + \frac{1+x}{x}h(x)\\ &= -\left(\frac{1}{x}+2+x\right)\sum_{n=1}^{\infty} n h_n x^{n-1} + \left(\frac{1}{x}+1\right)\sum_{n=0}^{\infty} h_n x^n\\ &= -\sum_{n=-1}^{\infty}(n+2)h_{n+2}x^n - 2\sum_{n=0}^{\infty}(n+1)h_{n+1}x^n - \sum_{n=1}^{\infty} n h_n x^n\\ &\quad + \sum_{n=-1}^{\infty} h_{n+1}x^n + \sum_{n=0}^{\infty} h_n x^n\\ &= \sum_{n=0}^{\infty}\bigl(-(n+2)h_{n+2} - (2n+1)h_{n+1} - (n-1)h_n\bigr)x^n\,.\end{aligned}$$

On the other hand,

$$h(x)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} h_k h_{n-k} x^n\,.$$

Comparing the coefficients, we get the result. □

5. Hyperharmonic numbers

The nth hyperharmonic number of order r, denoted by $H_n^{(r)}$, is recursively defined by the following relations:

$$H_n^{(0)} = \frac{1}{n}$$

and

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \quad (r>0)\,. \tag{5}$$

The generating function of hyperharmonic numbers is given by

$$\sum_{n=1}^{\infty} H_n^{(r)} z^n = -\frac{\ln(1-z)}{(1-z)^r}\,.$$

In [15], the exponential generating function of hyperharmonic numbers is given. In [16], it is shown that the sum of the series formed by hyperharmonic numbers can be expressed in terms of the Riemann zeta function.

When $r=1$, $H_n = H_n^{(1)}$ are the original Harmonic numbers.

For nonnegative integers n, define determinantal hyperharmonic numbers $h_n^{(r)}$ by

$$\frac{(1+x)^r}{(1+x)^r - \ln(1+x)} = \sum_{n=0}^{\infty} h_n^{(r)} x^n \quad (|x|<1)\,. \tag{6}$$

We have the list of the numbers $h_n^{(r)}$.

n	0	1	2	3	4	5	6	7	8	9	10
$h_n^{(0)}$	1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{7}{60}$	$\frac{19}{360}$	$\frac{3}{70}$	$\frac{5}{336}$	$\frac{13}{756}$	$\frac{199}{75600}$
h_n	1	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{2}{15}$	$-\frac{23}{360}$	$\frac{241}{2520}$	$-\frac{47}{1680}$	$-\frac{403}{15120}$	$\frac{139}{4725}$
$h_n^{(2)}$	1	1	$-\frac{3}{2}$	$\frac{1}{3}$	2	$-\frac{61}{20}$	$\frac{41}{72}$	$\frac{2687}{630}$	$-\frac{4537}{720}$	$\frac{7531}{7560}$	$\frac{17127}{18900}$
$h_n^{(3)}$	1	1	$-\frac{5}{2}$	$\frac{11}{6}$	$\frac{25}{6}$	$-\frac{1971}{15}$	$\frac{4003}{360}$	$\frac{5591}{280}$	$-\frac{118169}{1680}$	$\frac{1010273}{15120}$	$\frac{7085539}{75600}$
$h_n^{(4)}$	1	1	$-\frac{7}{2}$	$\frac{13}{3}$	$\frac{35}{6}$	$-\frac{2033}{60}$	$\frac{18811}{360}$	$\frac{226511}{630}$	$-\frac{552871}{1680}$	$\frac{2284103}{3780}$	$\frac{1322737}{10800}$
$h_n^{(5)}$	1	1	$-\frac{9}{2}$	$\frac{47}{6}$	6	$-\frac{339}{5}$	$\frac{55849}{360}$	$\frac{10567}{2520}$	$-\frac{1001705}{1008}$	$\frac{8674609}{3024}$	$-\frac{33243599}{18900}$

When $r=0$, the sequence of coefficients of the exponential generating function is given by

$$\{n!h_n^{(0)}\}_{n=0}^{\infty}=1,1,1,2,4,14,38,216,600,6240,9552,319296,-519312,\dots$$

from [19, A006252] and also studied in [18, p. 9]. It can be expressed as

$$n!h_n^{(0)}=\sum_{k=0}^n(-1)^{n-k}k!\begin{bmatrix}n\\k\end{bmatrix},$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ denotes the (unsigned) Stirling numbers of the first kind. Notice that Fubini numbers (or ordered Bell numbers) F_n are given by

$$F_n=\sum_{k=0}^n k!\begin{Bmatrix}n\\k\end{Bmatrix},$$

where $\begin{Bmatrix}n\\k\end{Bmatrix}$ denotes the Stirling numbers of the second kind.

Hyperharmonic numbers of order r can be obtained from those of order $r-1$, as seen in (5). Though there does not exist a similar simpler relation between $h_n^{(r)}$ and $h_n^{(r-1)}$, we can see some relations for small n.

Since

$$\begin{aligned}
h_1^{(r)}&=1\,,\\
h_2^{(r)}&=-\frac{2r-1}{2}\,,\\
h_3^{(r)}&=\frac{3r^2-6r+2}{6}\,,\\
h_4^{(r)}&=-\frac{(2r-1)(r^2-7r+2)}{6}=-\frac{2r^3-15r^2+11r-2}{6}\,,
\end{aligned}$$

we have

$$\begin{aligned}
h_2^{(r)}&=-h_1^{(r-1)}+h_2^{(r-1)}\,,\\
h_3^{(r)}&=-h_2^{(r-1)}+h_3^{(r-1)}\,,\\
h_4^{(r)}&=4-2h_3^{(r-2)}+h_4^{(r-1)}\,.
\end{aligned}$$

We also have

$$\begin{aligned}
&h_n^{(0)}=h_n^{(1)}=1,\quad h_n^{(2)}=-\frac{2n-1}{1},\\
&h_n^{(3)}=\frac{3n^2-6n+2}{6},\quad h_n^{(5)}=-\frac{(2n-1)(n^2-7n+2}{2}\,.
\end{aligned}$$

Similarly to the results of harmonic numbers, we can obtain some results for hyperharmonic numbers.

Lemma 4 *For any integer $n \ge 1$,*

$$h_n^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k-1} H_{n-k}^{(r)} h_k^{(r)}$$

with $h_0^{(r)} = 1$.

Theorem 7 *For any integer $n \ge 1$,*

$$h_n^{(r)} = \begin{vmatrix} H_1^{(r)} & 1 & 0 & & \\ H_2^{(r)} & H_1^{(r)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ H_{n-1}^{(r)} & H_{n-2}^{(r)} & \cdots & H_1^{(r)} & 1 \\ H_n^{(r)} & H_{n-1}^{(r)} & \cdots & H_2^{(r)} & H_1^{(r)} \end{vmatrix} . \tag{7}$$

Theorem 8 *For any integer $n \ge 1$,*

$$\begin{aligned} h_n^{(r)} &= \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} H_{i_1}^{(r)} \cdots H_{i_k}^{(r)} \\ &= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} \left(H_1^{(r)}\right)^{t_1} \cdots \left(H_n^{(r)}\right)^{t_n} . \end{aligned}$$

Theorem 9 *For $n \ge 1$, we have*

$$H_n^{(r)} = \begin{vmatrix} h_1^{(r)} & 1 & 0 & & \\ h_2^{(r)} & h_1^{(r)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ h_{n-1}^{(r)} & h_{n-2}^{(r)} & \cdots & h_1^{(r)} & 1 \\ h_n^{(r)} & h_{n-1}^{(r)} & \cdots & h_2^{(r)} & h_1^{(r)} \end{vmatrix} .$$

Theorem 10 *For $n \ge 1$, we have*

$$\begin{aligned} H_n^{(r)} &= \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} h_{i_1}^{(r)} \cdots h_{i_k}^{(r)} \\ &= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} \left(h_1^{(r)}\right)^{t_1} \cdots \left(h_n^{(r)}\right)^{t_n} . \end{aligned}$$

6. Shifted determinantal hyperharmonic numbers

For $m, n, r \geq 0$, define the shifted determinantal hyperharmonic numbers $h_{n,m}^{(r)}$ by

$$\sum_{n=0}^{\infty} h_{n,m}^{(r)} x^n$$

$$= \left(1 + \frac{1}{(-x)^{m-1}(1+x)^r}\bigl(-\ln(1+x) + F_{m-1}(x)\bigr) - x\sum_{j=1}^{r} \frac{H_{m-1}^{(j)}}{(1+x)^{r-j+1}}\right)^{-1}, \quad (8)$$

where

$$F_m(z) = z - \frac{z^2}{2} + \cdots + \frac{(-1)^{m-1}z^m}{m}$$

is the partial summation of $\ln(1+z)$. $F_m(z)$ has an important role to introduce incomplete Cauchy numbers [9]. When $r = 1$, $h_{n,m} = h_{n,m}^{(1)}$ are the shifted determinantal harmonic numbers. When $m = 1$, $h_n^{(r)} = h_{n,1}^{(r)}$ are the determinantal harmonic numbers. When $m = r = 1$, $h_n = h_{n,1}^{(1)}$ are the original determinantal harmonic numbers.

Then the fundamental determinantal results are obtained by the recurrence relation.

Lemma 5 *For $m, r \geq 0$, we have*

$$h_{n,m}^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k} H_{m+n-k-1}^{(r)} h_{k,m}^{(r)} \quad (n \geq 1) \quad (9)$$

and $h_{0,m}^{(r)} = 1$.

Proof Since

$$\sum_{n=1}^{\infty} H_{m+n-1} z^n = \sum_{n=1}^{\infty} \sum_{k=1}^{m+n-1} \frac{z^n}{k}$$

$$= \frac{1}{(1-z)z^{m-1}}\left(-\ln(1-z) + F_{m-1}(-z)\right) + \frac{zH_{m-1}}{1-z},$$

we have

$$1 = \left(\sum_{n=0}^{\infty} h_{n,m} x^n\right)$$

$$\times \left(1 + \frac{1}{(-x)^{m-1}(1+x)}\bigl(-\ln(1+x) + F_{m-1}(x)\bigr) - \frac{xH_{m-1}}{1+x}\right)$$

$$= \left(\sum_{n=0}^{\infty} h_{n,m} x^n\right)\left(1 + \sum_{l=1}^{\infty} (-1)^l H_{m+l-1} x^l\right)$$

$$= \sum_{n=0}^{\infty} h_{n,m} x^n + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-k} H_{m+n-k-1} h_{k,m} x^n.$$

By comparing the coefficients on both sides, we have

$$h_{n,m}=\sum_{k=0}^{n-1}(-1)^{n-k}H_{m+n-k-1}h_{k,m}\quad(n\ge 1)\tag{10}$$

and $h_{0,m}=1$.

By induction on r, together with the definition of hyperharmonic numbers in (5), we can prove that

$$\sum_{n=1}^\infty H_{m+n-1}^{(r)}z^n=\frac{1}{(1-z)^r z^{m-1}}\left(-\ln(1-z)+F_{m-1}(-z)\right)+z\sum_{j=1}^r\frac{H_{m-1}^{(j)}}{(1-z)^{r-j+1}}\,.$$

This is also valid for $r=0$. Then, analogous to (10), we obtain the desired result. □

By Lemma 5, in view of Trudi's formula, we obtain the determinantal results with their inversion forms. The proof is similar to that of Theorem 1.

Theorem 11 *For $m,n\ge 1$ and $r\ge 0$, we have*

$$h_{n,m}^{(r)}=\begin{vmatrix}H_m^{(r)} & 1 & 0 & & \\ H_{m+1}^{(r)} & H_m^{(r)} & & & \\ \vdots & \vdots & \ddots & 1 & 0\\ H_{m+n-2}^{(r)} & H_{m+n-3}^{(r)} & \cdots & H_m^{(r)} & 1\\ H_{m+n-1}^{(r)} & H_{m+n-2}^{(r)} & \cdots & H_{m+1}^{(r)} & H_m^{(r)}\end{vmatrix}$$

and

$$H_{m+n-1}^{(r)}=\begin{vmatrix}h_{1,m}^{(r)} & 1 & 0 & & \\ h_{2,m}^{(r)} & h_{1,m}^{(r)} & & & \\ \vdots & \vdots & \ddots & 1 & 0\\ h_{n-1,m}^{(r)} & h_{n-2,m}^{(r)} & \cdots & h_{1,m}^{(r)} & 1\\ h_{n,m}^{(r)} & h_{n-1,m}^{(r)} & \cdots & h_{2,m}^{(r)} & h_{1,m}^{(r)}\end{vmatrix}\,.$$

We have two kinds of explicit expressions of shifted determinantal hyperharmonic numbers in terms of hyperharmonic numbers. The shifted determinantal hyperharmonic numbers can be expressed explicitly together with Trudi's formula. There are several ways to prove them, one of which is similar to the proof in 2. Another proof using the Hasse–Teichmüller derivative can be seen in [12]. Once shifted determinantal hyperharmonic numbers can be expressed in terms of hyperharmonic numbers, hyperharmonic numbers can be expressed in terms of shifted determinantal hyperharmonic numbers because they have the inversion relations with each other.

Theorem 12 *For $m,n\ge 1$ and $r\ge 0$,*

$$\begin{aligned}h_{n,m}^{(r)}&=\sum_{k=1}^n(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}H_{m+i_1-1}^{(r)}\cdots H_{m+i_k-1}^{(r)}\\ &=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}\left(H_m^{(r)}\right)^{t_1}\cdots\left(H_{m+n-1}^{(r)}\right)^{t_n}\end{aligned}$$

and

$$
\begin{aligned}
H_{m+n-1}^{(r)} &= \sum_{k=1}^{n}(-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} h_{i_1,m}^{(r)}\cdots h_{i_k,m}^{(r)}\\
&= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}\bigl(h_{1,m}^{(r)}\bigr)^{t_1}\cdots\bigl(h_{n,m}^{(r)}\bigr)^{t_n}\,.
\end{aligned}
$$

7. Examples

Set $m=5$ and $r=1$. Then

$$
\begin{aligned}
&\left(1-\frac{1}{(-x)^4(1+x)}\bigl(-\ln(1+x)+F_4(x)\bigr)-\frac{xH_4}{1+x}\right)^{-1}\\
&\qquad = 1+\frac{137}{60}x+\frac{9949}{3600}x^2+\frac{5003111}{1512000}x^3+\frac{361705747}{90720000}x^4+\frac{26049679919}{5443200000}x^5+\cdots\,.
\end{aligned}
$$

We can get

$$
\begin{vmatrix} H_5 & 1\\ H_6 & H_5\end{vmatrix} = \begin{vmatrix} \frac{137}{60} & 1\\ \frac{49}{20} & \frac{137}{60}\end{vmatrix} = \frac{9949}{3600} = h_{2,5}
$$

and

$$
\begin{vmatrix} h_{1,5} & 1\\ h_{1,6} & h_{1,5}\end{vmatrix} = \begin{vmatrix} \frac{137}{60} & 1\\ \frac{9949}{3600} & \frac{137}{60}\end{vmatrix} = \frac{49}{20} = H_6\,,
$$

$$
\begin{vmatrix} H_5 & 1 & 0\\ H_6 & H_5 & 1\\ H_7 & H_6 & H_5\end{vmatrix} = \begin{vmatrix} \frac{137}{60} & 1 & 0\\ \frac{49}{20} & \frac{137}{60} & 1\\ \frac{363}{140} & \frac{49}{20} & \frac{137}{60}\end{vmatrix} = \frac{5003111}{1512000} = h_{3,5}
$$

and

$$
\begin{vmatrix} h_{1,5} & 1 & 0\\ h_{1,6} & h_{1,5} & 1\\ h_{1,7} & h_{1,6} & h_{1,5}\end{vmatrix} = \begin{vmatrix} \frac{137}{60} & 1 & 0\\ \frac{9949}{3600} & \frac{137}{60} & 1\\ \frac{5003111}{1512000} & \frac{9949}{3600} & \frac{137}{60}\end{vmatrix} = \frac{363}{140} = H_7\,.
$$

Set $m=5$, $n=2$, and $r=1$. Since $(i_1)=(2)$ and $(i_1,i_2)=(1,1)$ satisfy the condition $i_1+\cdots+i_k=2$ with $i_1,\dots,i_k\ge 1$ for $k\ge 1$, we get

$$
-H_6+(H_5)^2=\frac{9949}{3600}=h_{2,5}\,.
$$

Since $\{(t_1,t_2)|t_1+2t_2=2,\ t_1,t_2\ge 0\}=(2,0),\ (0,1)$, we get

$$
\frac{2!}{2!}(-1)^{2-2}(H_5)^2+\frac{2!}{1!1!}(-1)^{2-1}H_6=\frac{9949}{3600}=h_{2,5}\,.
$$

On the other hand, we get

$$
-h_{1,6}+(h_{1,5})^2=\frac{49}{20}=H_6
$$

and

$$\frac{2!}{2!}(-1)^{2-2}(h_{1,5})^2 + \frac{2!}{1!1!}(-1)^{2-1}h_{1,6} = \frac{49}{20} = H_6\,.$$

Set $m = 5$, $n = 3$, and $r = 1$. Since $(i_1) = (3)$, $(i_1, i_2) = (1,2), (2,1)$, and $(i_1, i_2, i_3) = (1,1,1)$ satisfy the condition $i_1 + \cdots + i_k = 3$ with $i_1, \dots, i_k \geq 1$ for $k \geq 1$, we get

$$H_7 - 2H_5H_6 + (H_5)^3 = \frac{5003111}{1512000} = h_{3,5}\,.$$

Since $\{(t_1,t_2,t_3)|t_1 + 2t_2 + 3t_3 = 3,\ t_1,t_2,t_3 \geq 0\} = (3,0,0),\ (1,1,0),\ (0,0,1)$, we get

$$\frac{3!}{3!}(-1)^{3-3}(H_5)^3 + \frac{2!}{1!1!}(-1)^{3-1-1}H_5H_6 + \frac{1!}{1!}(-1)^{3-1}H_7 = \frac{5003111}{1512000} = h_{3,5}\,.$$

On the other hand, we get

$$h_{1,7} - 2h_{1,5}h_{1,6} + (h_{1,5})^3 = \frac{363}{140} = H_7$$

and

$$\frac{3!}{3!}(-1)^{3-3}(h_{1,5})^3 + \frac{2!}{1!1!}(-1)^{3-1-1}h_{1,5}h_{1,6} + \frac{1!}{1!}(-1)^{3-1}h_{1,7} = \frac{363}{140} = H_7\,.$$

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References

[1] Agoh T, Dilcher K. Convolution identities and lacunary recurrences for Bernoulli numbers. J Number Theory 2007; 124: 105-122.

[2] Brioschi F. Sulle funzioni Bernoulliane ed Euleriane. Annali de Mat 1858; 1: 260-263 (in Italian).

[3] Cameron PJ. Some sequences of integers. Discrete Math 1989; 75: 89-102.

[4] Chu W. Summation formulae involving harmonic numbers. Filomat 2012; 26: 143-152.

[5] Comtet L. Advanced Combinatorics. Dordrecht, the Netherlands: Reidel, 1974.

[6] Glaisher JWL. Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants. Messenger 1875; 6: 49-63.

[7] Gross OA. Preferential arrangements. Am Math Mon 1962; 69: 4-8.

[8] Komatsu T. Complementary Euler numbers. Period Math Hungar 2017; 75: 302-314.

[9] Komatsu T, Mező I, Szalay L. Incomplete Cauchy numbers. Acta Math Hungar 2016; 149: 306-323.

[10] Komatsu T, Ohno Y. Lehmer's generalized Euler numbers. Preprint.

[11] Komatsu T, Ramirez JL. Some determinants involving incomplete Fubini numbers. An Ştiinţ Univ "Ovidius" Constanţa Ser Mat 2018; 26: 143-170.

[12] Komatsu T, Yuan P. Hypergeometric Cauchy numbers and polynomials. Acta Math Hungar 2017; 153: 382-400.

[13] Kronenburg MJ. On two types of Harmonic number identities. arXiv:1202.3981, 2012.

[14] Lehmer DH. Lacunary recurrence formulas for the numbers of Bernoulli and Euler. Ann Math 1935; 36: 637-649.

[15] Mező I. Exponential generating function of hyperharmonic numbers indexed by arithmetic progressions. Cent Eur J Math 2013; 11: 931-939.

[16] Mező I, Dil A. Hyperharmonic series involving Hurwitz zeta function. J Number Theory 2010; 130: 360-369.

[17] Muir T. The Theory of Determinants in the Historical Order of Development. Four Volumes. New York, NY, USA: Dover Publications, 1960.

[18] Polya G. Induction and Analogy in Mathematics. Princeton, NJ, USA: Princeton University Press, 1954.

[19] Sloane NJA. The On-line Encyclopedia of Integer Sequences. Available at www.oeis.org, 2017.

[20] SpießJ. Some identities involving harmonic numbers. Math Comp 1990; 55: 839-863.

[21] Trudi N. Intorno ad alcune formole di sviluppo. Rendic dell' Accad Napoli 1862: 135-143 (in Italian).