

A broken cycle theorem for the restrained chromatic function

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Abstract: A restraint r on a graph G is a function that assigns each vertex of the graph a finite subset of \mathbb{N} . For each vertex v of the graph, $r(v)$ is called the set of colors forbidden at v . A proper coloring of G is said to be permitted by a given restraint r if each vertex v of the graph receives a color that is not from its set of forbidden colors $r(v)$. The restrained chromatic function, denoted by $\pi_r(G, x)$, is a function whose evaluations at integer x values count the number of proper x -colorings of the graph G permitted by the restraint r and this function is known to be a polynomial function of x for large enough x . The restrained chromatic function $\pi_r(G, x)$ is a generalization of the well-known chromatic polynomial $\pi(G, x)$, as $\pi_r(G, x) = \pi(G, x)$ if $r(v) = \emptyset$ for each vertex v of the graph. Whitney's celebrated broken cycle theorem gives a combinatorial interpretation of the coefficients of the chromatic polynomial via certain subgraphs (the so-called broken cycles). We provide an extension of this result by finding combinatorial interpretations of the coefficients of the restrained chromatic function.

Key words: x -Coloring, restraint, chromatic polynomial, restrained chromatic function

1. Introduction

In this article all graphs are finite, simple, and undirected. Given a graph G , let $V(G)$ be the vertex set of G and $E(G)$ be the edge set of G . The order and size of G are $|V(G)|$ and $|E(G)|$, respectively. For a positive integer x , an x -coloring of G is a function $c : V(G) \rightarrow \{1, \dots, x\}$ and it is called a *proper x -coloring of G* if $c(u) \neq c(v)$ for every edge uv in $E(G)$. The well-known *chromatic polynomial of G* , denoted by $\pi(G, x)$, is the polynomial whose evaluations at positive integer x -values count the number of proper x -colorings of G .

In generalized graph coloring problems, each vertex has a list of available or forbidden colors and the goal is to properly color the graph obeying the restrictions. They arise naturally in the regular graph coloring problem because the assignment of some color to each vertex imposes restrictions on the neighbor vertices. Such problems are also useful tools in modeling real-world problems such as scheduling and timetabling [13] and they have been extensively studied in the literature; see, for example, [1, 19]. Here we shall focus on *restrained colorings*. In a restrained coloring, there is a *restraint function r* that assigns each vertex of the graph a finite subset of \mathbb{N} . For each vertex v of the graph, $r(v)$ is called *the set of colors forbidden at v* . An x -coloring c of a graph G is said to be permitted by the restraint r if $c(v) \notin r(v)$ for every vertex v in $V(G)$. The study of restrained colorings proved useful in the construction of critical graphs [17] and recently several extremal problems on this topic were studied in [4, 5, 11].

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Counting the number of generalized colorings has also been studied in a number of settings. Enumeration of list colorings was studied and relations to the chromatic polynomial were investigated in [10, 16, 20]. Recently, Knox and Mohar (arXiv:1708.01781) proved a long-standing conjecture on the chromatic polynomial by studying a list coloring function. Given a positive integer x , let $\pi_r(G, x)$ be the number of proper x -colorings of the graph G permitted by restraint r . It is known that $\pi_r(G, x)$ is a polynomial function of x when x is large enough, or more precisely, when $x > \max_{v \in V(G)} r(v)$ (see Theorem 4.1.2, p. 102 in [11]). Furthermore, $\pi_r(G, x)$ is a monic polynomial of degree $|V(G)|$ with integer coefficients that alternate in sign [11]. We call $\pi_r(G, x)$ the restraint chromatic function of G with respect to the restraint r . It is clear that the restraint chromatic function generalizes the chromatic polynomial because $\pi_r(G, x) = \pi(G, x)$ if $r(v) = \emptyset$ for every vertex v of G .

Since the restrained chromatic function $\pi_r(G, x)$ turns out to be a polynomial function of x for large enough x , a natural problem is to find a combinatorial interpretation of the coefficients and such interpretation is of crucial importance to study this function. A celebrated result due to Whitney gives a combinatorial interpretation of the coefficients of the chromatic polynomial via the so-called broken cycles (Theorem 2.1). Our main result (Theorem 2.2) generalizes this result to a restrained chromatic function by providing a combinatorial interpretation for its coefficients.

2. The broken cycle theorem

A subgraph H of a graph G is called a *spanning subgraph of G* if $V(H) = V(G)$. A *spanning forest* of a graph G is a spanning subgraph that is a forest. Given a graph G , let $\mathcal{C}(G)$ denote the family of all connected components of G . Given a set S and an integer k , let $\binom{A}{k}$ denote the family of all subsets of A with size k . Also, let $M_r(G)$ be equal to $\max_{v \in V(G)} r(v)$ if it is nonempty and 0 otherwise.

Let f be a labeling of the edges of the graph G such that $f : E(G) \rightarrow \{1, 2, \dots, |E(G)|\}$ is a bijection. A path P in the graph G is called a *broken cycle with respect to f* if P is obtained from a cycle by removing the edge of the maximum label in the cycle. In other words, there is a cycle C and an edge $e \in E(C)$ such that $P = C - e$ and $f(e) > f(e')$ for every edge $e' \in E(C) \setminus \{e\}$; see the Figure. The family of subgraphs of a graph that do not contain any broken cycle forms an abstract simplicial complex, known as the *broken circuit complex*. Such a complex has received considerable attention and has been investigated from combinatorial algebraic, topological, and geometric points of view [2, 6, 7, 14, 15].

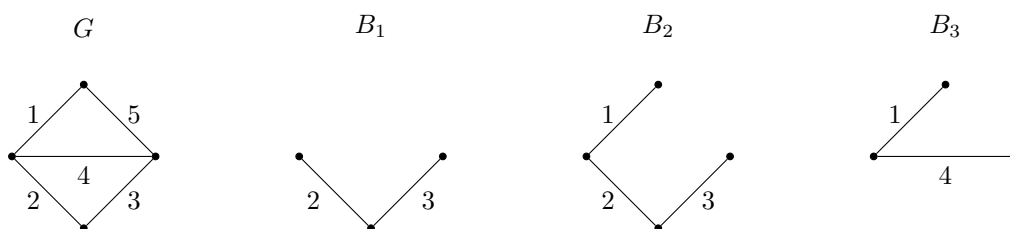


Figure. A graph G with an edge labeling f and its broken cycles B_1 , B_2 , and B_3 with respect to f .

Theorem 2.1 [21] [Whitney’s broken cycle theorem] *Let G be a graph on n vertices and m edges, and $f : E(G) \rightarrow \{1, 2, \dots, m\}$ be a bijection. Then*

$$\pi(G, x) = \sum_{k=1}^n (-1)^k a_k(G) x^{n-k},$$

where $a_k(G)$ is the number of spanning forests of G with size k that do not contain any broken cycle with respect to f .

Whitney’s broken cycle theorem is a classical result in the theory of chromatic polynomials and several generalizations of this theorem were obtained via various other structures, such as lattices [3], two-variable generalization of the chromatic polynomial [8], matroids (Grinberg, arXiv:1604.03063), and hypergraphs [18]; see [9] for more related work. Our main result is the following, which generalizes Theorem 2.1 via restrained colorings.

Theorem 2.2 *Let G be a graph on n vertices, r be a restraint on G , and $x > M_r(G)$. Then,*

$$\pi_r(G, x) = \sum_{k=0}^n (-1)^k a_k(G, r) x^{n-k}$$

and

$$a_k(G, r) = \sum_{t=0}^k \sum_{H \in \mathcal{BC}^t} \sum_{\mathcal{A} \in \binom{\mathcal{C}(H)}{k-t}} \prod_{C \in \mathcal{A}} \left| \bigcup_{v \in V(C)} r(v) \right|,$$

where \mathcal{BC}^t is the family of spanning forests of G with size t that do not contain any broken cycle with respect to f .

Observe that Theorem 2.2 reduces to Theorem 2.1 when $r(v) = \emptyset$ for every vertex v of G . In that case, $|\bigcup_{v \in V(C)} r(v)| = 0$ in the formula for $a_k(G, r)$. Note that $\prod_{C \in \mathcal{A}} 0$ is 1 if $\mathcal{A} = \emptyset$ and it is 0 otherwise. Also, $\mathcal{A} = \emptyset$ if and only if $t = k$. Thus, when $r(v) = \emptyset$ for each vertex v , $a_k(G, r) = \sum_{H \in \mathcal{BC}^k} 1$, which is precisely the number of spanning forests of size k that do not contain any broken cycle.

The broken cycle interpretation of the coefficients has been invaluable in the study of chromatic polynomials. It proved to be a useful tool to understand significant properties of chromatic polynomials. For example, a long-standing log-concavity conjecture was settled by applying algebro-geometric tools to the broken circuit complex [12]. The restrained chromatic function has not been studied as much as the chromatic polynomial and it has not been well understood yet. There are various aspects of this function to be explored and we believe that Theorem 2.2 may provide insights into a better understanding of the properties of the restrained chromatic function.

3. Proof of the main result

Lemma 3.1 *Let G be a graph, r be a restraint on G , and $x > M_r(G)$. Then,*

$$\pi_r(G, x) = \sum_{t=0}^m (-1)^t \sum_{H \in \mathcal{S}(G)^t} \prod_{C \in \mathcal{C}(H)} \left(x - \left| \bigcup_{v \in V(C)} r(v) \right| \right),$$

where $\mathcal{S}(G)^t$ is the family of spanning subgraphs of G that have size t .

Proof Let e_1, \dots, e_m be the edges of G . Also, let A_i denote the set of all x -colorings of G permitted by r such that the vertices of the edge e_i receive the same color. The total number of x -colorings of G permitted by r is $\prod_{v \in V(G)} (x - |r(v)|)$. The number of x -colorings of G permitted by r that are not proper is equal to $|A_1 \cup \dots \cup A_m|$. Therefore, the number of proper x -colorings of G permitted by r is

$$\pi_r(G, x) = \prod_{v \in V(G)} (x - |r(v)|) - |A_1 \cup \dots \cup A_m|.$$

By the principle of inclusion-exclusion, we get

$$\pi_r(G, x) = \prod_{v \in V(G)} (x - |r(v)|) + \sum_{t=1}^m (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq m} |A_{i_1} \cap \dots \cap A_{i_t}|.$$

Consider t edges of G , without loss, say e_1, \dots, e_t . Let H be the spanning subgraph of G whose edge set is $\{e_1, \dots, e_t\}$. Now $|A_1 \cap \dots \cap A_t|$ is precisely the number of x -colorings of G permitted by r such that each connected component of H is monochromatic, i.e. the vertices in the same connected component of H receive the same color. Thus,

$$|A_1 \cap \dots \cap A_t| = \prod_{C \in \mathcal{C}(H)} \left(x - \left| \bigcup_{v \in V(C)} r(v) \right| \right)$$

and

$$\sum_{1 \leq i_1 < \dots < i_t \leq m} |A_{i_1} \cap \dots \cap A_{i_t}| = \sum_{H \in \mathcal{S}(G)^t} \prod_{C \in \mathcal{C}(H)} \left(x - \left| \bigcup_{v \in V(C)} r(v) \right| \right).$$

Hence, the result follows. □

Now using Lemma 3.1 we shall derive the next result.

Lemma 3.2 *Let G be a graph of order n and size m , r be a restraint on G , and $x > M_r(G)$. Also, let $f : E(G) \rightarrow \{1, \dots, m\}$ be a bijection. Then,*

$$\pi_r(G, x) = \sum_{t=0}^{n-1} (-1)^t \sum_{H \in \mathcal{BC}^t} \prod_{C \in \mathcal{C}(H)} \left(x - \left| \bigcup_{v \in V(C)} r(v) \right| \right),$$

where \mathcal{BC}^t is the family of spanning forests of G with size t that do not contain any broken cycle with respect to f .

Proof Let B_1, \dots, B_k be broken cycles in G for some $k \geq 0$. Also let e_1, \dots, e_k be the edges of G such that $e_i \notin E(B_i)$ and $B_i + e_i$ is a cycle in G for each $i = 1, \dots, k$. Given a broken cycle B , we define $\mu(B) = \max\{f(e) : e \in E(B)\}$. We may assume that $\mu(B_1) \leq \dots \leq \mu(B_k)$. Now we partition the family of spanning subgraphs of G into $k + 1$ subfamilies as follows. For $1 \leq i \leq k$, let \mathcal{F}_i be the family of spanning subgraphs of G that contain B_i but do not contain any of B_1, \dots, B_{i-1} (if $i \geq 2$). Also, let \mathcal{F}_{k+1} be the

family of spanning subgraphs of G that contains none of B_1, \dots, B_k as a subgraph. It is clear that $\mathcal{F}_1, \dots, \mathcal{F}_{k+1}$ are mutually disjoint and every spanning subgraph of G belongs to some subfamily \mathcal{F}_i . Let $1 \leq i \leq k$ be fixed. For every subgraph H in \mathcal{F}_i that does not contain the edge e_i , there exists a unique subgraph $H + e_i$. It is clear that $H + e_i$ also contains B_i . Moreover, for $2 \leq i \leq k$, the subgraph $H + e_i$ does not contain any of B_1, \dots, B_{i-1} because $f(e_i) > \mu(B_i)$ by the definition of broken cycles, and $\mu(B_i) \geq \mu(B_j)$ for each $j = 1, \dots, i - 1$ by the assumption. Note that H and $H + e_i$ have exactly the same connected components and their sizes differ by one. Hence, in the formula for $\pi_r(G, x)$ given in Lemma 3.1, the contribution of H and $H + e_i$ cancel each other. Thus, the total contribution of each subfamily \mathcal{F}_i is zero for $1 \leq i \leq k$. Thus, it suffices to consider the contributions of the subgraphs in \mathcal{F}_{k+1} . Since the subgraphs in \mathcal{F}_{k+1} do not contain any broken cycle, they cannot contain any cycle at all. Therefore, \mathcal{F}_{k+1} consists of spanning forests of G that do not contain any broken cycle with respect to f and we obtain the result by Lemma 3.1. \square

We are now ready to prove our main result.

Proof [Proof of Theorem 2.2] Let $H \in \mathcal{BC}^t$ for some t . Then $|\mathcal{C}(H)| = n - t$ as H is a spanning forest. Observe that

$$\prod_{C \in \mathcal{C}(H)} \left(x - \left| \bigcup_{v \in V(C)} r(v) \right| \right) = \sum_{j=0}^{n-t} (-1)^j \sum_{\mathcal{A} \in \binom{\mathcal{C}(H)}{j}} \prod_{C \in \mathcal{A}} \left| \bigcup_{v \in V(C)} r(v) \right| x^{n-t-j}.$$

By Lemma 3.2, $\pi_r(G, x)$ is equal to

$$\sum_{t=0}^{n-1} (-1)^t \sum_{H \in \mathcal{BC}^t} \sum_{j=0}^{n-t} (-1)^j \sum_{\mathcal{A} \in \binom{\mathcal{C}(H)}{j}} \prod_{C \in \mathcal{A}} \left| \bigcup_{v \in V(C)} r(v) \right| x^{n-t-j},$$

which equals

$$\sum_{t=0}^{n-1} \sum_{j=0}^{n-t} (-1)^{t+j} \sum_{H \in \mathcal{BC}^t} \sum_{\mathcal{A} \in \binom{\mathcal{C}(H)}{j}} \prod_{C \in \mathcal{A}} \left| \bigcup_{v \in V(C)} r(v) \right| x^{n-t-j}.$$

Now, letting $k = t + j$, we see that the latter is equal to

$$\sum_{k=0}^n (-1)^k \sum_{t=0}^k \sum_{H \in \mathcal{BC}^t} \sum_{\mathcal{A} \in \binom{\mathcal{C}(H)}{k-t}} \prod_{C \in \mathcal{A}} \left| \bigcup_{v \in V(C)} r(v) \right| x^{n-k}$$

and the result follows. \square

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