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A discrete chaotic dynamical system on the Sierpinski gasket

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Abstract: The Sierpinski gasket (also known as the Sierpinski triangle) is one of the fundamental models of self-similar sets. There have been many studies on different features of this set in the last decades. In this paper, initially we construct a dynamical system on the Sierpinski gasket by using expanding and folding maps. We then obtain a surprising shift map on the code set of the Sierpinski gasket, which represents the dynamical system, and we show that this dynamical system is chaotic on the code set of the Sierpinski gasket with respect to the intrinsic metric. Finally, we provide an algorithm to compute periodic points for this dynamical system.

Key words: Sierpinski gasket, intrinsic metric, code set, dynamical system, chaos

1. Introduction

Fractals were introduced as the geometry of nature by Mandelbrot [10] and studied in various fields including mathematics, social science, computer science, engineering, economics, physics, chemistry, and biology (for details see [1–3, 5, 7, 8, 12]). As typical examples for fractals can be given the Julia sets of the functions (or discrete dynamical systems) $f : \mathbb{C} \to \mathbb{C}$, $f_c(z) = z^2 + c$ $(c \in \mathbb{C})$ since they generally have the properties of self-similarity and noninteger fractal dimension (as an example, see Figure 1). Discrete dynamical system f_c on Julia set J_c is chaotic in the sense of Devaney and thus chaotic dynamical systems on J_c emerge naturally (for details see [5]).

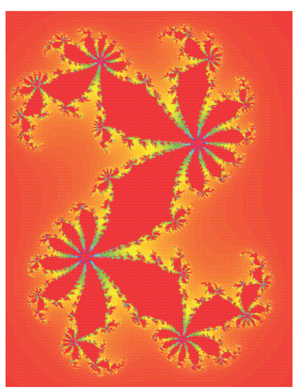

Figure 1. The filled Julia set for $c = 0.360284 + 0.100376i$.

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Barnsley considered the right Sierpinski gasket (or briefly S) as the Julia set of the following dynamical system:

$$f(x,y)=\begin{cases}(2x,2y-1), & y\geq 0.5\\ (2x-1,2y), & x\geq 0.5,\ y<0.5\\ (2x,2y), & \text{otherwise.}\end{cases} \tag{1.1}$$

The relationship between the dynamical system $\{\mathbb{R}^2;f\}$ and the iterated function system (IFS), whose attractor is S with vertices $(0,0),(0,1),(1,0)$, is that $\{S;f\}$ is a shift dynamical system associated with the IFS. Note that f is not continuous. Moreover, Barnsley did not examine in detail whether this transformation is chaotic (for details see [2]).

An interesting question is how to construct chaotic dynamical systems on the different fractals. It is known that there are several definitions of chaos in the sense of Devaney, Li and Yorke, Block and Coppel, etc. In [6], Ercai investigated the chaotic behavior of the Sierpinski carpet in the sense of Li and Yorke.

In this paper, we construct a discrete dynamical system $\{S;F\}$ by using the code representations of the points on the Sierpinski gasket and then we show that this dynamical system is chaotic in the sense of Devaney with respect to the intrinsic metric d in Proposition 2.2. To this end, we first obtain a dynamical system on S by using expanding and folding maps. As seen in (2.1), since F is quite complicated for some computations, the code representation of this dynamical system on S is determined in Proposition 2.1. In order to show that this dynamical system is chaotic with respect to the intrinsic metric of S, the formula that is expressed by the codes of the points on S is used ([11]). Note that this dynamical system is also chaotic in the sense of Li and Yorke since (S,d) is a compact metric space and F is a continuous map (for details, see [9]).

Using a similar method to our approach, chaotic dynamical systems can be defined on different fractals such as the Sierpinski carpet, Sierpinski tetrahedron, box fractal, etc.

First of all, let us recall basic concepts such as the code set, the code representation of the points on S, and the chaotic dynamical system in the sense of Devaney.

Construction of the Sierpinski gasket: The Sierpinski gasket can be defined in multiple ways. One of them is the IFS. The attractor of the iterated function system $\{\mathbb{R}^2;f_0,f_1,f_2\}$ such that $f_i:\mathbb{R}^2\to\mathbb{R}^2$ $(i=0,1,2)$,

$$\begin{aligned} f_0(x,y) &= \left(\frac{x}{2},\frac{y}{2}\right),\\ f_1(x,y) &= \left(\frac{x}{2}+\frac{1}{2},\frac{y}{2}\right),\\ f_2(x,y) &= \left(\frac{x}{2}+\frac{1}{4},\frac{y}{2}+\frac{\sqrt{3}}{4}\right),\end{aligned}$$

is the (classical) Sierpinski gasket whose vertices are $P_0=(0,0)$, $P_1=(1,0)$, and $P_2=(\frac{1}{2},\frac{\sqrt{3}}{2})$ (throughout this paper, we use these vertices).

Another definition of S can be given as follows: first consider the equilateral triangle denoted by T_0. Subdivide T_0 into four smaller congruent equilateral triangles and remove the central one, except for external points. We thus obtain three smaller equilateral subtriangles and denote this by T_1. After one more step, we get nine smaller equilateral subtriangles and denote this by T_2. Continuing the process gives us the Sierpinski gasket in Figure 2; in other words,

$$S = \bigcap_{k \geq 0} T_k.$$

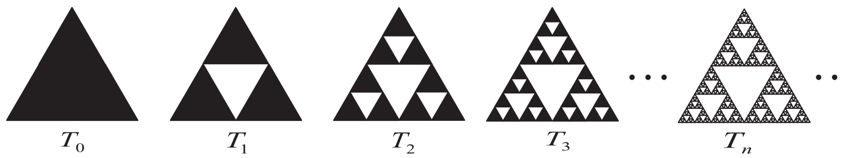

Figure 2. The construction of the Sierpinski gasket.

Coding process of S: Throughout this paper, we will be more concerned with the definition of the code set of S. The bottom-left part of S is denoted by

$$S_0 = \{0a_2a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 2,3,4,\ldots\},$$

the bottom-right part of S is denoted by

$$S_1 = \{1a_2a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 2,3,4,\ldots\},$$

and the upper part of S is denoted by

$$S_2 = \{2a_2a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 2,3,4,\ldots\}.$$

Obviously, the code set of S is the union of the code sets S_0, S_1, and S_2 (Figure 3).

Similarly, denote the bottom-left part, the bottom-right part, and the upper part of S_0 as

$$S_{00} = \{00a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 3,4,5,\ldots\},$$

$$S_{01} = \{01a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 3,4,5,\ldots\},$$

$$S_{02} = \{02a_3a_4\ldots \mid a_i \in \{0,1,2\} \text{ and } i = 3,4,5,\ldots\},$$

respectively. It is clear that the code set of S_0 is the union of the code sets S_{00}, S_{01}, and S_{02}. The subtriangles of S_1 and S_2 can be defined in a similar fashion. For the general case, fix $w_1, w_2, \ldots, w_n$ where $w_i \in \{0,1,2\}$ for $i = 1,2,\ldots,n$. $S_{w_1w_2\ldots w_n0}$, $S_{w_1w_2\ldots w_n1}$, and $S_{w_1w_2\ldots w_n2}$ are the bottom-left part, the bottom-right part, and the upper part of $S_{w_1w_2\ldots w_n}$ respectively. Thus, we have the subtriangle

$$S_{w_1\ldots w_nw} = \{w_1\ldots w_nwa_{n+2}a_{n+3}\ldots \mid w, a_i \in \{0,1,2\} \text{ and } i = n+2, n+3, \ldots\}$$

(see Figure 4).

Let us now define the code representation of a point on S by using these code sets. Consider the code sets

$$S_{a_1}, S_{a_1a_2}, S_{a_1a_2a_3}, \ldots, S_{a_1a_2\ldots a_n}, \ldots.$$

Due to the construction above, it is obvious that $S_{a_1} \supset S_{a_1a_2} \supset S_{a_1a_2a_3} \supset \ldots \supset S_{a_1a_2\ldots a_n} \supset \ldots$. By the Cantor intersection theorem, the infinite intersection

$$\bigcap_{n=1}^{\infty} S_{a_1a_2\ldots a_n}$$

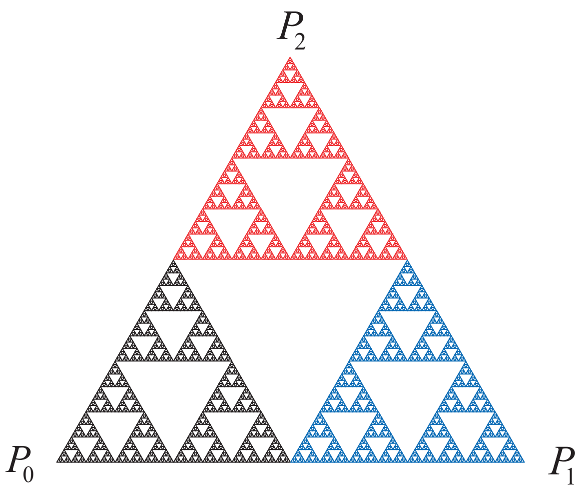

Figure 3. The subtriangles S_0 (with black color), S_1 (with blue color), and S_2 (with red color) of S.

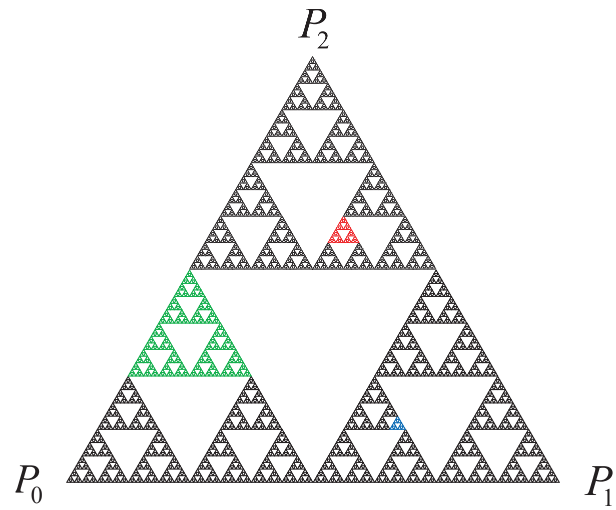

Figure 4. The subtriangles S_{02} (with green color) of S_0, S_{2102} (with red color) of S_{210}, and S_{10211} (with blue color) of S_{1021}.

is a point on S, say $\{a\}$. Let us denote the point $a \in S$ by $a_1a_2\ldots a_n\ldots$ where $a_n \in \{0,1,2\}, n \in \mathbb{N}$. The infinite word $a_1a_2\ldots a_n\ldots$ is called a code representation of a. Note that if $a \in S$ is the intersection point of any two subtriangles of $S_{a_1a_2\ldots a_k}$, then a is called a junction point of S. In this case, a has two different code representations such that $a_1a_2\ldots a_k\beta\alpha\alpha\alpha\alpha\ldots$ and $a_1a_2\ldots a_k\alpha\beta\beta\beta\beta\ldots$ where $\alpha, \beta \in \{0,1,2\}$ and $\alpha \neq \beta$. If any point is not in this form, then it has a unique code representation. For example, the points $\overline{01} = 010101\ldots$, $\overline{0} = 000\ldots$, $\overline{012} = 012012012\ldots$ and the sequences whose elements do not repeat regularly such as $011010210212010112\ldots$ have a unique code representation.

Intrinsic metric on the code set of S: The intrinsic metric (or geodesic metric) on the Sierpinski gasket is defined as follows:

$$d(x,y) = \inf\{\delta \mid \delta \text{ is the length of a rectifiable curve in } S \text{ joining } x \text{ and } y\}$$

(see [4]).

In [11], an explicit formula that gives the intrinsic distance between any two points of the code set of S is defined as follows:

Definition 1.1 *Let $a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots$ and $b_1b_2\ldots b_{k-1}b_kb_{k+1}\ldots$ be two representations respectively of the points $a \in S$ and $b \in S$ such that $a_i = b_i$ for $i = 1,2,\ldots,k-1$ and $a_k \neq b_k$. The distance $d(a,b)$ between a and b is determined by the following formula:*

$$d(a,b) = \min\left\{\sum_{i=k+1}^{\infty}\frac{\alpha_i+\beta_i}{2^i}, \frac{1}{2^k}+\sum_{i=k+1}^{\infty}\frac{\gamma_i+\delta_i}{2^i}\right\} \tag{1.2}$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \quad and\ a_i \neq b_k \\ 1, & otherwise \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i \neq b_k \quad and\ b_i \neq a_k \\ 1, & otherwise \end{cases}.$$

Note that the metric d defined in (1.2) is a strictly intrinsic metric on the code set of S [11]. That is, there exists a shortest path between any two points.

Chaotic dynamical system in the sense of Devaney: Chaos theory constitutively studies the behavior of dynamical systems being sensitive to initial conditions. That is, small differences in initial conditions can change the long-term behavior of a system. The tent map and the doubling map are well-known examples of chaotic dynamical systems. Devaney gives the definition of chaos as follows:

A dynamical system is a transformation $f : X \longrightarrow X$ on a metric space (X, d) and is denoted by $\{X, f\}$. If $\{X, f\}$ satisfies the following three conditions, then it is chaotic:

- f depends sensitively on the initial condition: there exists $\epsilon > 0$ such that, for any $x \in X$ and any ball $B(x, \delta)$ with radius $\delta > 0$, there is $y \in B(x, \delta)$ and an integer $n \geq 0$ satisfying $d(f^n(x), f^n(y)) > \epsilon$.
- f is topologically transitive if, whenever U and V are open subsets of the metric space (X, d), there exists a finite integer n such that $U \cap f^n(V) \neq \emptyset$.
- Periodic points of f are dense in X: there exist periodic points of f that are sufficiently close to any point of X

  (see [2, 5]).

We can now construct a chaotic dynamical system on the Sierpinski gasket that is equipped with metric d defined in (1.2).

2. The construction of a chaotic dynamical system on the Sierpinski gasket

In this section, we first define the function F on S as the composition of the expanding and folding maps. In Proposition 2.1, F is formulated on the code set of S both to show that F is chaotic and to compute the periodic points of F more easily. To avoid unnecessary repetitions, first let us denote the filled equilateral triangle whose vertices are $(0,0)$, $(1,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ by $\blacktriangle$ and the filled equilateral triangle with vertices $(0,0)$, $(2,0)$, $(1, \sqrt{3})$ by $\blacktriangle'$. We also denote the isosceles trapezoid whose vertices are $(0,0)$, $(2,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ by IT and the parallelogram with vertices $(0,0)$, $(1,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ by $\blacklozenge$. We now define the composition function

$$F = f_4 \circ f_3 \circ f_2 \circ f_1 \tag{2.1}$$

where $f_1, f_2, f_3, f_4 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\begin{aligned}
f_1(x,y) &= (2x, 2y), \\
f_2(x,y) &= \left(x, \tfrac{\sqrt{3}}{2} - \left|y - \tfrac{\sqrt{3}}{2}\right|\right), \\
f_3(x,y) &= \left(-\tfrac{\sqrt{3}}{2}\left|\tfrac{\sqrt{3}}{2}(x-1) - \tfrac{y}{2}\right| + \tfrac{(x-1)+y\sqrt{3}}{4} + 1, \tfrac{\sqrt{3}}{2}\left(\tfrac{x-1}{2} + \tfrac{y\sqrt{3}}{2}\right) + \tfrac{1}{2}\left|\tfrac{\sqrt{3}(x-1)-y}{2}\right|\right), \\
f_4(x,y) &= \left(-\tfrac{\sqrt{3}}{2}\left|\tfrac{\sqrt{3}}{2}(x-1) + \tfrac{y}{2}\right| + \tfrac{(x-1)-y\sqrt{3}}{4} + 1, \tfrac{\sqrt{3}}{2}\left(\tfrac{y\sqrt{3}}{2} - \tfrac{(x-1)}{2}\right) - \tfrac{1}{2}\left|\tfrac{\sqrt{3}(x-1)+y}{2}\right|\right).
\end{aligned}$$

These functions might seem a bit complicated and strange at first, but the geometry of the action of these functions is intelligible to construct a dynamical system on $\blacktriangle$. Namely, the function f_1 is an expanding map such that $f_1(\blacktriangle) = \blacktriangle'$. The function f_2 is a folding map with respect to the line $y = \frac{\sqrt{3}}{2}$ such that $f_2(\blacktriangle') = IT$. The function f_3 is a folding map with respect to the line $y = \sqrt{3}(x-1)$ such that $f_3(IT) = \blacklozenge$. The function f_4 is a folding map with respect to the line $y = -\sqrt{3}(x-1)$ such that $f_4(\blacklozenge) = \blacktriangle$. As seen in Figure 5, we obtain the function

$$F = f_4 \circ f_3 \circ f_2 \circ f_1$$

such that $F(\blacktriangle) = \blacktriangle$.

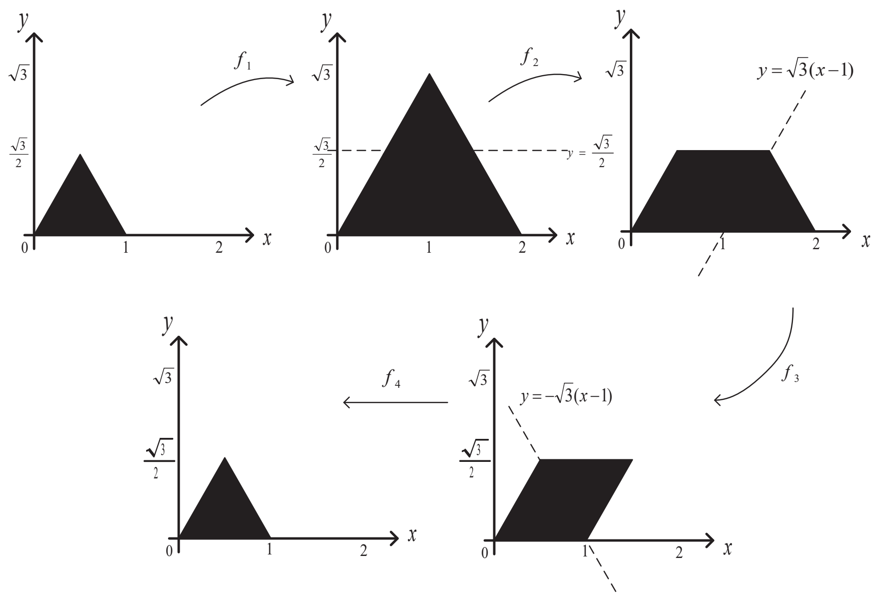

Figure 5. The action of the functions f_1, f_2, f_3, f_4 on $\blacktriangle$.

Let us now consider the restriction of F to S. Obviously, $\{S, F\}$ is a dynamical system. It is difficult to show whether F is chaotic or not since finding periodic points of F requires tedious processing. To eliminate this problem, we express this function on the code set of the Sierpinski gasket in the following proposition:

Proposition 2.1 *If $x_1x_2x_3\ldots$ is the code representation of an arbitrary point x of S, then the function $F : S \to S$ defined in (2.1) is expressed by $F(x) = y$ such that the code representation of y is $y_1y_2y_3\ldots$ where $y_i \equiv x_{i+1} + x_1 \pmod{3}$ for $x_i, y_i \in \{0,1,2\}$ and $i = 1,2,3,\ldots$.*

Proof Initially, we show that F is well defined on the code set of S. If the point x has a unique code representation, the desired result is obtained. Suppose that x has two different code representations such that $x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots$ and $x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots$ where $\alpha, \beta \in \{0,1,2\}$ and $\alpha \neq \beta$,

$$F(x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots) = y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots$$

$$F(x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots) = z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots.$$

Let us now show that $y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots$ and $z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots$ are the code representations of the same point. Due to the definition of F, we obtain

$$y_i \equiv z_i \equiv x_1 + x_{i+1} \ (\text{mod}3)$$

for $i = 1,2,3,\dots,n-1$. Moreover, we have

$$y_n \equiv x_1 + \alpha \ (\text{mod}3),$$

$$y_{n+i} \equiv x_1 + \beta \ (\text{mod}3),$$

$$z_n \equiv x_1 + \beta \ (\text{mod}3),$$

$$z_{n+i} \equiv x_1 + \alpha \ (\text{mod}3),$$

for $i = 1,2,3,\dots$ and $\gamma \neq \delta$. Let $x_1 + x_{i+1} \equiv s_i \ (\text{mod}3)$ for $i = 1,2,3,\dots,n-1$ and let $x_1 + \alpha \equiv \gamma \ (\text{mod}3)$ and $x_1 + \beta \equiv \delta \ (\text{mod}3)$. It follows that

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\gamma\delta\delta\delta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots$$

, which shows that these points are different code representations of the same point.

We now reconstruct the function F by using the code representations of the points on S. At each step we obtain the images of the code sets S_0, S_1, and S_2 under F, respectively. Notice that if we know x_1, then the image of $x_1x_2x_3\dots$ can be acquired. We use different colors on each code set to make the proof more understandable and give a sketch of the proof using the following figures.

Case 1: Let $x_1 = 0$. Observe that $F(S_0) = S$, $F(S_{00}) = S_0$, $F(S_{01}) = S_1$, and $F(S_{02}) = S_2$. Likewise, the images of S_{000}, S_{001}, S_{002}, S_{010}, S_{011}, S_{012}, S_{020}, S_{021}, S_{022} under F are S_{00}, S_{01}, S_{02}, S_{10}, S_{11}, S_{12}, S_{20}, S_{21}, S_{22}, respectively (Figure 6). Note that if the first term of the point $x_1x_2x_3x_4\dots$ is zero, then $F(0x_2x_3x_4\dots)$ equals $x_2x_3x_4\dots$. Hence, we get $F(0x_2x_3x_4\dots) = y_1y_2y_3\dots$ where $y_i \equiv x_{i+1} \ (\text{mod}3)$ for $i = 1,2,3,\dots$.

Case 2: Let $x_1 = 1$. As seen in Figure 7, we have $F(S_1) = S$, $F(S_{10}) = S_1$, $F(S_{11}) = S_2$, and $F(S_{12}) = S_0$. In addition, the images of S_{100}, S_{101}, S_{102}, S_{110}, S_{111}, S_{112}, S_{120}, S_{121}, S_{122} under F are S_{11}, S_{12}, S_{10}, S_{21}, S_{22}, S_{20}, S_{01}, S_{02}, S_{00}, respectively. In the general case, we obtain $F(1x_2x_3x_4\dots) = y_2y_3y_4\dots$ where $y_i \equiv x_i + 1 \ (\text{mod}3)$.

Case 3: Let $x_1 = 2$. Notice that $F(S_2) = S$, $F(S_{20}) = S_2$, $F(S_{21}) = S_0$, and $F(S_{22}) = S_1$. Furthermore, the images of S_{200}, S_{201}, S_{202}, S_{210}, S_{211}, S_{212}, S_{220}, S_{221}, S_{222} under F are S_{22}, S_{20}, S_{21}, S_{02}, S_{00}, S_{01}, S_{12}, S_{10}, S_{11}, respectively (Figure 8). We thus have $F(2x_2x_3x_4\dots) = y_2y_3y_4\dots$ where $y_i \equiv x_i + 2 \ (\text{mod}3)$.

Hence, the proof is completed. □

To prove Proposition 2.2 and to compute the periodic points of F, we will use the definition of F given in Proposition 2.1

Proposition 2.2 *The dynamical system $\{S;F\}$ is chaotic in the sense of Devaney.*

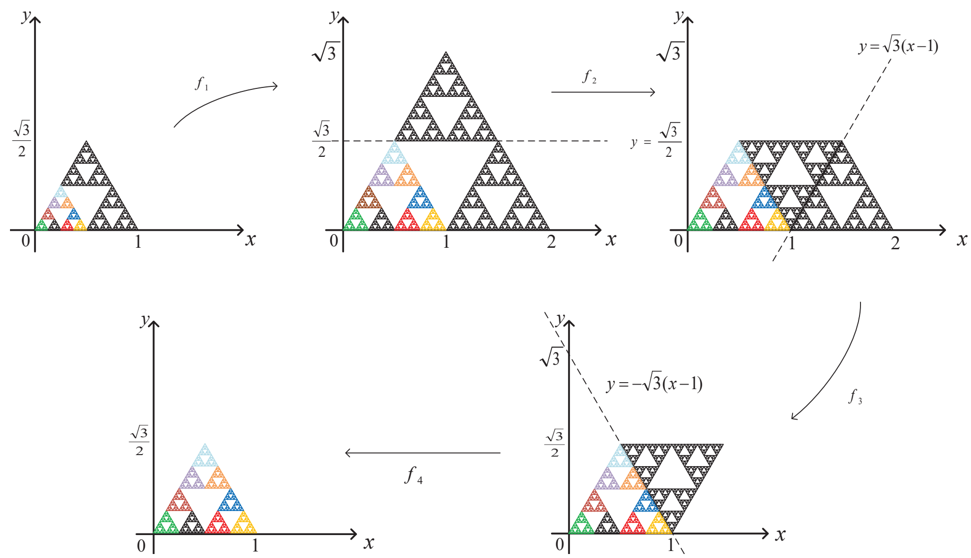

Figure 6. The action of the function F on code sets S_0, S_{0x_2}, and $S_{0x_2x_3}$.

Proof

- We first prove that F is sensitive to initial conditions. Let us take an arbitrary point a of S such that

$$a = a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots.$$

For any δ, there exists a large enough natural number k such that $\frac{1}{2^{k-2}} < \delta$. Let us choose the point

$$b = a_1a_2\ldots a_{k-1}b_kb_{k+1}\ldots$$

where $b_i \neq a_k$ for $i = k, k+1, k+2, \ldots$. Using the definition of metric d, one can compute that

$$d(a,b) < \delta.$$

On the other hand, we have

$$F^{k-1}(a) = c_kc_{k+1}c_{k+2}\ldots$$

$$F^{k-1}(b) = d_kd_{k+1}d_{k+2}\ldots$$

such that $c_t \equiv s + a_t \pmod 3$ and $d_t \equiv s + b_t \pmod 3$ where $t = k, k+1, k+2, \cdots$ and $s = (2^{k-1}a_1 + 2^{k-2}a_2 + \ldots + 2a_{k-1})$. It follows that $s + a_k \neq s + b_{k+i}$ for $i = 1, 2, 3, \ldots$, and therefore

$$d(F^{k-1}(a), F^{k-1}(b)) > \frac{1}{4},$$

which gives us the desired result.

- We now give the proof that F is topologically transitive. Let U and V be arbitrary nonempty open sets of S. Therefore, there exists an element $a = a_1a_2\ldots a_{k-1}a_ka_{k+1}\ldots$ of S and a natural number k such that $B(a, \frac{1}{2^{k-1}}) \subset U$. It is clear that $B(a, \frac{1}{2^{k-1}})$ contains a subtriangle of the Sierpinski gasket such that

$$U' = \{a_1a_2\ldots a_kx_{k+1}x_{k+2}x_{k+2}\ldots \mid a_1, \ldots, a_k \text{ are fixed}\}$$

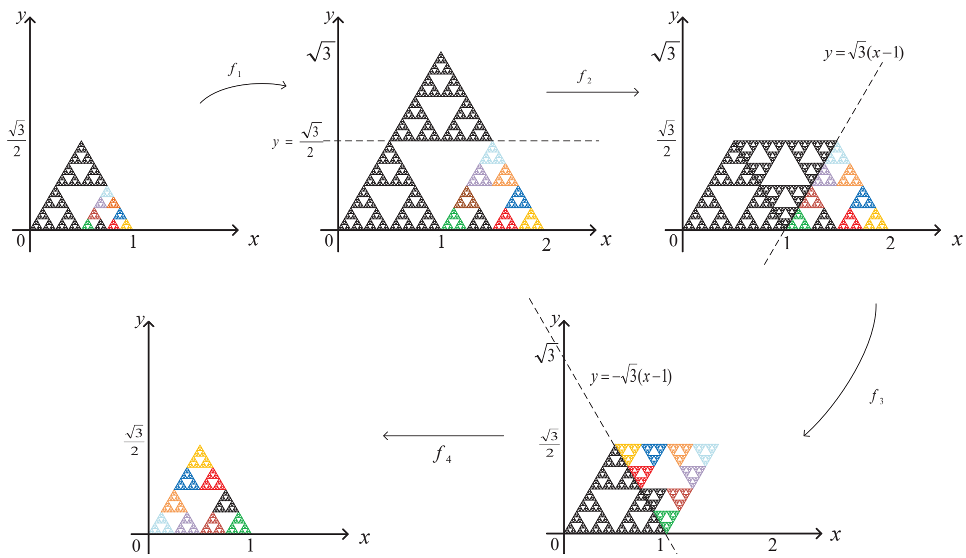

Figure 7. The action of the function F on code sets S_1, S_{1x_2}, and $S_{1x_2x_3}$.

where any $x_i \in \{0,1,2\}$ is arbitrary for $i = k+1, k+2, k+3, \ldots$. Due to the definition of F, we get

$$F^k(U') = \{y_1y_2y_3\ldots \mid y_i \equiv s + x_{k+i} \ (\text{mod}3),\ x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ are arbitrary}\}$$

for $i = 1,2,3,\ldots$ and $s = 2^{k-1}a_1 + 2^{k-2}a_2 + \ldots + a_k$. This shows that $F^k(U') = S$ and $F^k(U) = S$. Hence, there is a natural number k such that

$$F^k(U) \cap V \neq \emptyset$$

for any nonempty open subset U and V of S.

- To show that the periodic points of F are dense in S, we must find periodic points sufficiently close to any point of S. Let us take an arbitrary point a of S and open set U of S. There is $k \in \mathbb{N}$ such that $B(a, \frac{1}{2^{k-1}}) \subset U$ for large enough k. Because of the definition of metric d, we have

$$U' = \{a_1a_2\ldots a_kx_{k+1}x_{k+2}\ldots \mid a_i \text{ is fixed}, x_j \text{ is arbitrary}\} \subset B\Big(a, \frac{1}{2^{k-1}}\Big)$$

for $i = 1,2,\ldots,k$ and $j = k+1, k+2, \ldots$. We now show that U' contains a k-period point:

$$F^k(\{a_1a_2\ldots a_kx_{k+1}x_{k+2}\ldots\}) = \{y_{k+1}y_{k+2}y_{k+3}\ldots\}$$

where $y_{k+l} \equiv s + x_{k+l}$ (mod3) for $l = 1,2,3,\ldots$ and $s = 2^{k-1}a_1 + 2^{k-2}a_2 + \ldots + a_k$, which gives us the following equations:

$$s + x_{k+1} \equiv a_1 \ (\text{mod}3)$$

$$s + x_{k+2} \equiv a_2 \ (\text{mod}3)$$

$$\vdots$$

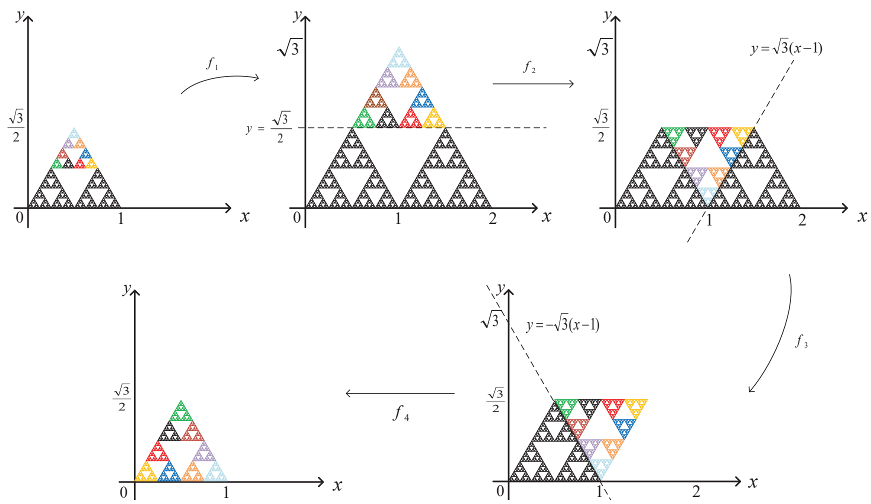

Figure 8. The actions of the function F on code sets S_2, S_{2x_2}, and $S_{2x_2x_3}$.

$$s + x_{2k} \equiv a_k \ (\text{mod}3)$$

$$\vdots$$

Since any x_j is arbitrary for $j = k+1, k+2, k+3, \ldots$, there always exists a solution of this system of equations. That is, we can find a periodic point sufficiently close to any point of S.

$\square$

2.1. An algorithm for the computation of periodic points of F

In this part, we give a general formula that computes the periodic points of F. Let us take an arbitrary point a on the Sierpinski gasket whose code representation is $a_1a_2a_3\ldots a_k\ldots$. If we apply the function F to this point, then we get

$$\begin{aligned} F(a) &= b_1b_2b_3\ldots b_k\ldots \\ &= (a_1+a_2)(a_1+a_3)(a_1+a_4)\ldots(a_1+a_{k+1})\ldots \end{aligned}$$

where $b_k \equiv a_1 + a_{k+1}$ (mod3) for $k = 1,2,3\ldots$. With simple calculations, we get three fixed points of F such that

$$\bullet\overline{0} = 000000\ldots, \quad \bullet\overline{201} = 201201201\ldots, \quad \bullet\overline{102} = 102102102\ldots.$$

To obtain the 2-periodic points of F, observe that

$$\begin{aligned} F^2(a) &= c_1c_2\ldots c_k\ldots \\ &= (2a_1+a_2+a_3)(2a_1+a_2+a_4)\ldots(2a_1+a_2+a_{k+2})\ldots \end{aligned}$$

where $c_k \equiv 2a_1 + a_2 + a_{k+2}$ (mod3) for $k = 1,2,3\ldots$. Thus, 2-periodic points of F are:

$$\bullet\overline{1} = 111111\ldots, \quad \bullet\overline{2} = 2222\ldots, \quad \bullet\overline{012} = 012012012\ldots,$$

$$\bullet\overline{120} = 120120120\ldots, \quad \bullet\overline{210} = 210210210\ldots, \quad \bullet\overline{021} = 021021021\ldots.$$

Continuing in this manner, the image of point a under F^n can be obtained as

$$\begin{aligned} F^n(a) &= d_1 d_2 \ldots d_k \ldots \\ &= (2^{n-1}a_1 + 2^{n-2}a_2 + \ldots + a_n + a_{n+1})(2^{n-1}a_1 + 2^{n-2}a_2 + \ldots + a_n + a_{n+2}) \\ & \quad \ldots (2^{n-1}a_1 + 2^{n-2}a_2 + \ldots + a_n + a_{n+k}) \ldots \end{aligned}$$

where

$$d_k \equiv 2^{n-1}a_1 + 2^{n-2}a_2 + \ldots + a_n + a_{n+k} \ (\text{mod} 3)$$

for $k = 1, 2, 3 \ldots$. In order to compute n-periodic points of F, we solve the equation $F^n(a) = a$, which gives

$$2^{n-1}a_1 + 2^{n-2}a_2 + \ldots + a_n + a_{n+k} \equiv a_k \quad (\text{mod} 3)$$

where $n = 1, 2, 3, \ldots$ for $k = 1, 2, 3 \ldots$.

By using the general case, we compute some cycles of period 3 as follows:

$$\{\overline{011}, \overline{110}, \overline{212}\}, \{\overline{022}, \overline{220}, \overline{121}\}, \{\overline{001220112}, \overline{012201120}, \overline{122011200}\},$$

$$\{\overline{010121202}, \overline{101212020}, \overline{120201012}\}, \{\overline{100022211}, \overline{111000222}, \overline{221110002}\},$$

$$\{\overline{200011122}, \overline{222000111}, \overline{112220001}\}.$$

Using similar calculations, some of the cycles of period 4 are as follows:

$$\{\overline{1100}, \overline{2112}, \overline{0011}, \overline{0110}\}, \{\overline{2200}, \overline{1221}, \overline{0022}, \overline{0220}\}, \{\overline{1122}, \overline{2002}, \overline{2211}, \overline{1001}\}.$$

Some of the cycles of period 5 are

$$\begin{aligned} &\{\overline{00011}, \overline{00110}, \overline{01100}, \overline{11000}, \overline{21112}\}, \{\overline{100000222221111}, \overline{111110000022222}, \\ &\overline{222211111000002}, \overline{111000002222211}, \overline{221111100000222}\} \end{aligned}$$

and some of the cycles of period 6 are

$$\{\overline{20}, \overline{21}, \overline{01}, \overline{10}, \overline{12}, \overline{02}\}, \{\overline{100}, \overline{112}, \overline{202}, \overline{211}, \overline{001}, \overline{010}\},$$

$$\{\overline{002211}, \overline{022110}, \overline{221100}, \overline{100221}, \overline{110022}, \overline{211002}\}.$$

3. Conclusion

As seen in the example of the right Sierpinski gasket in (1.1), a dynamical system can be naturally defined on any fractal that can be expressed by the attractor of an IFS. Thus, dynamical systems defined in this way are directly related to their IFSs. This paper presents an alternative method to construct a dynamical system on the equilateral Sierpinski gasket by using expanding and folding mappings. Our construction in the present paper does not depend on its IFS and is completely related to the structure of the Sierpinski gasket. Moreover, our method is very convenient to express this dynamical system via ternary numbers. In Proposition 2.2, we prove that $\{S, F\}$ is chaotic in the sense of Devaney. Since (S, d) is a compact metric space and F is a continuous map, this dynamical system is also chaotic in the sense of Li and Yorke. Furthermore, the proof of $\{S, F\}$ being chaotic in the sense of Devaney becomes easier by using the intrinsic metric formula on the code set of the Sierpinski gasket. This method will also guide the construction of chaotic dynamical systems on different fractals by using appropriate folding and expanding mappings.

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