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From ordered semigroups to ordered hypersemigroups

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Abstract: In an attempt to show the way we pass from ordered semigroups to ordered hypersemigroups, we examine some well known results of regular and intraregular ordered semigroups in case of ordered hypersemigroups. The corresponding results on hypersemigroups (without order) can also be obtained as application of the results of the present paper. The sets we use in our investigation shows the pointless character of the results.

Key words: Ordered hypersemigroup, right (left) ideal, bi-ideal, quasi-ideal, regular, intraregular

1. Introduction

An ordered groupoid is a groupoid $(S,\cdot)$ endowed with an order relation "$\le$" in which the multiplication is compatible with the ordering; and it is denoted by $(S,\cdot,\le)$. In particular, if the multiplication on S is associative, then $(S,\cdot,\le)$ is called an ordered semigroup [1]. For an ordered groupoid $(S,\cdot,\le)$ and a subset A of S, we denote by $(A]$ the subset of S defined by $(A] := \{t \in S \mid t \le a \text{ for some } a \in A\}$. If $(S,\cdot,\le)$ is an ordered groupoid, a nonempty subset A of S is called a right (resp. left) ideal of S if (1) it is a right (resp. left) ideal of $(S,\cdot)$ and (2) if $a \in A$ and $S \ni b \le a$, then $b \in A$, that is if $(A] = A$ (cf. e.g., [5]). An ordered semigroup $(S,\cdot,\le)$ is called regular [4] if for every $a \in S$ there exists $x \in S$ such that $a \le axa$; it is called intraregular [5, 6] if for every $a \in S$ there exist $x, y \in S$ such that $a \le xa^2y$. A subset B of an ordered semigroup $(S,\cdot,\le)$ is called a bi-ideal of S [3] if (1) $BSB \subseteq B$ and (2) if $a \in B$ and $S \ni b \le a$, then $b \in B$. A subset Q of an ordered groupoid S is called a quasi-ideal of S [7] if (1) $(QS] \cap (SQ] \subseteq Q$ and (2) if $a \in Q$ and $S \ni b \le a$, then $b \in Q$.

We have seen in [9] that an ordered semigroup S is regular if and only if for every right ideal A and every left ideal B of S, we have $A \cap B = (AB]$, equivalently $A \cap B \subseteq (AB]$. In [15], we showed that an ordered semigroup S is intraregular if and only if for every right ideal A and every left ideal B of S, we have $A \cap B \subseteq (BA]$. In [8], we proved that if S is a regular ordered semigroup, then B is a bi-ideal of S if and only if there exist a right ideal R and a left ideal L of S such that $B = (RL]$. In [9], we also showed that an ordered semigroup S is regular if and only if the ideals of S are idempotent and for every right ideal A and every left ideal B of S, the set $(AB]$ is a quasi-ideal of S. In [10], we studied some of the above results for hypersemigroups. As a continuation of the investigation on hypersemigroups (without order) [12], in the present paper, we examine the results on ordered semigroups mentioned above for ordered hypersemigroups. In the proofs, we tried to use sets instead of elements to show the pointless character of the results. This is in the

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same spirit with the abstract formulation of general topology (the so-called topology without points) initiated by Koutsý and Nöbeling [16, 17].

As an application of the results of the present paper, the corresponding results on hypersemigroup (without order) can be obtained, and this is because every hypersemigroup endowed with the equality relation "$=$" is an ordered hypersemigroup. Further interesting information related to hypersemigroups (without order) will be given in a forthcoming paper. For the sake of completeness, we will give some definitions already given in [10–14].

2. Prerequisites

An hypergroupoid is a nonempty set H with an hyperoperation "$\circ$" on H and an operation "$*$" on the set $\mathcal{P}^*(H)$ of nonempty subsets of H induced by "$\circ$" such that $A*B = \bigcup\limits_{(a,b)\in A\times B} (a\circ b)$ for every $A,B\in\mathcal{P}^*(H)$. For any $x,y\in H$, we have $\{x\}*\{y\} = x\circ y$. For an hypergroupoid, we have (1) if $x\in A*B$, then $x\in a\circ b$ for some $a\in A$, $b\in B$ and (2) if $a\in A$ and $b\in B$, then $a\circ b\subseteq A*B$. If $(H,\circ)$ is an hypergroupoid and $A,B,C,D\in\mathcal{P}^*(H)$, then $A\subseteq B$ implies $A*C\subseteq B*C$ and $C*A\subseteq C*B$; we also have $H*A\subseteq H$ and $A*H\subseteq H$. In an hypergroupoid, the following hold: (1) $(\bigcup\limits_{i\in I} A_i)*B = \bigcup\limits_{i\in I}(A_i*B)$ and (2) $B*(\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I}(B*A_i)$. An hypergroupoid $(H,\circ)$ is called hypersemigroup if $\{x\}*(y\circ z) = (x\circ y)*\{z\}$ for all $x,y,z\in H$. In an hypersemigroup, the operation "$*$" is associative; hence, $(\mathcal{P}^*(H),*)$ is a semigroup. If $(H,\circ)$ is an hypergroupoid and "$\le$" an order relation on H, we denote by $(A]$ the subset of H defined by $(A] := \{t\in H \mid t\le a \text{ for some } a\in A\}$. Exactly as in ordered groupoids we have: $A\subseteq (A]$; if $A\subseteq B$, then $(A]\subseteq(B]$; $(A\cup B] = (A]\cup(B]$; $(H]=H$; $((A]] = (A]$. We denote by "$\preceq$" the preorder (that is the reflexive and transitive relation) on $\mathcal{P}^*(H)$ defined by $A\preceq B$ if for every $a\in A$ there exists $b\in B$ such that $a\le b$. By an ordered hypergroupoid, we mean an hypergroupoid $(H,\circ)$ endowed with an order relation "$\le$" such that $a\le b$ implies $a\circ c\preceq b\circ c$ and $c\circ a\preceq c\circ b$ for every $c\in H$ (cf. also [2]). If $(H,\circ)$ is an ordered hypergroupoid, a nonempty subset A of H is called a *left* (resp. *right*) *ideal* of H if (1) $H*A\subseteq A$ (resp. $A*H\subseteq A$) and (2) if $a\in A$ and $H\ni b\le a$, then $b\in A$, that is if $(A]=A$. For further information related to ordered hypersemigroups, we refer to [11, 13, 14].

3. Main results

Proposition 3.1 *[13] Let $(H,\circ,\le)$ be an ordered hypergroupoid, $a\le b$, and $c\le d$. Then we have $a\circ c\preceq b\circ d$.*

Definition 3.2 *An ordered hypersemigroup $(H,\circ,\le)$ is called regular if for every $a\in H$ there exists $x\in H$ such that $\{a\}\preceq(a\circ x)*\{a\}$.*

Which means that for every $a\in H$ there exist $x,t\in H$ such that $t\in(a\circ x)*\{a\}$ and $a\le t$.

If no confusion is possible, we write in short $a*H$ instead of $\{a\}*H$, and $H*a$ instead of $H*\{a\}$.

Proposition 3.3 *Let H be an ordered hypersemigroup. If H is regular, then for every $A\in\mathcal{P}^*(H)$, we have $A\subseteq(A*H*A]$. Conversely, if $a\in\Big(\{a\}*H*\{a\}\Big]$ for every $a\in H$, then H is regular.*

Proof Let H be regular, $A \in \mathcal{P}^*(H)$ and $a \in A$. Since $a \in H$ and H is regular, there exist $x, t \in H$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$. We have $a \leq t \in \{a\} * \{x\} * \{a\}$ so $a \in \Big(\{a\} * H * \{a\}\Big] \subseteq (A * H * A]$. For the converse statement, let $a \in H$. By hypothesis, we have $a \in \Big((a * H) * \{a\}\Big]$. Then $a \leq t$ for some $t \in (a * H) * \{a\}$, $t \in u \circ a$ for some $u \in a * H$ and $u \in a \circ x$ for some $x \in H$. We have $t \in u \circ a = \{u\} * \{a\} \subseteq (a \circ x) * \{a\}$ and $a \leq t$, so H is regular. $\square$

Lemma 3.4 *[11] If H is an ordered hypergroupoid then, for any nonempty subsets A, B of H, we have $(A] * (B] \subseteq (A * B]$.*

For an hypergroupoid H and a nonempty subset A of H, we denote by $R(A)$ (resp. $L(A)$), the right (resp. left) ideal of H generated by A. For $A = \{a\}$, we write $R(a)$, $L(a)$ in short instead of $R(\{a\})$, $L(\{a\})$.

Lemma 3.5 *For an hypersemigroup H and a nonempty subset A of H, we have*

$$R(A) = \Big(A \cup (A * H)\Big] \text{ and } L(A) = \Big(A \cup (H * A)\Big].$$

Proof The set $\Big(A \cup (A * H)\Big]$ is a nonempty subset of H containing A, and

$$\begin{aligned}
\Big(A \cup (A * H)\Big] * H &= \Big(A \cup (A * H)\Big] * (H] \\
&\subseteq \left(\Big(A \cup (A * H)\Big) * H\right] \text{ (by Lemma 3.4)} \\
&= \Big((A * H) \cup (A * H * H)\Big] = (A * H] \text{ (since } H * H \subseteq H) \\
&\subseteq A \cup (A * H] \subseteq \Big(A \cup (A * H)\Big].
\end{aligned}$$

We also have $\left(\Big(A \cup (A * H)\Big]\right] = \Big(A \cup (A * H)\Big]$ (as $((X]] = (X]$ holds for any $X \subseteq H$). If now T is a right ideal of H such that $T \supseteq A$, then we have

$$\Big(A \cup (A * H)\Big] \subseteq \Big(T \cup (T * H)\Big] = (T] = T.$$

So the set $\Big(A \cup (A * H)\Big]$ is the right ideal of H generated by A. The rest of the lemma can be proved in a similar way. $\square$

Lemma 3.6 *(see also [13]) If H is an ordered hypergroupoid then, for any nonempty subsets A, B of H, we have*

$$(A * B] = \Big((A] * (B]\Big] = \Big((A] * B\Big] = \Big(A * (B]\Big].$$

Proof For the sake of completeness, let us give the proof of the first equality; for the rest, see [13]. By Lemma 3.4, we have $(A]*(B] \subseteq (A*B]$, then we have $\Big((A]*(B]\Big] \subseteq \Big((A*B]\Big] = (A*B]$. Since $A \subseteq (A]$ and $B \subseteq (B]$, we have $A*B \subseteq (A]*(B]$, then $(A*B] \subseteq \Big((A]*(B]\Big]$, thus $(A*B] = \Big((A]*(B]\Big]$. $\square$

Lemma 3.7 *[13] $(H,\circ,\leq)$ is an ordered hypersemigroup and A, B, C nonempty subsets of H, then we have*

$$\Big(A*(B]*C\Big) = (A*B*C].$$

Lemma 3.8 *[10] Let H be an hypergroupoid. If A is a right ideal and B is a left ideal of H, then $A \cap B \neq \emptyset$.*

Theorem 3.9 *An ordered hypersemigroup H is regular if and only if for every right ideal A and every left ideal B of H, we have*

$$A \cap B = (A*B], \text{ equivalently, } A \cap B \subseteq (A*B].$$

Proof Let H be regular, A be a right ideal, and B be a left ideal of H. By Lemma 3.8, we have $A \cap B \neq \emptyset$. Thus, by Proposition 3.3, we have

$$\begin{aligned} A \cap B &\subseteq \Big((A \cap B)*H*(A \cap B)\Big] \subseteq \Big((A*H)*B)\Big] \subseteq (A*B] \\ &\subseteq (A*H] \cap (H*B] \subseteq (A] \cap (B] = A \cap B, \end{aligned}$$

thus we have $A \cap B = (A*B]$.

Suppose now that $A \cap B \subseteq (A*B]$ for every right ideal A and every left ideal B of H, and let $A \in \mathcal{P}^*(H)$. Then we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq \Big(R(A)*L(A)\Big] = \bigg(\Big(A \cup (A*H)\Big]*\Big(A \cup (H*A)\Big]\bigg] \\ &= \bigg(\Big(A \cup (A*H)\Big)*\Big(A \cup (H*A)\Big)\bigg] \text{ (by Lemma 3.6)} \\ &= \Big((A*A) \cup (A*H*A) \cup (A*H*H*A)\Big] \\ &= \Big((A*A) \cup (A*H*A)\Big]. \end{aligned}$$

Then

$$\begin{aligned} A*A &\subseteq \Big((A*A) \cup (A*H*A)\Big]*(A] \\ &\subseteq \Big((A*A*A) \cup (A*H*A*A)\Big] \text{ (by Lemma 3.4)} \\ &\subseteq (A*H*A]. \end{aligned}$$

Then we have

$$A \subseteq \Big((A*H*A] \cup (A*H*A)\Big] = \Big((A*H*A]\Big] = (A*H*A]$$

and, by Proposition 3.3, H is regular. $\square$ $\square$

Definition 3.10 *[11] An ordered hypersemigroup H is called intraregular if for every $a \in H$ there exist $x, y \in H$ such that $\{a\} \preceq (x \circ a) * (a \circ y)$.*

That is, for every $a \in H$ there exist $x, y, t \in H$ such that

$$t \in (x \circ a) * (a \circ y) \text{ and } a \leq t.$$

Proposition 3.11 *[11] Let H be an ordered hypersemigroup. If H is intraregular then for every nonempty subset A of H, we have $A \subseteq (H * A * A * H]$. Conversely, if $a \in (H * \{a\} * \{a\} * H]$ for every $a \in H$, then H is intraregular.*

Theorem 3.12 *An ordered hypersemigroup H is intraregular if and only if for every right ideal A and every left ideal B of H, we have*

$$A \cap B \subseteq (B * A].$$

Proof $\Longrightarrow$. Let A be a right ideal and B be a left ideal of H. By Lemma 3.8, $A \cap B \in \mathcal{P}^*(H)$. Since H is intraregular, by Proposition 3.11, we have

$$\begin{aligned} A \cap B &\subseteq \Big(H * (A \cap B) * (A \cap B) * H\Big] \\ &\subseteq \Big((H * B) * (A * H)\Big] \\ &\subseteq (B * A]. \end{aligned}$$

$\Longleftarrow$. Let $a \in H$. By hypothesis, we have

$$\begin{aligned} a \in R(a) \cap L(a) \subseteq & \Big(L(a) * R(a)\Big] \\ = & \bigg(\Big(\{a\} \cup (H * \{a\})\Big] * \Big(\{a\} \cup (\{a\} * H)\Big]\bigg] \\ = & \bigg(\Big(\{a\} \cup (H * \{a\})\Big) * \Big(\{a\} \cup (\{a\} * H)\Big)\bigg] \text{ (by Lemma 3.6)} \\ = & \Big((a \circ a) \cup (H * \{a\} * \{a\}) \cup (\{a\} * \{a\} * H) \cup (H * \{a\} * \{a\} * H)\Big]. \end{aligned}$$

Then $a \in (a \circ a]$ or $a \in (H * \{a\} * \{a\}]$ or $a \in (\{a\} * \{a\} * H]$ or $a \in (H * \{a\} * \{a\} * H]$. If $a \in (a \circ a]$, then

$$\begin{aligned} a &\in \Big((a \circ a] * (a \circ a]\Big] = \Big((a \circ a) * (a \circ a)\Big] \text{ (by Lemma 3.6)} \\ &\subseteq (H * \{a\} * \{a\} * H] \end{aligned}$$

then, by Proposition 3.11, H is intraregular. If $a \in (H * \{a\} * \{a\}]$, then

$$\begin{aligned} a &\in \Big(H * (H * \{a\} * \{a\}] * \{a\}\Big] = (H * H * \{a\} * \{a\} * \{a\}] \text{ (by Lemma 3.7)} \\ &\subseteq (H * \{a\} * \{a\} * H]. \end{aligned}$$

If $a \in (\{a\} * \{a\} * H]$ or $a \in (H * \{a\} * \{a\} * H]$ in a similar way, we have $a \in (H * \{a\} * \{a\} * H]$. In any case, H is intraregular. $\square$

Although this nice short proof seems to be enough at first, it is far from indicating the pointless character of the result. That is why we avoided using a similar proof in Theorem 3.9. Hence, we will also give another proof which, though more technical, is much more interesting as it shows that Theorem 3.12 can be also proved using only sets and not elements, which is very important for further investigation on hypersemigroups in general.

Second proof Let $A \in \mathcal{P}^*(H)$. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq \Big(L(A) * R(A)\Big] = \Bigg(\Big(A \cup (H * A)\Big] * \Big(A \cup (A * H)\Big]\Bigg] \\ &= \Bigg(\Big(A \cup (H * A)\Big) * \Big(A \cup (A * H)\Big)\Bigg] \text{ (by Lemma 3.6)} \\ &= \Big((A * A) \cup (H * A * A) \cup (A * A * H) \cup (H * A * A * H)\Big]. \end{aligned}$$

Then we have

$$\begin{aligned} A * A &\subseteq \Big((A * A) \cup (H * A * A) \cup (A * A * H) \cup (H * A * A * H)\Big] * (A] \\ &\subseteq \Big((A * A * A) \cup (H * A * A * A) \cup (A * A * H * A) \cup (H * A * A * H * A)\Big] \\ &\subseteq \Big((A * A * H) \cup (H * A * A * H)\Big]. \end{aligned}$$

Then

$$\begin{aligned} H * A * A &\subseteq (H] * \Big((A * A * H) \cup (H * A * A * H)\Big] \\ &\subseteq \Big((H * A * A * H) \cup (H * H * A * A * H)\Big] \text{ (by Lemma 3.4)} \\ &= (H * A * A * H]. \end{aligned}$$

Hence we obtain

$$\begin{aligned} A &\subseteq \Bigg(\Big((A * A * H) \cup (H * A * A * H)\Big] \cup (A * A * H) \cup (H * A * A * H]\Bigg] \\ &= \Bigg(\Big((A * A * H) \cup (H * A * A * H)\Big]\Bigg] = \Big((A * A * H) \cup (H * A * A * H)\Big], \end{aligned}$$

from which

$$\begin{aligned} A * A &\subseteq (A] * \Big((A * A * H) \cup (H * A * A * H)\Big] \\ &\subseteq \Bigg(A * \Big((A * A * H) \cup (H * A * A * H)\Big)\Bigg] \text{ (by Lemma 3.4)} \\ &= \Big((A * A * A * H] \cup (A * H * A * A * H)\Big] \\ &\subseteq (H * A * A * H], \end{aligned}$$

and then

$$\begin{aligned} A*A*H &\subseteq (H*A*A*H]*(H] \subseteq (H*A*A*H*H] \\ &\subseteq (H*A*A*H]. \end{aligned}$$

Thus we get

$$\begin{aligned} A &\subseteq \Big((A*A*H)\cup(H*A*A*H)\Big] \subseteq \Big((H*A*A*H]\cup(H*A*A*H)\Big] \\ &= \Big((H*A*A*H]\Big] = (H*A*A*H] \end{aligned}$$

and, by Proposition 3.11, H is intraregular. □

Definition 3.13 *Let $(H,\circ,\leq)$ be an ordered hypersemigroup. A nonempty subset B of H is called a bi-ideal of H if*

(1) $B*H*B\subseteq B$ and

(2) if $a\in B$ and $H\ni b\leq a$, then $b\in B$; that is if $(B]=B$.

Proposition 3.14 *Let H be an ordered hypersemigroup. If C is a right (resp. left) ideal of H and $D\in\mathcal{P}^*(H)$, then the set $B=(C*D]$ is a bi-ideal of H. "Conversely", if H is regular and B is a bi-ideal of H, then there exist a right ideal C and a left ideal D of H such that $B=(C*D]$.*

Proof $\Longrightarrow$. Let C be a right ideal of H and $D\in\mathcal{P}^*(H)$. First of all, the set $B=(C*D]$ is a nonempty subset of H. Moreover, we have

$$\begin{aligned} B*H*B &= (C*D]*(H]*(C*D] \\ &\subseteq \Big((C*D)*H\Big]*(C*D] \text{ (by Lemma 3.4)} \\ &\subseteq \Big((C*D)*H*(C*D)\Big] \text{ (by Lemma 3.4)} \\ &= \Big((C*(D*H*C)*D\Big] \subseteq \Big((C*H)*D\Big] \\ &\subseteq (C*D]=B, \end{aligned}$$

so B is a bi-ideal of H.

$\Longleftarrow$. Let B be a bi-ideal of H. Then we have $B*H*B\subseteq B$; thus $(B*H*B]\subseteq(B]=B$. Since H is regular, by Proposition 3.3, we have $B\subseteq(B*H*B]$; thus we have $B=(B*H*B]$. On the other hand,

$$\begin{aligned} R(B)*L(B) &= \Big((B\cup(B*H)\Big]*\Big((B\cup(H*B)\Big] \\ &\subseteq \Bigg(\Big(B\cup(B*H)\Big)*\Big(B\cup(H*B)\Big)\Bigg] \text{ (by Lemma 3.6)} \\ &= \Big((B*B)\cup(B*H*B)\cup(B*H*H*B)\Big] \\ &= \Big((B*B)\cup(B*H*B)\Big]. \end{aligned}$$

Since

$$B * B = (B] * (B * H * B] \subseteq (B * B * H * B] \subseteq (B * H * B],$$

we have

$$\begin{aligned} R(B) * L(B) &\subseteq \Big((B * H * B] \cup (B * H * B)\Big] = \Big((B * H * B]\Big] \\ &= (B * H * B] = B, \end{aligned}$$

so $\Big(R(B) * L(B)\Big] \subseteq (B] = B$. In addition, we have

$$\begin{aligned} B = (B * H * B] &\subseteq \left(\Big(R(B) * H\Big) * L(B)\right] \text{ (since } B \subseteq R(B), L(B)) \\ &\subseteq \Big(R(B) * L(B)\Big] \text{ (since } R(B) \text{ is a right ideal of } H). \end{aligned}$$

Hence we obtain $B = \Big(R(B) * L(B)\Big]$, where $R(B)$ is a right ideal and $L(B)$ is a left ideal of H. □

By Proposition 3.14, we have the following.

Theorem 3.15 *Let H be a regular ordered hypersemigroup. Then B is a bi-ideal of H if and only if there exist a right ideal C and a left ideal D of H such that $B = (C * D]$.*

We finally characterize the regular hypersemigroups in terms of right ideals, left ideals, and quasi-ideals.

Definition 3.16 *Let $(H, \circ, \leq)$ be an ordered hypergroupoid. A nonempty subset Q of H is called a quasi-ideal of H if*

(1) $(Q * H] \cap (H * Q] \subseteq Q$ and

(2) if $a \in Q$ and $H \ni b \leq a$, then $b \in Q$; that is if $(Q] = Q$.

If H is an ordered hypergroupoid, an element $A \in \mathcal{P}^*(H)$ is called subidempotent if $(A * A] \subseteq A$. It is called idempotent if $(A * A] = A$.

Remark 3.17 *If H is an ordered hypergroupoid, then the right ideals and the left ideals of H are subidempotent. In particular, if H is a regular ordered hypersemigroup, then the right ideals and the left ideals of H are idempotent.*

Indeed, if A is a right ideal of an hypergroupoid H, then $(A * A] \subseteq (A * H] \subseteq (A] = A$, so A is subidempotent. Let now H be regular hypersemigroup and A a right ideal of H. Then $A \subseteq \Big((A * H) * A\Big] \subseteq (A * A] \subseteq A$, then $(A * A] = A$, and A is idempotent.

Theorem 3.18 *An ordered hypersemigroup H is regular if and only if the right ideals and the left ideals of H are idempotent and for every right ideal A and every left ideal B of H, the set $(A * B]$ is a quasi-ideal of H.*

Proof $\Longrightarrow$. Let H be regular. By Remark 3.17, the right and the left ideals of H are idempotent. Let now A be a right ideal and B a left ideal of H. Since H is regular, by Theorem 3.9, we have $A \cap B = (A * B]$. It is enough to prove that $A \cap B$ is a quasi-ideal of H. First of all, by Lemma 3.8, $A \cap B \in \mathcal{P}^*(H)$. Moreover,

$$\begin{aligned} \Big((A \cap B) * H\Big] \cap \Big(H * (A \cap B)\Big] &\subseteq (A * H] \cap (H * B] \\ &\subseteq (A] \cap (B] = A \cap B, \end{aligned}$$

and if $x \in A \cap B$ and $H \ni y \leq x$ then, since $y \leq x \in A$ and A is a right ideal of H, we have $y \in A$ and since $y \leq x \in B$ and B is a left ideal of H, we have $y \in B$, then $y \in A \cap B$. Thus, $A \cap B$ is a quasi-ideal of H, and so is $(A * B]$.

$\Longleftarrow$. Let $A \in \mathcal{P}^*(H)$. Since $R(A)$ is a right ideal of H, by Remark 3.17, it is idempotent, and we have

$$\begin{aligned} A &\subseteq R(A) = \Big(R(A) * R(A)\Big] = \Big(\Big(A \cup (A * H)\Big] * \Big(A \cup (A * H)\Big]\Big] \\ &= \Big(\Big(A \cup (A * H)\Big) * \Big(A \cup (A * H)\Big)\Big] \text{ (by Lemma 3.6)} \\ &= \Big((A * A) \cup (A * H * A) \cup (A * A * H) \cup (A * H * A * H)\Big] \\ &\subseteq (A * H]. \end{aligned}$$

Since $L(A)$ is a left ideal of H, in a similar way, we have $A \subseteq (H * A]$. Thus, we have

$$A \subseteq (A * H] \cap (H * A] \tag{a}$$

Since $(A * H]$ is a right ideal and $(H * A]$ is a left ideal of H, they are idempotent, so we have

$$\begin{aligned} (A * H] \cap (H * A] &= \Big((A * H] * (A * H]\Big] \cap \Big((H * A] * (H * A]\Big] \\ &= \Big((A * H) * (A * H)\Big] \cap \Big((H * A) * (H * A)\Big] \text{ (by Lemma 3.6)} \\ &= \Big((A * H * A) * H\Big] \cap \Big(H * (A * H * A)\Big] \\ &= \Big((A * H * A] * H\Big] \cap \Big(H * (A * H * A]\Big] \text{ (by Lemma 3.6)} \end{aligned} \tag{b}$$

On the other hand,

$$\begin{aligned} (A * H * A] &= \Big(A * (H * H] * A\Big] \text{ (since } H \text{ as a right ideal of } H \text{ is idempotent)} \\ &= \Big(A * (H * H) * A\Big] \text{ (by Lemma 3.7)} \\ &= \Big((A * H) * (H * A)\Big] \\ &= \Big((A * H] * (H * A]\Big] \text{ (by Lemma 3.6).} \end{aligned}$$

Since $(A * H]$ is a right ideal and $(H * A]$ is a left ideal of H, by hypothesis, the set $\Big((A * H] * (H * A]\Big]$ is a quasi-ideal of H, and so is $(A * H * A]$, which means that

$$\Big((A * H * A] * H\Big] \cap \Big(H * (A * H * A]\Big] \subseteq (A * H * A] \tag{c}$$

By (a)–(c), we have $A \subseteq (A * H * A]$ and by Proposition 3.3, H is regular. □

From Theorem 3.15, we have the following corollary.

Corollary 3.19 *[10, Theorem 2.9] If H is a regular hypersemigroup, then B is a bi-ideal of H if and only if there exists a right ideal C and a left ideal D of H such that $B = C * D$.*

From Theorem 3.18, we have the following:

Corollary 3.20 *[10, Theorem 2.12] An hypersemigroup H is regular if and only if the right ideals and the left ideals of H are idempotent, and for every right ideal A and every left ideal B of H, the product $A * B$ is a quasi-ideal of H.*

To prove the two corollaries, we observe the following: Let $(S, \circ)$ be an hypersemigroup. We endow it with the equality relation, that is, with the order "$\leq$" defined by $\leq := \{(a, b) \mid a = b\}$ and consider the ordered hypersemigroup $(S, \circ, \leq)$. It is enough to observe that the hypersemigroup $(S, \circ)$ is regular if and only if the ordered hypersemigroup $(S, \circ, \leq)$ is regular; the set A is a right (left) ideal, bi-ideal, or quasi-ideal of $(S, \circ)$ if and only if it is a right (left) ideal, bi-ideal, or quasi-ideal of $(S, \circ, \leq)$. The set A is an idempotent set in $(S, \circ)$ if and only if it is an idempotent set in $(S, \circ, \leq)$.

From Theorems 3.9 and 3.12, we can also get the corresponding results for hypersemigroups without order in the similar way.

As an application of the theorems of the paper, we give the following examples.

Example 3.21 *(see also [7, Example 1]) We consider the ordered semigroup $H = \{a, b, c, d, e\}$ with the multiplication "$\cdot$" given by Table 1 and the order "$\leq$" below.*

Table 1. Multiplication table of the ordered semigroup.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

$$\leq = \{a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (c, c), (d, d), (e, e)\}.$$

We give the covering relation and the figure of H.

$$\prec = \{(a, b), (a, c), (a, d), (a, e)\}$$

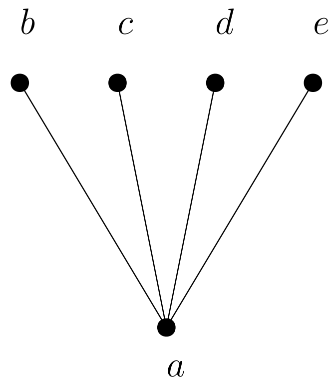

Figure 1. Figure corresponding to the order of the ordered semigroup.

As we have seen in [7], this is a regular (and not intraregular) ordered semigroup. We have also seen in [7] that:
The right ideals of H are the sets: $\{a\}, \{a, b, c\}, \{a, d, e\}$ and H.
The left ideals of H are the sets: $\{a\}, \{a, b, d\}, \{a, c, e\}$ and H.
The bi-ideals of H are the sets: $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{a, d\}$, $\{a, b, d\}$, $\{a, e\}$, $\{a, c, e\}$, $\{a, d, e\}$ and H.

Using the methodology described in [14], to this ordered semigroup corresponds an ordered hypersemigroup with the hyperoperation given in Table 2 having the same order with the ordered semigroup, the same right, left ideals, bi-ideals, and quasi-ideals, and it is a regular ordered hypersemigroup.

Table 2. The hyperoperation corresponding to the ordered semigroup.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, c\}$
c	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, d\}$	$\{a, e\}$
e	$\{a\}$	$\{a, d\}$	$\{a, e\}$	$\{a\}$	$\{a\}$

Independently, we can see that the ordered hypersemigroup $(H, \circ, \leq)$ is regular as follows. We have
$\{a\} \preceq (a \circ a) * \{a\} = \{a\}$ as $(a \circ a) * \{a\} = \{a\} * \{a\} = a \circ a = \{a\}$;
$\{b\} \preceq (b \circ e) * \{b\}$ as
$(b \circ e) * \{b\} = \{a, c\} * \{b\} = (a \circ b) \cup (c \circ b) = \{a\} \cup \{a, b\} = \{a, b\}$; similarly
$\{c\} \preceq (c \circ c) * \{c\} = \{a, c\}$ and $\{d\} \preceq (d \circ d) * \{d\} = \{a, d\}$.

Theorem 3.9 can be applied: For the right ideal $\{a, b, c\}$ and the left ideal $\{a, b, c\}$, for example, we have

$$\Big(\{a, b, c\} * \{a, b, d\}\Big] = \Big(\{a, b\}\Big] = \{a, b\} = \{a, b, c\} \cap \{a, b, d\}.$$

In a similar way, for every right ideal A and every left ideal B of H, we have $A \cap B = (A * B]$. Theorem 3.15 can also be applied. The set of quasi-ideals of H coincide with the set of bi-ideals of H (as in the corresponding ordered semigroup is so) and Theorem 3.18 can also be applied. It might be noted that this is not intraregular as the ordered semigroup $(H, \cdot, \leq)$ is not intraregular [7, 14]; independently, it is not intraregular as there are no $x, y \in H$ such that $\{b\} \preceq (x \circ b) * (b \circ y)$.

The examples that follow also come from ordered semigroups using the methodology described in [14]. We will give them directly on ordered hypersemigroups and we will refer to the paper where the corresponding examples of ordered semigroups appeared.

Example 3.22 *(see also [7, Example 2]) We consider the ordered hypersemigroup $H = \{a, b, c, d, e\}$ given by Table 3 and the order below:*

Table 3. The hyperoperation of Example 3.22.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a,b,c,d\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b,c,d\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a,b,c,d\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a,b,c,d\}$	$\{a\}$	$\{a\}$	$\{a\}$
e	$\{a\}$	$\{a,b,c,d\}$	$\{a\}$	$\{a\}$	$\{a,c,d,e\}$

$$\leq = \{(a,a), (a,b), (a,e), (b,b), (c,b), (c,c), (c,e), (d,b), (d,d), (d,e), (e,e)\}.$$

We give the covering relation and the figure of H.

$$\prec = \{(a,b), (a,e), (c,b), (c,e), (d,b), (d,e)\}.$$

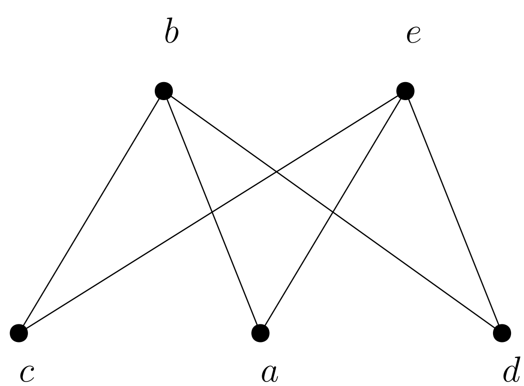

Figure 2. Figure corresponding to the order of Example 3.22.

This is intraregular, indeed
$\{a\} \preceq (a \circ a) * (a \circ a) = \{a\}$, $\{b\} \preceq (b \circ b) * (b \circ b) = \{a, b, c, d\}$,
$\{c\} \preceq (a \circ a) * (a \circ b) = \{a, b, c, d\}$, $\{d\} \preceq (b \circ d) * (a \circ b) = \{a, b, c, d\}$,
$\{e\} \preceq (e \circ b) * (e \circ e) = \{a, c, d, e\}$.
The right ideals of H are the sets: $\{a, b, c, d\}$ and H.
The left ideals of H are the sets: $\{a\}$, $\{a, c\}$, $\{a, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$ $\{a, c, d, e\}$ and H.
The bi-ideals of H are the sets: $\{a\}$, $\{a, c\}$, $\{a, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and H.

Theorems 3.12 and 3.15 can be applied. As an example, for the right ideal $\{a,b,c,d\}$ and the left ideal $\{a,c\}$, we have

$$\begin{aligned}\{a,b,c,d\}\cap\{a,c\} &= \{a,c\}\subseteq\{a,b,c,d\}=\Big(\{a,b,c,d\}\Big]\\ &= \Big(\{a,c\}*\{a,b,c,d\}\Big].\end{aligned}$$

It might be noted that this is not regular, there is no $x\in H$ such that $\{c\}\preceq(c\circ x)*\{c\}$.

Example 3.23 *(see also [7, Example 3]) The ordered hypersemigroup $H=\{a,b,c,d,e\}$ defined by Table 4 and Figure 3 is both regular and intraregular.*

Table 4. The hyperoperation of Example 3.23.

$\circ$	a	b	c	d	e
a	$\{a,c,d\}$	$\{a,b,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
b	$\{a,c,d\}$	$\{a,b,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
c	$\{a,c,d\}$	$\{a,b,c,d\}$	$\{c\}$	$\{a,c,d\}$	$\{a,c,d\}$
d	$\{a,c,d\}$	$\{a,b,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{d\}$
e	$\{a,c,d\}$	$\{a,b,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{e\}$

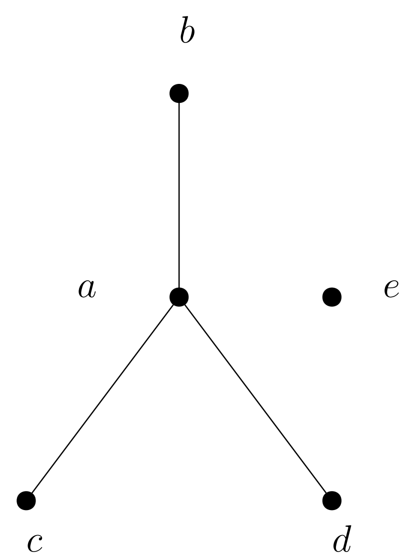

Figure 3. Figure that shows the order of Example 3.23.

The right ideals of H are the sets: $\{a,b,c,d\}$ and H.
The left ideals of H are the sets: $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,c,d,e\}$ and H.
The right and the left ideals of H are bi-ideals of H and there is no other bi-ideal of H except them. The quasi-ideals of H coincide with the bi-ideals of H.

Theorems 3.9, 3.12, and 3.18 can be applied.

Perhaps it is interesting to give the following example as well. Theorem 3.15 can also be applied to this example.

Example 3.24 *(see also [7, Example 4]) The ordered hypersemigroup* $H = \{a, b, c, d, e\}$ *defined by Table 5 and Figure 4 is not regular and not intraregular.*

Table 5. The hyperoperation of Example 3.24.

$\circ$	a	b	c	d	e
a	$\{b\}$	$\{d\}$	$\{a,d\}$	$\{b\}$	$\{b,d,e\}$
b	$\{d\}$	$\{b\}$	$\{b\}$	$\{d\}$	$\{b,d,e\}$
c	$\{d\}$	$\{b\}$	$\{b,c\}$	$\{d\}$	$\{b,d,e\}$
d	$\{b\}$	$\{d\}$	$\{d\}$	$\{b\}$	$\{b,d,e\}$
e	$\{b,d,e\}$	$\{b,d,e\}$	$\{b,d,e\}$	$\{b,d,e\}$	$\{b,d,e\}$

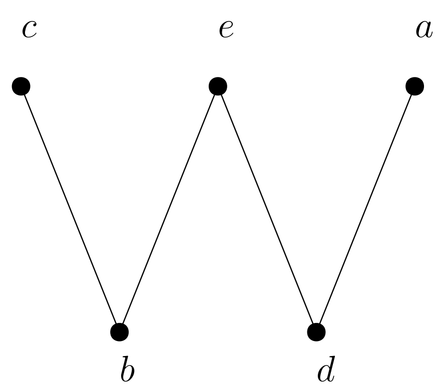

Figure 4. Figure that shows the order of Example 3.24.

The right ideals of H are the sets: $\{b, d, e\}$, $\{a, b, d, e\}$, $\{b, c, d, e\}$ and H.
The left ideals of H are the sets: $\{b, d, e\}$, $\{a, b, d, e\}$ and H.
The right and the left ideals of H are bi-ideals of H and there is no other bi-ideal of H. The quasi-ideals of H and the bi-ideals of H are the same.

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