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Properties in L_p of root functions for a nonlocal problem with involution

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Abstract: The spectral problem $-u''(x)+\alpha u''(-x)=\lambda u(x)$, $-1<x<1$, with nonlocal boundary conditions $u(-1)=\beta u(1)$, $u'(-1)=u'(1)$, is studied in the spaces $L_p(-1,1)$ for any $\alpha\in(-1,1)$ and $\beta\neq\pm1$. It is proved that if $r=\sqrt{(1-\alpha)/(1+\alpha)}$ is irrational then the system of its eigenfunctions is complete and minimal in $L_p(-1,1)$ for any $p>1$, but does not form a basis. In the case of a rational value of r, the way of supplying this system with associated functions is specified to make all the root functions a basis in $L_p(-1,1)$.

Key words: ODE with involution, nonlocal boundary-value problem, basicity, root functions

1. Introduction

Consider the problem

$$\begin{aligned} &Lu\equiv -u''(x)+\alpha u''(-x)=\lambda u(x), \quad -1<x<1,\\ &u(-1)=\beta u(1), \quad u'(-1)=u'(1), \end{aligned} \tag{1.1}$$

with the differential expression that contains the involution transformation of the argument x in its highest derivative, and an arbitrary parameter $\alpha\in(-1,1)$.

As the periodic case with $\beta=1$ was thoroughly discussed in [28] and the value $\beta=-1$ leads to a degenerate problem, we study the problem (1.1) for any real $\beta\neq\pm1$.

If $\alpha=\beta=0$ then one faces the well-known nonlocal problem of the Samarskii–Ionkin type [10] which delivers an example of a spectral problem for ODEs with an infinite number of associated functions. Il'in [8] called these spectral problems essentially nonself-adjoint and pointed out their intrinsic instability with respect to small perturbations of the differential expression and the choice of the associated functions [9, 12, 21–23].

This paper continues the research of [16] and shows that the considered boundary-value problem (1.1) encapsulates the same instability but with respect to its parameter α in all spaces $L_p(-1,1)$, $p>1$.

Proposition 1.1 *([16]) Denote*

$$r=\sqrt{(1-\alpha)/(1+\alpha)}, \tag{1.2}$$

and let $\beta\neq\pm1$. Then:

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1) for any positive r, the system of root functions of (1.1) is complete and minimal in $L_2(-1,1)$;

2) if r is irrational then there are no associated functions while the eigenfunctions of (1.1) do not constitute a basis in $L_2(-1,1)$;

3) if r is rational then there is an infinite number of associated functions that could be chosen to make the whole system of root functions of (1.1) an unconditional basis in $L_2(-1,1)$.

In this paper, we obtain an analogous result for any Lebesgue space $L_p(-1,1)$, $p>1$.

Theorem 1.2 *Let r in (1.2) be a positive irrational number and $\beta \neq \pm 1$. Then the system of eigenfunctions of (1.1) is complete and minimal in $L_p(-1,1)$ for all $p>1$, but is not uniformly minimal, and therefore does not constitute a basis in $L_p(-1,1)$.*

Theorem 1.3 *Let r in (1.2) be rational and $\beta \neq \pm 1$. Then the system of root functions of (1.1) is complete and minimal in $L_p(-1,1)$ for any $p>1$, and the associated functions could be chosen in such a way that the whole system forms a basis in $L_p(-1,1)$.*

We note that the case $\beta = 0$ was studied in detail in [17, 18].

It is also worth mentioning that, for all Lebesgue spaces $L_p(-1,1)$, $p>1$, the problem (1.1) demonstrates an unexpected stability of its characteristics with respect to the values of $\beta \neq \pm 1$. Both the absence of uniform minimality in Theorem 1.2 and the presence of basis property in Theorem 1.3 stay unchanged when the parameter $\alpha \in (-1,1)$ is fixed. This is a new effect, which was obscured in [16–18].

Since the 1970s the qualitative theory of differential equations with involution has been cultivated rather extensively (see, e.g., books by Przeworska-Rolewicz [26], Wiener [36], and Cabada and Tojo [5]). Spectral topics for first- and second-order operations that have involution in their main terms are discussed in [13–15, 32, 33]. Applications of the spectral approach for PDEs with involution and/or nonlocal boundary conditions are discussed in [1, 2, 25, 27, 29–31]. For the spectral properties of conventional differential operators in non-Hilbert spaces, one could refer to [4, 6, 7, 19, 20, 34].

2. The case of irrational number r

One can easily calculate the spectrum of (1.1):

$$\sigma(L) = \left\{0; (1 \pm \alpha)\pi^2 n^2 \,|\, n \in \mathbb{N}\right\}, \tag{2.1}$$

and the corresponding eigenfunctions:

$$\begin{aligned} &\lambda_0 = 0: \; u_0(x) = (1-\beta)x + (1+\beta), \qquad \lambda_l' = (1+\alpha)\,\pi^2 l^2: \; u_l^{(1)}(x) = \sin(\pi l x), \\ &\lambda_k'' = (1-\alpha)\,\pi^2 k^2: \; u_k^{(2)}(x) = \cos(\pi k x) + \frac{(1-\beta)\cos\pi k}{(1+\beta)\sin(\pi r k)}\,\sin(\pi r k x), \qquad l,k \in \mathbb{N}. \end{aligned} \tag{2.2}$$

The dual system is formed by eigenfunctions of the adjoint problem

$$\begin{aligned} &Lv(x) = \lambda v(x), \quad -1 < x < 1, \\ &v(-1) = v(1), \quad (\alpha - \beta)v'(-1) = (\alpha\beta - 1)v'(1), \end{aligned} \tag{2.3}$$

namely,

$$\begin{aligned} &\lambda_0 = 0: \ v_0(x) = 1/(2+2\beta), \qquad \lambda''_k = (1-\alpha)\,\pi^2 k^2: \ v_k^{(2)}(x) = \cos(\pi k x), \\ &\lambda'_l = (1+\alpha)\,\pi^2 l^2: \ v_l^{(1)}(x) = \sin(\pi l x) + \frac{(1-\beta)\cos\pi l}{(1+\beta) r \sin(\pi l/r)} \cos\Big(\frac{\pi l x}{r}\Big), \qquad l,k \in \mathbb{N}. \end{aligned} \tag{2.4}$$

In order to study the basicity of systems (2.2) and (2.4), we start with the following result.

Lemma 2.1 *For any $\beta \neq \pm 1$, both systems (2.2) and (2.4) are complete and minimal in $L_p(-1,1)$ for any $p > 1$.*

Proof Recall that the system $\{e_n\}$ in a Banach space $\mathcal{B}$ is complete in $\mathcal{B}$ if it spans $\mathcal{B}$ and is minimal if neither element in this system belongs to the span of others. It is known that the minimality of a system is provided by existence of a dual system. Therefore, in our case it is sufficient to prove completeness of (2.2) and (2.4) in $L_p(-1,1)$ which is equivalent to their totality in $L_q(-1,1)$, $q^{-1} + p^{-1} = 1$.

For instance, consider an element $f \in L_q(-1,1)$, which is orthogonal to each function in (2.2). Then, as $f(x)$ is orthogonal to the sine-functions and due to the fact that the trigonometric system forms a basis in L_q [11, p. 128], the function $f(x)$ a.e. coincides with an even function. Thus, we have

$$0 = \int_{-1}^{1} f(x) u_k^{(2)}(x)\, dx = \int_{-1}^{1} f(x) \cos(\pi k x)\, dx,$$

and therefore $f(x)$ is a.e. constant on $[-1,1]$. The relations $\int_{-1}^{1} f(x) u_0(x)\, dx = 0$ and $\int_{-1}^{1} u_0(x)\, dx = 2+2\beta \neq 0$ provide that $f(x)$ vanishes a.e. on $[-1,1]$. System (2.4) is analyzed similarly. □

A system $\{e_n\} \subset \mathcal{B}$ is called a basis in $\mathcal{B}$ if, for any $f \in \mathcal{B}$, there exists a unique convergent to f series: $\sum_{n=1}^{\infty} \alpha_n e_n = f$. Any basis in $\mathcal{B}$ is complete and minimal and thus has a unique dual system $\{e_n^*\} \subset \mathcal{B}^*$. Moreover, $\alpha_n = e_n^*(f)$ for any n and the series for f is called biorthogonal.

Let us prove now that, in the case of irrational r, both systems (2.2) and (2.4) do not form bases in $L_p(-1,1)$. It actually follows from the fact [24] that these systems are not uniformly minimal in $\mathcal{B} = L_p(-1,1)$, i.e. the property

$$\sup_n \Big(\|e_n\| \cdot \|e_n^*\| \Big) < \infty \tag{2.5}$$

is violated.

Lemma 2.2 *For any $\beta \neq \pm 1$, neither system (2.2) nor system (2.4) is uniformly minimal in $L_p(-1,1)$, $p > 1$.*

Proof Let us consider the system (2.2) in the space $L_p(-1,1)$. The L_q-norms of the functions $v_k^{(2)}(x)$ in (2.4) satisfy the estimates

$$2^{1/q} \geq \|v_k^{(2)}\|_q \geq 2^{-1/p} \|v_k^{(2)}\|_1 \geq 2^{-1/p} \|v_k^{(2)}\|_2^2 = 2^{-1/p}. \tag{2.6}$$

We show that there exists a sequence k_n of positive integers such that the norms $\|u_{k_n}^{(2)}\|_p$ tend to infinity. Indeed, the L_1-norm of the function $u_k^{(2)}(x)$ satisfies the inequalities

$$\int_{-1}^{1} |u_k^{(2)}(x)|\, dx \geq \Big| \frac{1-\beta}{(1+\beta)\sin(\pi r k)} \Big| \int_{-1}^{1} |\sin(\pi r k x)|\, dx - 2 \geq \Big| \frac{1-\beta}{(1+\beta)\sin(\pi r k)} \Big| \Big(1 - \frac{\sin(2\pi k r)}{2\pi k r} \Big) - 2. \tag{2.7}$$

It follows from [11, p.25] that the inequality $\left|\frac{1}{r}-\frac{k}{s}\right|<\frac{1}{s^2}$ has infinitely many solutions $k=k_n, s=s_n\in\mathbb{N}$. Hence, $|\pi r k_n-\pi s_n|<\frac{\pi r}{s_n}$ and $|\sin(\pi r k_n)|<|\sin(\pi r/s_n)|$. Therefore, the factor on the right-hand side of (2.7) tends to infinity as $k=k_n\to\infty$, which means that the norm

$$\|u^{(2)}_{k_n}\|_p\geq 2^{(1-p)/p}\|u^{(2)}_{k_n}\|_1$$

blows up. □

Lemmas 2.1 and 2.2 yield the result of Theorem 1.2.

3. The case of rational number r

Now let r be equal to some irreducible fraction $\frac{m_1}{m_2}$ $(m_1,m_2\in\mathbb{N})$. Then the spectrum (2.1) contains two merging subsequences:

$$\lambda^*_n\equiv\lambda'_{m_1n}=\lambda''_{m_2n}\qquad\forall n\in\mathbb{N}. \tag{3.1}$$

The corresponding eigenfunctions are no more linearly independent, all eigenvalues λ^*_n have multiplicity 2, and each of their root subspaces is the linear span of one eigenfunction and one associated function. Straightforward calculation shows that the biorthogonal pairs are formed by the functions (compare with (2.2) and (2.4)):

$$\begin{aligned}
&u_0(x),\quad u^{(1)}_l(x),\ l\not\equiv 0\,(\mathrm{mod}\ m_1),\quad u^{(2)}_k(x),\ k\not\equiv 0\,(\mathrm{mod}\ m_2),\\
&u^*_n(x)=\sin(\pi m_1 nx),\\
&u^*_{n,1}(x)=\Big(2(1+\alpha)\pi m_1 n\Big)^{-1}\Big[x\cos(\pi m_1 nx)+\frac{1+\beta}{1-\beta}(-1)^{(m_1+m_2)n}\cos(\pi m_2 nx)\Big]+a_n u^*_n(x),\quad n\in\mathbb{N},
\end{aligned} \tag{3.2}$$

for the direct problem (1.1) and

$$\begin{aligned}
&v_0(x),\quad v^{(1)}_l(x),\ l\not\equiv 0\,(\mathrm{mod}\ m_1),\quad v^{(2)}_k(x),\ k\not\equiv 0\,(\mathrm{mod}\ m_2),\\
&v^*_n(x)=2(1+\alpha)\pi m_1 n\frac{1-\beta}{1+\beta}(-1)^{(m_1+m_2)n}\cos(\pi m_2 nx),\\
&v^*_{n,1}(x)=-\frac{1-\beta}{r(1+\beta)}(-1)^{(m_1+m_2)n}x\sin(\pi m_2 nx)+\sin(\pi m_1 nx)-a_n v^*_n(x),\quad n\in\mathbb{N},
\end{aligned} \tag{3.3}$$

for the adjoint problem (2.3) (the constants $a_n\in\mathbb{R}$ could be chosen arbitrarily).

Lemma 3.1 *Let $\beta\neq\pm1$. Then systems (3.2) and (3.3) are complete and minimal in $L_p(-1,1)$ for any $p>1$.*

The proof of Lemma 3.1 is similar to the proof of Lemma 2.1.

Lemma 3.2 *Let $\beta\neq\pm1$. If $a_n=O(1/n)$, $n\to\infty$, then both systems (3.2) and (3.3) are uniformly minimal in $L_p(-1,1)$, $p>1$. If $\lim\limits_{n\to\infty} na_n=\infty$ then these systems are not uniformly minimal and therefore do not form bases.*

Proof Consider the biorthogonal pair $u_l^{(1)}(x)$ and $v_l^{(1)}(x)$, $\forall l \not\equiv 0\,(\text{mod } m_1)$. It is clear that

$$\|u_l^{(1)}\|_p \leq 2^{1/p}, \quad \|v_l^{(1)}\|_q \leq 2^{1/q}\Big(1 + \Big|\frac{1-\beta}{(1+\beta) r \sin(\pi l/r)}\Big|\Big)$$

and the estimate $|\sin(\pi l/r)| \geq \sin(\pi/m_1)$ holds as the number $l/r = lm_2/m_1$ is not an integer. Therefore, the product $\|u_l^{(1)}\|_p \cdot \|v_l^{(1)}\|_q$ is bounded for all $l \not\equiv 0\,(\text{mod } m_1)$.

Similarly, one can evaluate the product $\|u_k^{(2)}\|_p \cdot \|v_k^{(2)}\|_q$ for all $k \not\equiv 0\,(\text{mod } m_2)$.

For the eigenvalues $\lambda = \lambda_n^*$ and the related eigenfunctions, the estimates

$$c_1 \leq \|u_n^*\|_p \leq c_2, \quad c_1 n \leq \|v_n^*\|_q \leq c_2 n$$

are apparently valid with some positive constants c_1, c_2. As for the associated functions, the condition $a_n = O(1/n)$ yields the estimates

$$\|u_{n,1}^*\|_p = O(1/n), \quad \|v_{n,1}^*\|_q = O(1),$$

and therefore the uniform minimality condition (2.5) is satisfied.

If $\lim\limits_{n\to\infty} n a_n = \infty$ then the situation alters and we get the estimates

$$\|u_{n,1}^*\|_p \geq c_3|a_n| > 0, \quad \|v_{n,1}^*\|_q \geq c_3|a_n| n,$$

which show that the condition (2.5) breaks. □

Further, we consider systems (3.2) and (3.3) only with $a_n \equiv 0$ for all n. Gaposhkin's theorem [24] states that, for any $p > 1$, $p \neq 2$, these systems could form only conditional bases in $L_p(-1,1)$. Therefore, we need to arrange them before studying their basicity. The arrangement naturally corresponds to that of the classical trigonometric system.

The ordered biorthogonal system starts with the pair $\begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix}$, which is followed by the juxtaposed blocks of coupled pairs

$$\begin{bmatrix} u_k^{(1)}(x) & u_k^{(2)}(x) \\ v_k^{(1)}(x) & v_k^{(2)}(x) \end{bmatrix}, \quad k = 1, 2, \ldots. \tag{3.4}$$

However, if $k \equiv 0\,(\text{mod } m_1)$, then the first column of the block (3.4) should be replaced by the column

$$\begin{bmatrix} u_n^*(x) \\ v_{n,1}^*(x) \end{bmatrix},$$

and if $k \equiv 0\,(\text{mod } m_2)$ then also the second column is replaced by the column

$$\begin{bmatrix} u_{n,1}^*(x) \\ v_n^*(x) \end{bmatrix}.$$

In order to analyze the biorthogonal series, we split its partial sum $S_N(x,f)$ in the following way (here $\mathbb{K}_1 = m_1\mathbb{N}$ and $\mathbb{K}_2 = m_2\mathbb{N}$):

$$\begin{aligned} S_N(x,f) = (f,v_0)\,u_0(x) &+ \sum_{\substack{1\le k\le N\\ k\notin\mathbb{K}_1}} \left(f, v_k^{(1)}\right) u_k^{(1)}(x) + \sum_{\substack{1\le k\le N\\ k\notin\mathbb{K}_2}} \left(f, v_k^{(2)}\right) u_k^{(2)}(x) \\ &+ \sum_{\substack{1\le k\le N\\ k\in\mathbb{K}_1}} \left(f, v_{n,1}^*\right) u_n^*(x) + \sum_{\substack{1\le k\le N\\ k\in\mathbb{K}_2}} \left(f, v_n^*\right) u_{n,1}^*(x). \end{aligned} \tag{3.5}$$

This sum apparently contains the partial sum of Fourier trigonometric series:

$$S_N^{(0)}(x,f) = (f(t),1/2) + \sum_{k=1}^{N} \Big\{ \Big(f(t),\cos(\pi k t)\Big)\, \cos(\pi k x) + \Big(f(t),\sin(\pi k t)\Big)\, \sin(\pi k x) \Big\}, \tag{3.6}$$

and the remaining items are

$$\begin{aligned} &\left(f(t), \frac{1-\beta}{2+2\beta}\right) x, \\ &S_N^{(1)}(x,f) = \frac{1-\beta}{(1+\beta)r} \sum_{\substack{1\le k\le N\\ k\notin\mathbb{K}_1}} \frac{\cos \pi k}{\sin(\pi k/r)} \left(f(t), \cos\Big(\frac{\pi k t}{r}\Big)\right)\, \sin(\pi k x), \\ &S_N^{(2)}(x,f) = \frac{1-\beta}{1+\beta} \sum_{\substack{1\le k\le N\\ k\notin\mathbb{K}_2}} \frac{\cos \pi k}{\sin(\pi k r)} \Big(f(t), \cos(\pi k t)\Big)\, \sin(\pi k r x), \\ &S_N^{(3)}(x,f) = -\frac{1-\beta}{(1+\beta)r} \sum_{\substack{1\le k\le N\\ k\in\mathbb{K}_1}} (-1)^{(1+r)k/r} \left(f(t), t\sin\Big(\frac{\pi k t}{r}\Big)\right) \sin(\pi k x), \\ &S_N^{(4)}(x,f) = \frac{1-\beta}{1+\beta} x \sum_{\substack{1\le k\le N\\ k\in\mathbb{K}_2}} (-1)^{(1+r)k} \Big(f(t), \cos(\pi k t)\Big)\, \cos(\pi k r x). \end{aligned} \tag{3.7}$$

To analyze these sums for a given $f \in L_p(-1,1)$, we decompose $f(x)$ into the sum of its even and odd components, $f(x) = f_+(x) + f_-(x)$ and note that for the odd component $f_-(x)$ all the sums in (3.7) vanish.

In $S_N^{(3)}(x,f_+)$, we substitute $k = m_1 n$ and, without loss of generality, suppose that $m_1 + m_2$ is even. Then it takes the form

$$S_N^{(3)}(x,f_+) = -\frac{2-2\beta}{(1+\beta)r} \sum_{n:\,1\le m_1 n\le N} \int_0^1 f_+(t) t \sin(\pi m_2 n t)\, dt \cdot \sin(\pi m_1 n x),$$

which, under the transform $\tau = m_2 t, y = m_1 x$, equals

$$-\frac{2-2\beta}{(1+\beta) r m_2^2} \sum_{n:\,1\le m_1 n\le N} \int_0^{m_2} f_+\Big(\frac{\tau}{m_2}\Big) \tau \sin(\pi n \tau)\, d\tau \cdot \sin(\pi n y).$$

The latter sum is clearly a sum of m_2 partial sums of Fourier trigonometric series for functions with L_p-norms, which are $O(1)\|f\|_p$.

The same reasoning gives a similar representation for $S_N^{(4)}(x, f_+)$.

The sum $S_N^{(2)}(x, f_+)$ could be split into $m_2 - 1$ items in accordance with the remainder $k_1 = k \pmod{m_2}$. Then, for instance, the item $S_N^{(2,k_1)}(x, f_+)$ with even k_1 and $m_1 + m_2$ equals

$$\frac{1-\beta}{(1+\beta)\sin(\pi k_1 r)} \sum_{\substack{1 \le k \le N \\ k = k_1 + m_2 n}} \left\{ \int_0^1 f(t)\cos(\pi k_1 t)\cos(\pi m_2 n t)\,dt \Big[\cos(\pi m_1 n x)\sin(\pi k_1 r x) + \sin(\pi m_1 n x)\cos(\pi k_1 r x)\Big] \right.$$
$$\left. - \int_0^1 f(t)\sin(\pi k_1 t)\sin(\pi m_2 n t)\,dt \Big[\cos(\pi m_1 n x)\sin(\pi k_1 r x) + \sin(\pi m_1 n x)\cos(\pi k_1 r x)\Big] \right\}.$$

This expression consists of four items, which are linear combinations of the partial sums of Fourier trigonometric series for some functions for which L_p-norms are $O(1)\|f\|_p$, and of the partial sums of conjugate trigonometric series, which converge in $L_p(0,1)$ to functions with L_p-norms that are also $O(1)\|f\|_p$ due to Riesz's theorem [3, p. 566]. The sum $S_N^{(1)}(x, f_+)$ is considered similarly.

It follows from [3, pp. 593–594] that if $F(x) \in L_p$ then the partial sums $\sigma_N(x, F)$ of its Fourier trigonometric series and the partial sums $\sigma_N^*(x, F)$ of its conjugate series satisfy the estimates

$$\|\sigma_N(x, F)\|_p \le c\|F\|_p, \quad \|\sigma_N^*(x, F)\|_p \le c\|F\|_p$$

uniformly with respect to N with some positive constant c.

Relations (3.5)–(3.7) and the above reasoning thus prove that

$$\|S_N(x, f)\|_p \le \left\| \left(f, \frac{1-\beta}{2+2\beta} \right) x \right\|_p + \left\| S_N^{(0)}(x, f) \right\|_p + \sum_{j=1}^{4} \left\| S_N^{(j)}(x, f_+) \right\|_p = O(1)\|f\|_p \tag{3.8}$$

uniformly with respect to N.

The system of root functions (3.2) is complete and minimal in $L_p(-1,1)$ by Lemma 3.1; therefore (see, e.g., [11, p. 11]), the estimate (3.8) is sufficient for its basicity in $L_p(-1,1)$. Theorem 1.3 is proved.

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References

[1] Aleroev TS, Kirane M, Malik SA. Determination of a source term for a time fractional diffusion equation with an integral type over-determining condition. Electron J Differ Eq 2013; 2013: 1-16.

[2] Ashyralyev A, Sarsenbi AM. Well-posedness of a parabolic equation with nonlocal boundary condition. Bound Value Probl 2015; 2015: 38.

[3] Bari NK. A Treatise on Trigonometric Series. New York, NY, USA: Pergamon Press, 1964.

[4] Benzinger HE. The L^p behavior of eigenfunction expansions. T Am Math Soc 1972; 174: 333-344.

[5] Cabada A, Tojo FAF. Differential Equations with Involutions. Amsterdam, the Netherlands: Atlantis Press, 2015.

[6] Gomilko AM, Radzievskii GV. Equivalence in $L_p[0,1]$ of the system $e^{i2\pi kx}$ $(k = 0, \pm 1, \dots)$ and the system of the eigenfunctions of an ordinary functional-differential operator. Math Notes 1991; 49: 34-40.

[7] Huseynli A. On the stability of basisness in L_p $(1 < p < \infty)$ of cosines and sines. Turk J Math 2011; 35: 47-54.

[8] Il'in VA. Basis property and equiconvergence with the trigonometric series of root function expansions and the form of boundary conditions for a nonself-adjoint differential operator. Diff Equat 1994; 30: 1402-1413.

[9] Il'in VA, Kritskov LV. Properties of spectral expansions corresponding to nonself-adjoint differential operators. J Math Sci NY 2003; 116: 3489-3550.

[10] Ionkin NI. Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition. Diff Equat 1977; 13: 204-211.

[11] Kashin BS, Saakyan AA. Orthogonal Series. New York, NY, USA: AMS Translations of Mathematics Monographs, 1989.

[12] Kerimov NB. On a boundary value problem of N. I. Ionkin type. Diff Equat 2013; 49: 1233-1245.

[13] Khromov AP, Kurdyumov VP. Riesz bases formed by root functions of a functional-differential equation with a reflection operator. Differ Equat 2008; 44: 203-212.

[14] Kopzhassarova AA, Lukashov AL, Sarsenbi AM. Spectral properties of non-self-adjoint perturbations for a spectral problem with involution. Abstr Appl Anal 2012; 2012: 590781.

[15] Kopzhassarova AA, Sarsenbi AM. Basis properties of eigenfunctions of second-order differential operators with involution. Abstr Appl Anal 2012; 2012: 576843.

[16] Kritskov LV, Sadybekov MA, Sarsenbi AM. Nonlocal spectral problem for a second-order differential equation with an involution. B Karaganda Univ Math 2018; 3: 53-60.

[17] Kritskov LV, Sarsenbi AM. Spectral properties of a nonlocal problem for the differential equation with involution. Diff Equat 2015; 51: 984-990.

[18] Kritskov LV, Sarsenbi AM. Basicity in L_p of root functions for differential equations with involution. Electron J Differ Eq 2015; 2015: 1-9.

[19] Kurbanov VM. On an analog of the Riesz theorem and the basis property of the system of root functions of a differential operator in L_p I. Diff Equat 2013; 49: 7-19.

[20] Kurbanov VM. On an analog of the Riesz theorem and the basis property of the system of root functions of a differential operator in L_p II. Diff Equat 2013; 49: 437-449.

[21] Kurbanov VM, Safarov RA. On the influence of the potential on the convergence rate of expansions in root functions of the Schrödinger operator. Diff Equat 2010; 46: 1077-1084.

[22] Lomov IS. Nonsmooth eigenfunctions in problems of mathematical physics. Diff Equat 2011; 47: 355-362.

[23] Makin AS. A class of essentially nonself-adjoint boundary value problems. Dokl Math 2014; 89: 351-353.

[24] Mil'man VD. Geometric theory of Banach spaces. Part I. Russian Math Surveys 1970; 25: 111-170.

[25] Mokin AYu. On a family of initial-boundary value problems for the heat equation. Diff Equat 2009; 45: 126-141.

[26] Przeworska-Rolewicz D. Equations with Transformed Argument. Algebraic Approach. Amsterdam, the Netherlands: Elsevier, 1973.

[27] Sadybekov M, Dildabek G, Ivanova M. Spectral properties of a Frankl type problem for parabolic-hyperbolic equations. Electron J Differ Eq 2017; 65: 1-11.

[28] Sadybekov M, Dildabek G, Ivanova M. On an inverse problem of reconstructing a heat conduction process from nonlocal data. Adv Math Phys 2018; 2018: 8301656.

[29] Sadybekov MA, Dildabek G, Tengayeva A. Constructing a basis from systems of eigenfunctions of one not strengthened regular boundary value problem. Filomat 2017; 31: 981-987.

[30] Sadybekov MA, Oralsyn G. Nonlocal initial boundary value problem for the time-fractional diffusion equation. Electron J Differ Eq 2017; 201: 1-7.

[31] Sadybekov MA, Oralsyn G, Ismailov M. Determination of a time-dependent heat source under not strengthened regular boundary and integral overdetermination conditions. Filomat 2018; 32: 809-814.

[32] Sadybekov MA, Sarsenbi AM. Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution. Diff Equat 2012; 48: 1112-1118.

[33] Sadybekov MA, Sarsenbi AM. Mixed problem for a differential equation with involution under boundary conditions of a general form. In: AIP Conference Proceedings vol. 1470; 2012. pp. 225-227.

[34] Serov V. Green's function and convergence of Fourier series for elliptic differential operators with potential from Kato space. Abstr Appl Anal 2010; 2010: 902638.

[35] Shidlovskii AB. Transcendental Numbers. Berlin, Germany: de Gruyter, 1989.

[36] Wiener J. Generalized Solutions of Functional Differential Equations. Singapore: World Scientific, 1993.